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Spectral asymptotics and metastability for the linear relaxation Boltzmann equation

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Abstract. We consider the linear relaxation Boltzmann equation in a semiclassical framework. We construct a family of sharp quasimodes for the associated operator which yields sharp spectral asymptotics for its small spectrum in the low temperature regime. We deduce some information on the long time behavior of the solutions with a sharp estimate on the return to equilibrium as well as a quantitative metastability result. The main novelty is that the collision operator is a pseudo-differential operator in the critical class $S^{1/2}$ and that its action on the Gaussian quasimodes yields a superposition of exponentials.

1. Introduction

1.1. Motivations

We are interested in the linear Boltzmann equation

$$\begin{cases} h\partial_t u + v \cdot h\partial_x u - \partial_x V \cdot h\partial_v u + Q_{\mathcal{H}}(h,u) = 0, \\ u_{|t=0} = u_0, \end{cases} \tag{1.1}$$

in a semiclassical framework (i.e., in the limit $h \to 0$), where h is a *semiclassical parameter* and corresponds to the temperature of the system. Here for shortness we denoted by ∂_x and ∂_v the partial gradients with respect to x and v. This equation is used to model the evolution of a system of charged particles in a gas on which acts an electrical force associated to the real valued potential V that only depends on the space variable x. The operator $Q_{\mathcal{H}}$ is called *collision operator* and models the interactions between the particles. Here the unknown is the function $u: \mathbb{R}_+ \to L^1(\mathbb{R}^{2d})$ giving the probability density of the system of particles at time $t \in \mathbb{R}_+$, position $x \in \mathbb{R}^d$ and velocity $v \in \mathbb{R}^d$. For our purpose, we introduce the square roots of the usual

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Maxwellian distributions

$$\mu_h(v) = \frac{\mathrm{e}^{-\frac{v^2}{4h}}}{(2\pi h)^{d/4}} \quad \text{and} \quad \mathcal{M}_h = \mathrm{e}^{-\frac{V}{2h}} \mu_h. \tag{1.2}$$

This paper is devoted to the study of the linear BGK model for which the collision operator is

$$Q_{\mathcal{H}}(h,u) = h\Big(u - \int_{v'\in\mathbb{R}^d} u(x,v')\,\mathrm{d}v'\mu_h^2\Big) \tag{1.3}$$

and corresponds to a simple relaxation towards the Maxwellian. Denoting by $Q_{\mathcal{H}}^*(h,\cdot)$ the formal adjoint of $Q_{\mathcal{H}}(h,\cdot)$, one can easily compute

$$Q_{\mathcal{H}}(h,\mathcal{M}_h^2) = 0 \quad \text{and} \quad Q_{\mathcal{H}}^*(h,1) = 0; \tag{1.4}$$

so, in particular, $\mathcal{M}_h^2$ is a stable state of (1.1) and $Q_{\mathcal{H}}$ features the local conservation of mass. In order to do a perturbative study of the time independent operator associated to (1.1) near $\mathcal{M}_h^2$, we introduce the natural Hilbert space

$$\mathcal{H} = \{u \in \mathcal{D}' : \mathcal{M}_h^{-1}u \in L^2(\mathbb{R}^{2d})\}.$$

It is clear from the Cauchy–Schwarz inequality that $\mathcal{H}$ is indeed a subset of $L^1(\mathbb{R}^{2d})$ provided that $\mathrm{e}^{-\frac{V}{2h}} \in L^2(\mathbb{R}^d_x)$. In view of (1.4) and the definition of $\mathcal{H}$, it is more convenient to work with the new unknown

$$f = \mathcal{M}_h^{-1}u \colon \mathbb{R}_+ \to L^2(\mathbb{R}^{2d})$$

for which the new equation becomes

$$\begin{cases} h\partial_t f + v\cdot h\partial_x f - \partial_x V\cdot h\partial_v f + Q_h(f) = 0, \\ f_{|t=0} = f_0, \end{cases} \tag{1.5}$$

where

$$Q_h = \mathcal{M}_h^{-1} \circ Q_{\mathcal{H}}(h,\cdot) \circ \mathcal{M}_h. \tag{1.6}$$

With the notation (1.2), denoting by

$$\Pi_h \colon L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^{2d}),$$

the orthogonal projection on $\mu_h L^2(\mathbb{R}^d_x)$, we have, by (1.3) and (1.6),

$$Q_h = h(\mathrm{Id} - \Pi_h). \tag{1.7}$$

Our study will be focused on the spectral properties of the new time independent operator

$$P_h = v \cdot h\partial_x - \partial_x V \cdot h\partial_v + h(\mathrm{Id} - \Pi_h) = X_0^h + Q_h$$

where the notation X_0^h will stand for the operator $v \cdot h\partial_x - \partial_x V \cdot h\partial_v$, but also for the vector field $(x, v) \mapsto h(v, -\partial_x V(x))$.

This type of questions has recently known some major progress on the impulse of microlocal methods. The operator P_h was already studied in 2016 in [13], where the use of hypocoercive techniques enabled to get some resolvent estimates and establish a rough localization of its small spectrum which consists of exponentially small eigenvalues in correspondance with the minima of the potential V. This type of result is similar to the one obtained for example for the Witten Laplacian by Helffer and Sjöstrand in [5] in the 1980's. Such a localization already leads to return to equilibrium and metastability results which can be improved as the description of the small spectrum becomes more precise. For example, sharp asymptotics of the small eigenvalues of the Witten Laplacian were obtained later in the 2000's in [2, 4], and later again for Kramers–Fokker–Planck-type operators by Hérau et al. in [6]. In these papers, the idea was to exhibit a supersymmetric structure for the operator and then study both the derivative acting from 0-forms into 1-forms and its adjoint with the help of basic quasimodes. However, these methods do not apply to the Boltzmann equation as in that case the matrix appearing in the modification of the inner product does not obey good estimates with respect to the semiclassical parameter h (see for instance [12] for the case of the *mild relaxation* collision operator).

This is why our goal in this paper will be to give precise spectral asymptotics for the operator P_h through a more recent approach which consists in directly constructing a family of accurate *Gaussian quasimodes* for our operator in the spirit of [1,8] for Fokker–Planck-type differential operators and [11] for the mild relaxation Boltzmann equation. Here the first difficulty is that like in [11], the operator that we consider is non-local and hence it is harder to compute its action on the constructed quasimodes. This will be overcome thanks to the factorization result stated in Proposition 2.2. The second and main difficulty is that, unlike in [11], the bad microlocal properties of Q_h are such that its action on a Gaussian quasimode as used in [1, 8, 11] does not yield a precise exponential, but rather a superposition of exponentials (see Lemma 2.4) wich will lead to the introduction of some new quasimodes given by a superposition of "usual" Gaussian quasimodes. The result that we manage to establish is similar to the one from [4] for the Witten Laplacian as well as the ones from [6, 7] with recent improvements by Bony et al. in [1] for the Fokker–Planck equation.

1.2. Setting and main results

For $d' \in \mathbb{N}^*$ and $Z \in \mathbb{C}^{d'}$, we use the standard notation $\langle Z \rangle = (1+|Z|^2)^{1/2}$. Let us introduce a few notations of semiclassical microlocal analysis which will be used in all this paper. These are mainly extracted from [14, Chapter 4]. For our purpose, it is sufficient to consider pseudo-differential operators acting only in the variable v. We will denote $\eta \in \mathbb{R}^d$ the dual variable of v and use the semiclassical Fourier transform

$$\mathcal{F}_h(f)(\eta) = \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{i}{h} v\cdot\eta} f(v)\,\mathrm{d}v.$$

We consider the space of semiclassical symbols

$$S^\kappa(\langle (v,\eta)\rangle^k) = \{a_h \in \mathcal{C}^\infty(\mathbb{R}^{2d}) : \text{for all } \alpha \in \mathbb{N}^{2d} \text{ there exists } C_\alpha > 0 \text{ such that } |\partial^\alpha a_h(v,\eta)| \leq C_\alpha h^{-\kappa|\alpha|}\langle (v,\eta)\rangle^k\}$$

where $k \in \mathbb{R}$ and $\kappa \in \left[0, \frac{1}{2}\right]$. Given a symbol $a_h \in S^\kappa(\langle (v,\eta)\rangle^k)$, we define the associated semiclassical pseudo-differential operator for the Weyl quantization acting on functions $u \in \mathcal{S}(\mathbb{R}^d)$ by

$$\mathrm{Op}_h(a_h)u(v) = (2\pi h)^{-d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{e}^{\frac{i}{h}(v-v')\cdot\eta}\, a_h\Big(\frac{v+v'}{2}, \eta\Big) u(v')\,\mathrm{d}v'\,\mathrm{d}\eta$$

where the integrals may have to be interpreted as oscillating integrals. We will denote by $\Psi^\kappa(\langle (v,\eta)\rangle^k)$ the set of such operators. Note that the operator $\mathrm{Op}_h(a_h)$ admits the distributional kernel

$$K_h(v,v') = \mathcal{F}_h^{-1}\Big(a_h\Big(\frac{v+v'}{2},\cdot\Big)\Big)(v-v').$$

Conversely, if an operator $\mathrm{Op}_h(a_h) \in \Psi^\kappa(\langle (v,\eta)\rangle^k)$ admits the distributional kernel $K_h(v,v')$, then its symbol is given by

$$a_h(v,\eta) = \mathcal{F}_h((K_h \circ A)(v,\cdot))(\eta) \tag{1.8}$$

where A denotes the change of variables

$$A(v,v') = \Big(v + \frac{v'}{2}, v - \frac{v'}{2}\Big).$$

We will also make a few confining assumptions on the function V, assuring for instance that the bottom spectrum of the associated Witten Laplacian is discrete. In particular, our potential will satisfy [8, Assumption 2] and [13, Hypothesis 1.1].

Hypothesis 1.1. The potential V is a smooth Morse function depending only on the space variable $x \in \mathbb{R}^d$ with values in $\mathbb{R}$ which is bounded from below and such that

$$|\partial_x V(x)| \geq \frac{1}{C} \quad \text{for } |x| > C.$$

Moreover, for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \geq 2$, there exists C_α such that

$$|\partial_x^\alpha V| \leq C_\alpha.$$

In particular, for every $0 \leq k \leq d$, the set of critical points of index k of V that we denote $\mathtt{U}^{(k)}$ is finite and we set

$$n_0 = \#\mathtt{U}^{(0)}. \tag{1.9}$$

Finally, we will suppose that $n_0 \geq 2$.

The last assumption comes from the fact that when $n_0 = 1$, the so-called *small spectrum* of the operator P_h (i.e., its eigenvalues with exponentially small modulus) is trivial, so there is nothing to study. It is shown in [9, Lemma 3.14] that, for a function V satisfying Hypothesis 1.1, we have $V(x) \geq \frac{|x|}{C}$ outside of a compact. In particular, under Hypothesis 1.1, it holds $\mathrm{e}^{-V/2h} \in L^2(\mathbb{R}^d_x)$. Moreover, in our setting, X_0^h is a smooth vector field whose differential is bounded on $\mathbb{R}^{2d}$, so the operator X_0^h endowed with the domain

$$D = \{u \in L^2(\mathbb{R}^{2d}) : X_0^h u \in L^2(\mathbb{R}^{2d})\}$$

is skew-adjoint on $L^2(\mathbb{R}^{2d})$ and the set $\mathcal{S}(\mathbb{R}^{2d})$ is a core for this operator. Since moreover the collision operator Q_h defined in (1.7) is bounded and self-adjoint, we have $(P_h, D)^* = (-X_0^h + Q_h, D)$ and (P_h, D) is m-accretive on $L^2(\mathbb{R}^{2d})$.

For an operator such as P_h, which is not for instance self-adjoint with compact resolvent, we do not have any information a priori on its spectrum (except here that it is contained in $\{z \in \mathbb{C} : \mathrm{Re}\ z \geq 0\}$). In [13], the use of hypocoercive techniques enabled to establish a first description of the spectrum of P_h near 0 which, in the spirit of the case of other non-self-adjoint operators studied in [6], appears in particular to be discrete. More precisely, the following result is shown in [13].

Theorem 1.2. *Assume that Hypothesis 1.1 is satisfied and recall the notation (1.9). Then the operator (P_h, D) admits 0 as a simple eigenvalue. Moreover, there exists $c > 0$ and $h_0 > 0$ such that, for all $0 < h \leq h_0$, the low lying spectrum* $\mathrm{Spec}(P_h) \cap \{\mathrm{Re}\, z \leq ch\}$ *consists of exactly n_0 eigenvalues (counted with algebraic multiplicity) which are real and exponentially small with respect to $\frac{1}{h}$. Finally, for all $0 < \tilde{c} \leq c$, the resolvent estimate*

$$(P_h - z)^{-1} = O(h^{-1})$$

holds uniformly in $\{\mathrm{Re}\, z \leq ch\} \backslash B(0, \tilde{c}h)$.

In order to study the long time behavior of the solutions of (1.5), we need a precise description of the small spectrum of P_h. To this aim, we construct in Section 3 a family of accurate quasimodes localized around the minima of V that enables us to establish sharp asymptotics of the small eigenvalues of P_h. This will lead to the following theorem which is the main result of this paper. Before we can state it, let us introduce a few notations that we will use throughout the paper. We denote by

$$W(x,v) = \frac{V(x)}{2} + \frac{v^2}{4} \tag{1.10}$$

the global potential on $\mathbb{R}^{2d}$ and, for $x \in \mathbb{R}^d$, by

$$\mathcal{V}_x \text{ (resp. } \mathcal{W}_x\text{) the Hessian of } V \text{ at } x \text{ (resp. the Hessian of } W \text{ at } (x,0)). \tag{1.11}$$

When $\mathbf{s} \in \mathbb{R}^d$ is a saddle point of V (i.e., $\mathbf{s} \in \mathtt{U}^{(1)}$), we also denote by

$$\tau_{\mathbf{s}} \text{ the only negative eigenvalue of } \mathcal{V}_{\mathbf{s}}. \tag{1.12}$$

For the sake of simplicity, we will make in the statement of the theorem an additional assumption (Hypothesis 2.8) on the topology of the potential V that could actually be omitted (see [10] or [1]). It implies in particular that V has a unique global minimum that we denote by $\underline{\mathbf{m}}$.

According to Theorem 1.2, we can associate to each $\mathbf{m} \in \mathtt{U}^{(0)} \backslash \{\underline{\mathbf{m}}\}$ a non-zero exponentially small eigenvalue of P_h that we denote by $\lambda(\mathbf{m},h)$.

Theorem 1.3. *Suppose that Hypotheses 1.1 and 2.8 are satisfied and recall the notations (1.11)–(1.12).*

The exponentially small eigenvalues of P_h satisfy the following equivalent in the limit $h \to 0$:

$$\lambda(\mathbf{m},h) \sim h\varrho(\mathbf{m})\, \mathrm{e}^{\frac{-2S(\mathbf{m})}{h}}$$

with

$$\varrho(\mathbf{m}) = \frac{1}{\pi} \sum_{\mathbf{s}\in \mathbf{j}(\mathbf{m})} \Big(\frac{2+\sqrt{2}}{2-\sqrt{2}}\Big)^{\frac{1}{\sqrt{|\tau_{\mathbf{s}}|}}} \Big(\frac{\det \mathcal{V}_{\mathbf{m}}}{|\det \mathcal{V}_{\mathbf{s}}|}\Big)^{1/2} \times \int_{\gamma_1 \le z \le \gamma < 1} k_0^{\mathbf{s}}(\gamma) k_0^{\mathbf{s}}(z) \ln\Big(2\frac{(1+z)(1+\gamma)}{1+3z+3\gamma+z\gamma}\Big)\, \mathrm{d}z\, \mathrm{d}\gamma$$

where

$$k_0^{\mathbf{s}}(z) = \frac{2\sqrt{2}}{\sqrt{|\tau_{\mathbf{s}}|}(z-\gamma_2)^2}\Big(\frac{z-\gamma_1}{z-\gamma_2}\Big)^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}-1}, \quad \gamma_1 = -3+2\sqrt{2},\ \gamma_2 = -3-2\sqrt{2},$$

and the maps S and $\mathbf{j}$ are defined in Definition 2.7.

In this theorem, we provide an Eyring–Kramers formula for the operator P_h which looks quite unusual, or at least quite different from the ones established in [11] for the mild relaxation Boltzmann operator, or in [1] in the case of Fokker–Planck-type differential operators. Indeed, here the prefactor $\varrho(\mathbf{m})$ obeys the above complicated integral formula and we do not provide a complete asymptotic expansion of the eigenvalues. Actually, these two remarks are both a consequence of the fact that the action of the collision operator Q_h on a Gaussian quasimode differs from the one of the operators from [1, 11] (see Lemma 2.4) and makes the whole analysis noticeably harder.

The plan of the proof is thus the following. In Section 2, we study the structure of the operator Q_h and show that its action on a Gaussian quasimode yields a integral superposition of exponentials. In Section 3, we therefore introduce some new quasimodes given by an integral superposition of Gaussian quasimodes, for which we show that the actions of the transport term and the collision term are of the same nature. This allows us to optimize the choice of our quasimodes in order to get the best possible compensations between these two terms. Section 4 is devoted to the computation of the *approximated eigenvalues* which is also made harder by the integral form of our optimized quasimodes and from which the expression of $\varrho(\mathbf{m})$ will result; while in Section 5 we prove that the true eigenvalues are equivalent when $h \to 0$ to the approximated ones.

Finally, following [13], we use the sharp localization obtained in Theorem 1.3 in order to discuss the phenomena of return to equilibrium and metastability for the solutions of (1.5). More precisely, we are able to give a sharp rate of convergence of the semigroup $\mathrm{e}^{-tP_h/h}$ towards $\mathbb{P}_1$, the orthogonal projector on Ker P_h: denoting by λ^* the smallest non-zero eigenvalue of P_h, we establish that the rate of return to equilibrium is essentially given by $\frac{\lambda^*}{h}$.

Corollary 1.4. *Under the assumptions of Theorem 1.3, there exists $h_0 > 0$ such that for all $0 < h \leq h_0$ and $t \geq 0$,*

$$\| \mathrm{e}^{-tP_h/h} - \mathbb{P}_1 \| \leq C\, \mathrm{e}^{-t\lambda^*/h} .$$

Besides, in the spirit of [1, 11], we also show the metastable behavior of the solutions of (1.5).

Corollary 1.5. *Suppose that the assumptions of Theorem 1.3 hold true. Let us consider some local minima $\mathbf{m}_1 = \underline{\mathbf{m}}, \mathbf{m}_2, \dots, \mathbf{m}_p$ such that*

$$S(\mathrm{U}^{(0)}) = \{+\infty = S(\mathbf{m}_1) > S(\mathbf{m}_2) > \cdots > S(\mathbf{m}_p)\}$$

for the map S from Definition 2.7. For $2 \leq k \leq p$, denote by $\mathbb{P}_k$ the spectral projection (which is not necessarily orthogonal) associated to the eigenvalues of P_h that are

$O(\mathrm{e}^{-2\frac{S(\mathbf{m}_k)}{h}})$. *Then, for any times* $(t_k^{\pm})_{1\leq k\leq p}$ *satisfying*

$$t_p^- \geq |\ln(h^\infty)| \quad \textit{and} \quad t_k^- \geq |\ln(h^\infty)|\,\mathrm{e}^{2\frac{S(\mathbf{m}_{k+1})}{h}} \quad \textit{for } k = 1,\dots,p-1,$$

as well as

$$t_1^+ = +\infty \quad \textit{and} \quad t_k^+ = O(h^\infty\,\mathrm{e}^{2\frac{S(\mathbf{m}_k)}{h}}) \qquad \textit{for } k = 2,\dots,p,$$

one has

$$\mathrm{e}^{-tP_h/h} = \mathbb{P}_k + O(h^\infty) \quad \textit{on } [t_k^-, t_k^+].$$

In other words, we have shown the existence of timescales on which, during its convergence towards the global equilibrium, the solution of (1.5) will essentially visit the metastable spaces associated to the small eigenvalues of P_h.

Another perspective would then be to study the case of collision operators satisfying the local conservation laws of physics, such as the *full linear Boltzmann operator*

$$Q_h^{FL} = h(\mathrm{Id} - \Pi_h^{FL})$$

with Π_h^{FL} the orthogonal projector on the *collision invariants* subspace

$$\mathrm{Vect}_{\mathbb{R}_v^d}\{\mathrm{e}^{-\frac{v^2}{4h}},\ v_1\,\mathrm{e}^{-\frac{v^2}{4h}},\dots,v_d\,\mathrm{e}^{-\frac{v^2}{4h}}, v^2\,\mathrm{e}^{-\frac{v^2}{4h}}\}L^2(\mathbb{R}_x^d),$$

which was recently studied in [3] at fixed temperature.

2. Preliminaries

From now on, the letter r will denote a small universal positive constant whose value may decrease as we progress in this paper (one can think of r as $\frac{1}{C}$).

2.1. Naive approach

In order to investigate a first natural approach to our problem, consisting in trying to reproduce the method from [11] which was itself inspired by [1, 8], let us make for simplicity and for this subsection only an additional assumption.

Hypothesis 2.1. The potential V has exactly one saddle point $\mathbf{s}$.

Roughly speaking, this approach consists in introducing a linear form $\ell(x,v) = \ell_x\cdot(x-\mathbf{s}) + \ell_v\cdot v$ in the variables $(x-\mathbf{s}, v)$ as well as a Gaussian cut-off θ which is essentially given by

$$\theta(x,v) = \int_0^{\ell(x,v)} \mathrm{e}^{-\frac{s^2}{2h}}\,\mathrm{d}s.$$

With the notation (1.10), the idea is then to introduce the so-called Gaussian quasi-mode

$$\varphi(x,v)=\theta(x,v)\,\mathrm{e}^{-\frac{W(x,v)}{h}}$$

and compute $P_h\varphi$ in order to then choose the linear form ℓ minimizing the norm of $P_h\varphi$. We already know from [11, proof of Proposition 3.13] that

$$X_0^h\varphi(x,v)=h\,p_\ell(x,v)\,\mathrm{e}^{-\frac{1}{h}(W(x,v)+\frac{1}{2}\ell^2(x,v))}(1+O(h)),\quad |x-\mathbf{s}|,\ |v|<r \tag{2.1}$$

with $p_\ell=O_{L^\infty}(1)$. It is also shown that the collision operator studied in this reference, that we denote by $Q_h^{S^0}$, satisfies a similar result:

$$Q_h^{S^0}\varphi(x,v)=h\,q_\ell(x,v)\,\mathrm{e}^{-\frac{1}{h}(W(x,v)+\frac{1}{2}\ell^2(x,v))}(1+O(h)),\quad |x-\mathbf{s}|,\ |v|<r \tag{2.2}$$

with $q_\ell=O_{L^\infty}(1)$, and it is then sufficient in that case to choose ℓ so that $p_\ell=-q_\ell$.

In our case, although Q_h may appear as a quite simple operator as it is just an orthogonal projection, in order to perform a computation similar to (2.2), it will be more convenient to adopt a microlocal point of view. This is the point of the two following results which are proven in Appendix A.

Proposition 2.2. *Let us set*

$$b_h=h\partial_v+\frac{v}{2}.$$

There exists a symbol $m_h\in S^{1/2}(\langle v,\eta\rangle^{-2})$ given by

$$m_h(v,\eta)=2\int_0^1(y+1)^{d-2}\,\mathrm{e}^{-\frac{y}{h}(\frac{v^2}{2}+2\eta^2)}\,\mathrm{d}y$$

such that

$$Q_h=b_h^*\circ\mathrm{Op}_h(m_h\,\mathrm{Id})\circ b_h.$$

Corollary 2.3. *One has*

$$Q_h=\mathrm{Op}_h(g_h)\circ b_h$$

with

$$g_h(v,\eta)=\int_0^1(y+1)^{d-1}\,\mathrm{e}^{-\frac{y}{h}(\frac{v^2}{2}+2\eta^2)}\,\mathrm{d}y\,(-2i\eta^\intercal+v^\intercal)\in S^{1/2}(\langle v,\eta\rangle^{-1}).$$

We are now in position to establish the following fundamental computation which shows that the balancing obtained between $X_0^h\varphi$ and $Q_h^{S^0}\varphi$ cannot happen between $X_0^h\varphi$ and $Q_h\varphi$. This will motivate the introduction of some new quasimodes later on.

Lemma 2.4. *Assume for simplicity that Hypothesis 2.1 holds true and let ℓ a linear form in the variables $(x-\mathbf{s}, v)$. We have*

$$Q_h\varphi(x,v) = -h\int_0^1 \partial_y(L_y)\,\mathrm{e}^{-\frac{W(x,v)+\frac{1}{2}L_y^2(x,v)}{h}}\,\mathrm{d}\,y\cdot\begin{pmatrix}x-\mathbf{s}\\ v\end{pmatrix}$$

where, with a slight abuse of notations, L_y denotes both the linear form

$$L_y(x,v) = \frac{(1+y)\ell_x\cdot(x-\mathbf{s})+(1-y)\ell_v\cdot v}{(4y\ell_v^2+(y+1)^2)^{1/2}}$$

and the vector representing it. Moreover, setting

$$m_{y,h}(v,\eta) = 2(y+1)^{d-2}\,\mathrm{e}^{-\frac{y}{h}(\frac{v^2}{2}+2\eta^2)}, \tag{2.3}$$

we have

$$\begin{aligned}\mathrm{Op}_h(m_{y,h})\circ b_h\varphi(x,v) &= 2h(2\pi h)^{-d/2}\,\mathrm{e}^{-\frac{V(x)}{2h}}\,\frac{(y+1)^{d-2}}{(4y)^{\frac{d}{2}}}\\ &\quad\times\int_{v'\in\mathbb{R}^d}\mathrm{e}^{-\frac{1}{h}(\frac{v'^2}{4}+\frac{y}{8}(v+v')^2+\frac{(v-v')^2}{8y}+\frac{1}{2}\ell^2(x,v'))}\,\mathrm{d}\,v'\ell_v.\end{aligned} \tag{2.4}$$

Proof. According to Corollary 2.3, we have

$$\begin{aligned}&Q_h\varphi(x,v)\\ &\quad= \mathrm{Op}_h(g_h)[h\partial_v\theta\,\mathrm{e}^{-W/h}](x,v)\\ &\quad= h\,\mathrm{Op}_h(g_h)[\mathrm{e}^{-\frac{1}{h}(W+\frac{1}{2}\ell^2)}\,\ell_v](x,v)\\ &\quad= h(2\pi h)^{-d}\int_{v'\in\mathbb{R}^d}\int_{\eta\in\mathbb{R}^d}\mathrm{e}^{\frac{i}{h}(v-v')\cdot\eta}\,g_h\Big(\frac{v+v'}{2},\eta\Big)\,\mathrm{e}^{-\frac{1}{h}(W(x,v')+\frac{1}{2}\ell^2(x,v'))}\,\mathrm{d}\,v'\,\mathrm{d}\,\eta\ell_v.\end{aligned}$$

Let us now compute the integral in η with the expression of g_h from Corollary 2.3:

$$\begin{aligned}&\int_{\eta\in\mathbb{R}^d}\mathrm{e}^{\frac{i}{h}(v-v')\cdot\eta}\,g_h\Big(\frac{v+v'}{2},\eta\Big)\,\mathrm{d}\,\eta\\ &\quad=\int_0^1(y+1)^{d-1}\,\mathrm{e}^{-\frac{y(v+v')^2}{8h}}\Bigg[\frac{(v+v')^{\mathsf{T}}}{2}\int_{\eta\in\mathbb{R}^d}\mathrm{e}^{\frac{i}{h}(v-v')\cdot\eta}\,\mathrm{e}^{-\frac{2y\eta^2}{h}}\,\mathrm{d}\,\eta\\ &\qquad\qquad-2i\int_{\eta\in\mathbb{R}^d}\eta^{\mathsf{T}}\,\mathrm{e}^{\frac{i}{h}(v-v')\cdot\eta}\,\mathrm{e}^{-\frac{2y\eta^2}{h}}\,\mathrm{d}\,\eta\Bigg]\mathrm{d}\,y\end{aligned}$$

$$= \int_0^1 (y+1)^{d-1} \, \mathrm{e}^{-\frac{y(v+v')^2}{8h}} \Big[\frac{(v+v')^\mathsf{T}}{2} + \frac{(v-v')^\mathsf{T}}{2y} \Big] \int_{\eta \in \mathbb{R}^d} \mathrm{e}^{\frac{i}{h}(v-v')\cdot\eta} \, \mathrm{e}^{-\frac{2y\eta^2}{h}} \, \mathrm{d}\eta \, \mathrm{d}y$$

$$= 2(2\pi h)^{d/2} \int_0^1 \frac{(y+1)^{d-1}}{(4y)^{\frac{d}{2}+1}} \big((v+v')y + v - v'\big)^\mathsf{T} \, \mathrm{e}^{-\frac{1}{8h}(y(v+v')^2 + \frac{(v-v')^2}{y})} \, \mathrm{d}y.$$

Hence, we get

$$Q_h\varphi(x,v) = 2h(2\pi h)^{-d/2} \, \mathrm{e}^{-\frac{V(x)}{2h}} \int_0^1 \frac{(y+1)^{d-1}}{(4y)^{\frac{d}{2}+1}} \int_{v' \in \mathbb{R}^d} ((v+v')y + v - v') \times \mathrm{e}^{-\frac{1}{h}(\frac{v'^2}{4} + \frac{y}{8}(v+v')^2 + \frac{(v-v')^2}{8y} + \frac{1}{2}\ell^2(x,v'))} \, \mathrm{d}v' \, \mathrm{d}y \cdot \ell_v \tag{2.5}$$

and (2.4) is now a straightforward adaptation of (2.5) with $m_{y,h}$ instead of g_h. Denoting $x_\mathbf{s} = x - \mathbf{s}$,

$$M_y = \frac{1}{2}\,\mathrm{Id} + \ell_v \ell_v^\mathsf{T} + \frac{y^2+1}{4y}\,\mathrm{Id}, \quad \text{and} \quad u_y(x_\mathbf{s}, v) = \ell_x \cdot x_\mathbf{s}\, \ell_v + \frac{y^2-1}{4y} v,$$

(2.5) becomes, by the change of variables $w = v' + M_y^{-1} u_y(x_\mathbf{s}, v)$,

$$Q_h\varphi(x,v)$$

$$= 2h(2\pi h)^{-d/2} \, \mathrm{e}^{-\frac{V(x)}{2h}} \int_0^1 \frac{(y+1)^{d-1}}{(4y)^{\frac{d}{2}+1}} \exp\Big[\frac{-1}{2h}\Big(\ell_x \ell_x^\mathsf{T} x_\mathbf{s} \cdot x_\mathbf{s} + \frac{y^2+1}{4y} v^2 - M_y^{-1} u_y(x_\mathbf{s}, v) \cdot u_y(x_\mathbf{s}, v)\Big)\Big]$$
$$\times \int_{w \in \mathbb{R}^d} \big[\big(v - M_y^{-1} u_y(x_\mathbf{s}, v)\big) y + v + M_y^{-1} u_y(x_\mathbf{s}, v)\big] \, \mathrm{e}^{-\frac{M_y w \cdot w}{2h}} \, \mathrm{d}w \, \mathrm{d}y \cdot \ell_v$$

$$= 2h \, \mathrm{e}^{-\frac{V(x)}{2h}} \int_0^1 \frac{(y+1)^{d-1}}{(4y)^{\frac{d}{2}+1}} \det(M_y)^{-1/2} \big((1+y)v + (1-y) M_y^{-1} u_y(x_\mathbf{s}, v)\big) \cdot \ell_v \times \exp\Big[\frac{-1}{2h}\Big(\ell_x \ell_x^\mathsf{T} x_\mathbf{s} \cdot x_\mathbf{s} + \frac{y^2+1}{4y} v^2 - M_y^{-1} u_y(x_\mathbf{s}, v) \cdot u_y(x_\mathbf{s}, v)\Big)\Big] \, \mathrm{d}y. \tag{2.6}$$

Now,

$$\frac{(y+1)^{d-1}}{(4y)^{\frac{d}{2}+1}} \det(M_y)^{-1/2} = \frac{1}{4y(4y\ell_v^2 + (y+1)^2)^{1/2}},$$

while

$$M_y^{-1} \ell_v = \frac{4y}{4y\ell_v^2 + (y+1)^2} \, \ell_v \tag{2.7}$$

so the prefactor in the integral from (2.6) becomes

$$\frac{1}{4y(4y\ell_v^2+(y+1)^2)^{1/2}}\Big[\frac{4y(1-y)\ell_v^2}{4y\ell_v^2+(y+1)^2}\ell_x\cdot x_{\mathbf{s}} + \Big((1+y)+\frac{(1-y)(y^2-1)}{4y\ell_v^2+(y+1)^2}\Big)\ell_v\cdot v\Big],$$

which is further equal to

$$\frac{(1-y)\ell_v^2\ell_x\cdot x_{\mathbf{s}}+(1+y)(1+\ell_v^2)\ell_v\cdot v}{(4y\ell_v^2+(y+1)^2)^{3/2}} = -\frac{1}{2}\partial_y(L_y)\cdot\begin{pmatrix}x_{\mathbf{s}}\\ v\end{pmatrix}. \tag{2.8}$$

Thus, it only remains to show that the exponentials coincide, i.e.,

$$\ell_x\ell_x^{\mathsf{T}}x_{\mathbf{s}}\cdot x_{\mathbf{s}}+\frac{y^2+1}{4y}v^2-M_y^{-1}u_y(x_{\mathbf{s}},v)\cdot u_y(x_{\mathbf{s}},v)=\frac{v^2}{2}+L_y^2(x,v)$$

or equivalently

$$\ell_x\ell_x^{\mathsf{T}}x_{\mathbf{s}}\cdot x_{\mathbf{s}}+\frac{(y-1)^2}{4y}v^2-M_y^{-1}u_y(x_{\mathbf{s}},v)\cdot u_y(x_{\mathbf{s}},v) = \frac{((1+y)\ell_x\cdot x_{\mathbf{s}}+(1-y)\ell_v\cdot v)^2}{4y\ell_v^2+(y+1)^2}. \tag{2.9}$$

Using (2.7), we already obtain

$$M_y^{-1}u_y(x_{\mathbf{s}},v)\cdot u_y(x_{\mathbf{s}},v) = \frac{4y\ell_v^2}{4y\ell_v^2+(y+1)^2}\ell_x\ell_x^{\mathsf{T}}x_{\mathbf{s}}\cdot x_{\mathbf{s}} + 2\frac{y^2-1}{4y\ell_v^2+(y+1)^2}\ell_x\cdot x_{\mathbf{s}}\ell_v\cdot v + \frac{(y^2-1)^2}{16y^2}M_y^{-1}v\cdot v$$

so the left-hand side of (2.9) becomes

$$\frac{(1+y)^2}{4y\ell_v^2+(y+1)^2}(\ell_x\cdot x_{\mathbf{s}})^2+2\frac{1-y^2}{4y\ell_v^2+(y+1)^2}\ell_x\cdot x_{\mathbf{s}}\ell_v\cdot v + \Big(\frac{(y-1)^2}{4y}-\frac{(y^2-1)^2}{16y^2}M_y^{-1}\Big)v\cdot v. \tag{2.10}$$

Finally, still using (2.7), one can easily check that

$$\frac{(y-1)^2}{4y}-\frac{(y^2-1)^2}{16y^2}M_y^{-1}=\frac{(1-y)^2}{4y\ell_v^2+(y+1)^2}\ell_v\ell_v^{\mathsf{T}},$$

so (2.10) equals the right-hand side of (2.9), and the proof is complete. ■

This result shows that, unlike in the case of some S^0 collisions operators as studied in [11] (or even in the case of differential operators [1, 8]), here the action of Q_h on the quasimode φ does not yield a precise exponential, but rather a superposition of exponentials with the linear form in the phase varying. This suggests the introduction of some new quasimodes given by a superposition of functions similar to φ with the linear form varying.

2.2. Labeling of the potential minima

We now drop Hypothesis 2.1. Before we can construct our quasimodes, we need to recall the general labeling of the minima which originates from [2, 4] and was generalized in [7], as well as the topological constructions that go with it. Here we only introduce the essential objects and omit the proofs. For more details, we refer to [11], where it is in particular shown that, roughly speaking, the constructions for the potential $\frac{V}{2}$ are the projections on $\mathbb{R}^d_x$ of the ones for the global potential W. Recall that we denote by

$$\mathtt{U}^{(k)} \text{ the critical points of } V \text{ of index } k. \tag{2.11}$$

For shortness, we will write "CC" instead of "connected component." The constructions rely on the following fundamental observation which is an easy consequence of the Morse lemma (see for instance [11, Lemma 3.1] for a proof).

Lemma 2.5. *If $x \in \mathtt{U}^{(1)}$, then there exists $r_0 > 0$ such that for all $0 < r < r_0$, $(x, 0)$ has a connected neighborhood $\mathcal{O}_r$ in $B_0(x, r)$ such that $\mathcal{O}_r \cap \{W < W(x, 0)\}$ has exactly two CCs.*

This motivates the following definition.

Definition 2.6. (1) We say that $x \in \mathtt{U}^{(1)}$ is a *separating saddle point* and we denote $x \in \mathtt{V}^{(1)}$ if for every $r > 0$ small enough, the two CCs of $\mathcal{O}_r \cap \{W < W(x, 0)\}$ are contained in different CCs of $\{W < W(x, 0)\}$.

(2) We say that $\sigma \in \mathbb{R}$ is a *separating saddle value* if $\sigma \in \frac{V}{2}(\mathtt{V}^{(1)})$.

It is known (see for instance [11, Lemma 3.4]) that $\mathtt{V}^{(1)} \neq \emptyset$ since $n_0 \geq 2$. Let us then denote $\sigma_2 > \cdots > \sigma_N$ where $N \geq 2$ the different separating saddle values of $\frac{V}{2}$ and for convenience we set $\sigma_1 = +\infty$. For $\sigma \in \mathbb{R} \cup \{+\infty\}$, let us denote $\mathcal{C}_\sigma$ the set of all the CCs of $\{W < \sigma\}$. We call *labeling* of the minima of V any injection $l : \mathtt{U}^{(0)} \to [\![1, N]\!] \times [\![1, \#\mathtt{U}^{(0)}]\!]$. If $l(\mathbf{m}) = (k, j)$, we denote for shortness $\mathbf{m} = \mathbf{m}_{k,j}$. Given a labeling l of the minima of V, we denote for $k \in [\![1, N]\!]$

$$\mathtt{U}^{(0)}_k = l^{-1}([\![1, k]\!] \times [\![1, \#\mathtt{U}^{(0)}]\!]) \cap \left\{\frac{V}{2} < \sigma_k\right\}$$

and we say that the labeling is *adapted* to the separating saddle values if for all $k \in [\![1, N]\!]$, each element of $l^{-1}(\{k\} \times [\![1, \#\mathtt{U}^{(0)}]\!])$ is a global minimum of V restricted to some CC of $\{\frac{V}{2} < \sigma_k\}$, and the map

$$T_k \colon \mathtt{U}_k^{(0)} \to \mathscr{C}_{\sigma_k} \tag{2.12}$$

sending $\mathbf{m} \in \mathtt{U}_k^{(0)}$ on the element of $\mathscr{C}_{\sigma_k}$ containing $(\mathbf{m}, 0)$ is bijective. In particular, it implies that $l^{-1}(\{k\} \times [\![1, \#\mathtt{U}^{(0)}]\!])$ is contained in $\mathtt{U}_k^{(0)}$. Such labelings exist, one can for instance easily check that the usual labeling procedure presented in [7] is adapted to the separating saddle values. From now on, we fix a labeling $(\mathbf{m}_{k,j})_{k,j}$ adapted to the separating saddle values of V.

Definition 2.7. Recall the notation (2.11) and Definition 2.6. We define the following mappings:

- $E \colon \mathtt{U}^{(0)} \to \mathscr{P}(\mathbb{R}^{2d})$, $\mathbf{m}_{k,j} \mapsto T_k(\mathbf{m}_{k,j})$, where T_k is the map defined in (2.12);
- $\mathbf{j}^W \colon \mathtt{U}^{(0)} \to \mathscr{P}((\mathtt{V}^{(1)} \cup \{\mathbf{s}_1\}) \times \{0\})$, given by
  - $\mathbf{j}^W(\mathbf{m}_{1,1}) = (\mathbf{s}_1, 0)$, where $\mathbf{s}_1$ is a fictive saddle point such that $V(\mathbf{s}_1) = \sigma_1 = +\infty$, and
  - for $2 \leq k \leq N$, $\mathbf{j}^W(\mathbf{m}_{k,j}) = \partial E(\mathbf{m}_{k,j}) \cap (\mathtt{V}^{(1)} \times \{0\})$, which is not empty (see for instance [11, Lemma 3.5]), finite, and included in $\{W = \sigma_k\}$;
- $\mathbf{j} \colon \mathtt{U}^{(0)} \to \mathscr{P}(\mathtt{V}^{(1)} \cup \{\mathbf{s}_1\})$ such that $\mathbf{j}(\mathbf{m}) \times \{0\} = \mathbf{j}^W(\mathbf{m})$;
- $\boldsymbol{\sigma} \colon \mathtt{U}^{(0)} \to \frac{V}{2}(\mathtt{V}^{(1)}) \cup \{\sigma_1\}$, $\mathbf{m} \mapsto \frac{V}{2}(\mathbf{j}(\mathbf{m}))$, where we allow ourselves to identify the set $\frac{V}{2}(\mathbf{j}(\mathbf{m}))$ and its unique element in $\frac{V}{2}(\mathtt{V}^{(1)}) \cup \{\sigma_1\}$;
- $S \colon \mathtt{U}^{(0)} \to]0, +\infty]$, $\mathbf{m} \mapsto \boldsymbol{\sigma}(\mathbf{m}) - \frac{V}{2}(\mathbf{m})$.

Following [2, 4, 7, 8], we can now state our last assumption that allows us to treat the generic case. As mentioned in the introduction, this assumption could actually be omitted (see [10] or [1, Section 6]), but this would introduce additionnal difficulties that are not the main concern of this paper.

Hypothesis 2.8. Recall that we fixed a labeling $(\mathbf{m}_{k,j})_{k,j}$ adapted to the separating saddle values of V. We assume the following:

(a) each $\mathbf{m}_{k,j}$ is the only global minimum of V on the CC of $\{\frac{V}{2} < \sigma_k\}$ to which it belongs;

(b) for all $\mathbf{m} \neq \mathbf{m}' \in \mathtt{U}^{(0)}$, the sets $\mathbf{j}(\mathbf{m})$ and $\mathbf{j}(\mathbf{m}')$ do not intersect.

According to [11], this hypothesis is equivalent to the facts that $(\mathbf{m}, 0)$ is the only global minimum of $W|_{E(\mathbf{m})}$ and $\mathbf{j}^W(\mathbf{m}) \cap \mathbf{j}^W(\mathbf{m}') = \emptyset$, which is what we use in practice.

3. Accurate quasimodes

3.1. Gaussian quasimodes superposition

By Hypothesis 2.8, the potential V has a unique global minimum that we denote $\underline{\mathbf{m}}$. For $r > 0$, denote $\tilde{r}$ a positive number such that for all $\mathbf{m} \in \mathtt{U}^{(0)} \backslash \{\underline{\mathbf{m}}\}$ and $\mathbf{s} \in \mathbf{j}(\mathbf{m})$,

$$W(x, v) \geq \boldsymbol{\sigma}(\mathbf{m}) + \frac{r^2}{8} + \frac{v^2 - r^2}{4} \quad \text{as soon as } |x - \mathbf{s}| < \tilde{r} \text{ and } |v| \geq r. \tag{3.1}$$

We also denote for $x \in \mathbb{R}^d$

$$B_0(x, r) = B(x, \tilde{r}) \times B(0, r) \subseteq \mathbb{R}^{2d}.$$

Let $\mathbf{m} \in \mathtt{U}^{(0)} \backslash \{\underline{\mathbf{m}}\}$; for each $\mathbf{s} \in \mathbf{j}(\mathbf{m})$ we introduce a vector $\ell^{\mathbf{s}} = (\ell^{\mathbf{s}}_x, \ell^{\mathbf{s}}_v) \in \mathbb{R}^{2d}$ which will represent a linear form involved in the construction of our quasimodes. Note that thanks to Hypothesis 2.8 (b), each $\ell^{\mathbf{s}}$ corresponds to a unique $\mathbf{m} \in \mathtt{U}^{(0)} \backslash \{\underline{\mathbf{m}}\}$. In the spirit of [1, 8, 11] and more precisely in view of (2.1)–(2.2), we want $\mathbf{s}$ to be a local minimum of the function

$$W(x, v) + \frac{1}{2}\big(\ell^{\mathbf{s}}_x \cdot (x - \mathbf{s}) + \ell^{\mathbf{s}}_v \cdot v\big)^2;$$

so according to Lemma B.1 and using the notation (1.11), we take $\ell^{\mathbf{s}}$ satisfying

$$-\mathcal{V}_{\mathbf{s}}^{-1} \ell^{\mathbf{s}}_x \cdot \ell^{\mathbf{s}}_x - |\ell^{\mathbf{s}}_v|^2 > \frac{1}{2}.$$

This condition would be sufficient to develop a framework for the construction of our quasimodes. However, it would appear later on; when establishing a result analogous to the one of Lemma 3.6; that the optimal choice of $\ell^{\mathbf{s}}$ would actually satisfy

$$-\mathcal{V}_{\mathbf{s}}^{-1} \ell^{\mathbf{s}}_x \cdot \ell^{\mathbf{s}}_x - |\ell^{\mathbf{s}}_v|^2 = 1.$$

Similarly, one could show in this framework from the analogous of (3.9) that our quasimodes would not depend on the norm of $\ell^{\mathbf{s}}$. Thus, we set

$$|\ell^{\mathbf{s}}_v|^2 = 1 \tag{3.2}$$

as well as

$$-\mathcal{V}_{\mathbf{s}}^{-1} \ell^{\mathbf{s}}_x \cdot \ell^{\mathbf{s}}_x = 2 \tag{3.3}$$

straight away as it leads to significant simplifications in the study.

We now introduce the polynomial

$$P(\gamma) = 4\gamma + (\gamma + 1)^2 \tag{3.4}$$

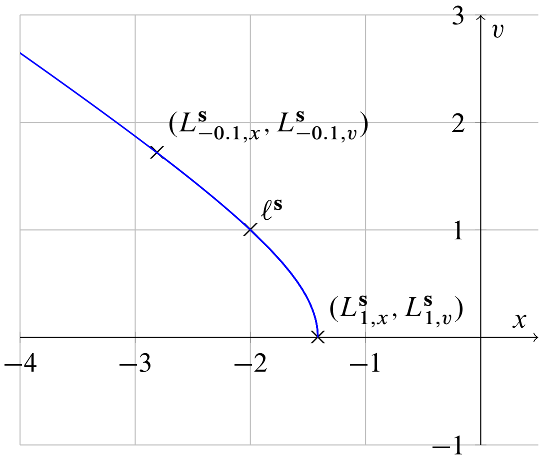
and its two roots

$$\gamma_1 = -3 + 2\sqrt{2} \in (-1, 0) \quad \text{and} \quad \gamma_2 = -3 - 2\sqrt{2} < -1.$$

In the spirit of Lemma 2.4, we also introduce for $\gamma \in (\gamma_1, 1]$ the vector $(L^{\mathbf{s}}_{\gamma;x}, L^{\mathbf{s}}_{\gamma;v}) \in \mathbb{R}^{2d}$ where

$$L^{\mathbf{s}}_{\gamma;x} = \frac{1+\gamma}{P(\gamma)^{1/2}}\,\ell^{\mathbf{s}}_x \quad \text{and} \quad L^{\mathbf{s}}_{\gamma;v} = \frac{1-\gamma}{P(\gamma)^{1/2}}\,\ell^{\mathbf{s}}_v. \tag{3.5}$$

Note that $(L^{\mathbf{s}}_{0;x}, L^{\mathbf{s}}_{0;v}) = \ell^{\mathbf{s}}$. Lemma 2.4 would actually suggest to consider only $\gamma \in [0, 1]$, but doing so it would appear with the notation (3.45) that (3.47) has no non-trivial solution, which is not true anymore when working on $(\gamma_1, 1]$. We do not consider γ outside $(\gamma_1, 1]$ as it would add a condition similar to (3.46) which would be incompatible with (3.46). Here is the picture of an example in the case $d = 1$:



Since, by Hypothesis 1.1, the potential V is a Morse function, there exists, according to the Morse lemma, a smooth diffeomorphism $\phi_{\mathbf{s}}$ defined on $B(\mathbf{s}, \tilde{r})$, sending $\mathbf{s}$ on 0, whose differential at $\mathbf{s}$ is the identity and such that

$$V \circ \phi_{\mathbf{s}}^{-1} = V(\mathbf{s}) + \frac{1}{2}\langle \mathcal{V}_{\mathbf{s}}\cdot, \cdot\rangle. \tag{3.6}$$

For shortness, we will use for $x \in B(\mathbf{s}, \tilde{r})$ the notation

$$\tilde{x}_{\mathbf{s}} = \phi_{\mathbf{s}}(x) \tag{3.7}$$

and we introduce the smooth function $L^{\mathbf{s}}_{\gamma}$ supported in $B(\mathbf{s}, 2\tilde{r}) \times \mathbb{R}^d_v$ and given when x is close to $\mathbf{s}$ by the twisted linear form:

$$L^{\mathbf{s}}_{\gamma}(x, v) = L^{\mathbf{s}}_{\gamma;x} \cdot \tilde{x}_{\mathbf{s}} + L^{\mathbf{s}}_{\gamma;v} \cdot v \quad \text{for } (x, v) \in B(\mathbf{s}, \tilde{r}) \times \mathbb{R}^d_v.$$

Now, let us denote by $\zeta \in \mathcal{C}_c^\infty(\mathbb{R}, [0,1])$ an even cut-off function supported in $[-\delta, \delta]$ that is equal to 1 on $\left[-\frac{\delta}{2}, \frac{\delta}{2}\right]$, where $\delta > 0$ is a parameter to be fixed later. As we will not be able to produce some remainder terms that are uniform with respect to $\gamma \in (\gamma_1, 1]$, we will work on $[\gamma_1 + \nu, 1]$ with

$$\nu > 0 \text{ that will be fixed small enough before letting } h \to 0.$$

Consider also a probability density $k_\nu^{\mathbf{s}}$ on $[\gamma_1 + \nu, 1]$ as well as the quantity

$$\begin{aligned} A_{\nu,h}^{\mathbf{s}} &= \int_{\gamma_1+\nu}^{1} k_\nu^{\mathbf{s}}(\gamma) \int_0^\infty \zeta\Big(\frac{s}{N^{\mathbf{s}}(\gamma)}\Big)\, \mathrm{e}^{-\frac{s^2}{2h}}\, \mathrm{d}s\, \mathrm{d}\gamma \\ &= \frac{\sqrt{\pi h}}{\sqrt{2}}(1 + O(\mathrm{e}^{-\alpha/h})) \quad \text{for some } \alpha > 0, \end{aligned} \tag{3.8}$$

where

$$N^{\mathbf{s}}(\gamma) = (|L_{\gamma;x}^{\mathbf{s}}|^2 + |L_{\gamma;v}^{\mathbf{s}}|^2)^{1/2} \geq \frac{1}{C}.$$

We will also use the notation

$$U_\gamma^{\mathbf{s}} = \frac{L_\gamma^{\mathbf{s}}}{N^{\mathbf{s}}(\gamma)}.$$

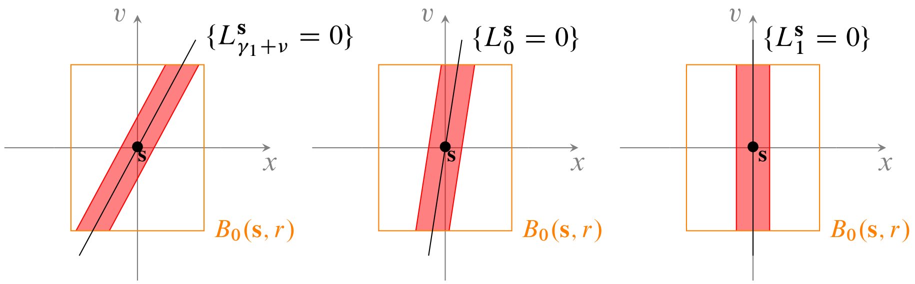
We now define for each $\mathbf{m} \in \mathsf{U}^{(0)} \backslash \{\underline{\mathbf{m}}\}$ the *Gaussian cut-off superposition* $\theta_{\nu,h}^{\mathbf{m}}$ as follows: if (x, v) belongs to

$$\bigcup_{\gamma \in [\gamma_1+\nu, 1]} \{|U_\gamma^{\mathbf{s}}| \leq 2\delta\} \cap B_0(\mathbf{s}, r)$$

for some $\mathbf{s} \in \mathbf{j}(\mathbf{m})$, then

$$\theta_{\nu,h}^{\mathbf{m}}(x, v) = \frac{1}{2}\Bigg(1 + (A_{\nu,h}^{\mathbf{s}})^{-1} \int_{\gamma_1+\nu}^{1} k_\nu^{\mathbf{s}}(\gamma) \int_0^{L_\gamma^{\mathbf{s}}(x,v)} \zeta\Big(\frac{s}{N^{\mathbf{s}}(\gamma)}\Big)\, \mathrm{e}^{-s^2/2h}\, \mathrm{d}s\, \mathrm{d}\gamma\Bigg). \tag{3.9}$$

Here are some pictures of the set $\{|U_\gamma^{\mathbf{s}}| \leq 2\delta\} \cap B_0(\mathbf{s}, r)$ for $\gamma = \gamma_1 + \nu$; $\gamma = 0$ and $\gamma = 1$:



Furthermore, we set

$$\theta^{\mathbf{m}}_{\nu,h} = 1 \quad \text{on } (E(\mathbf{m}) + B(0,\varepsilon)) \backslash \Big(\bigsqcup_{\mathbf{s}\in \mathbf{j}(\mathbf{m})} \Big(\bigcup_{\gamma\in[\gamma_1+\nu,1]} \{|U^{\mathbf{s}}_{\gamma}| \le 2\delta\} \cap B_0(\mathbf{s}, r) \Big)\Big) \tag{3.10}$$

with $\varepsilon = \varepsilon(r) > 0$ to be fixed later and

$$\theta^{\mathbf{m}}_{\nu,h} = 0 \quad \text{everywhere else.} \tag{3.11}$$

Note that $\theta^{\mathbf{m}}_{\nu,h}$ takes values in [0, 1] and that, thanks to (3.9), we also have

$$\theta^{\mathbf{m}}_{\nu,h} = 1 \quad \text{on } \Big(\bigcup_{\gamma\in[\gamma_1+\nu,1]} \{|U^{\mathbf{s}}_{\gamma}| \le 2\delta\} \cap B_0(\mathbf{s}, r)\Big) \bigcap \Big(\bigcap_{\gamma\in[\gamma_1+\nu,1]} \{U^{\mathbf{s}}_{\gamma} \ge \delta\}\Big)$$

and

$$\theta^{\mathbf{m}}_{\nu,h} = 0 \quad \text{on } \Big(\bigcup_{\gamma\in[\gamma_1+\nu,1]} \{|U^{\mathbf{s}}_{\gamma}| \le 2\delta\} \cap B_0(\mathbf{s}, r)\Big) \bigcap \Big(\bigcap_{\gamma\in[\gamma_1+\nu,1]} \{U^{\mathbf{s}}_{\gamma} \le -\delta\}\Big).$$

Denote by Ω the CC of $\{W \le \boldsymbol{\sigma}(\mathbf{m})\}$ containing $\mathbf{m}$. The CCs of $\{W \le \boldsymbol{\sigma}(\mathbf{m})\}$ are separated; so, for $\varepsilon > 0$ small enough, there exists $\tilde{\varepsilon} > 0$ such that

$$\min\{W(x, v) : \text{dist}((x, v), \Omega) = \varepsilon\} = \boldsymbol{\sigma}(\mathbf{m}) + 2\tilde{\varepsilon}.$$

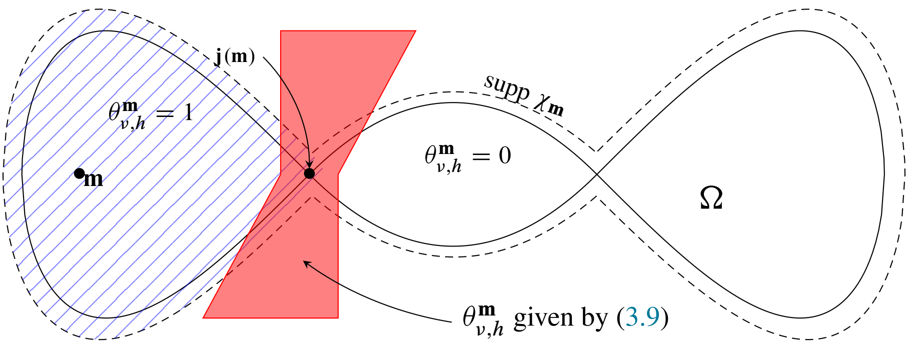
Thus, the distance between $\{W \le \boldsymbol{\sigma}(\mathbf{m}) + \tilde{\varepsilon}\} \cap (\Omega + B(0, \varepsilon))$ and $\partial(\Omega + B(0, \varepsilon))$ is positive and we can consider a cut-off function

$$\chi_{\mathbf{m}} \in \mathscr{C}^{\infty}_c(\mathbb{R}^{2d}, [0, 1])$$

such that

$$\chi_{\mathbf{m}} = 1 \text{ on } \{W \le \boldsymbol{\sigma}(\mathbf{m}) + \tilde{\varepsilon}\} \cap (\Omega + B(0, \varepsilon)) \quad \text{and} \quad \text{supp}\, \chi_{\mathbf{m}} \subset (\Omega + B(0, \varepsilon)). \tag{3.12}$$

To sum up, we have the following picture:



The following lemma will among other things help us discuss the regularity of $\theta^{\mathbf{m}}_{\nu,h}$.

Lemma 3.1. *Recall the notation* (1.11). *For all* $\gamma \in (\gamma_1, 1]$, *we have*

$$-\mathcal{V}_{\mathbf{s}}^{-1} L^{\mathbf{s}}_{\gamma,x} \cdot L^{\mathbf{s}}_{\gamma,x} - (L^{\mathbf{s}}_{\gamma,v})^2 = 1.$$

In particular, according to Lemma B.1, $(\mathbf{s}, 0)$ *is a non-degenerate minimum of* $W + \frac{1}{2}(L^{\mathbf{s}}_{\gamma})^2$ *and the associated hessian has determinant*

$$2^{-2d} |\det \mathcal{V}_{\mathbf{s}}|.$$

Proof. It suffices to use (3.2) and (3.3):

$$-\mathcal{V}_{\mathbf{s}}^{-1} L^{\mathbf{s}}_{\gamma,x} \cdot L^{\mathbf{s}}_{\gamma,x} - (L^{\mathbf{s}}_{\gamma,v})^2 = 2\frac{(1+\gamma)^2}{P(\gamma)} - \frac{(1-\gamma)^2}{P(\gamma)} = \frac{P(\gamma)}{P(\gamma)} = 1.$$

For the computation of the determinant, it is sufficient to notice that, with the notation (1.11), the hessian of $W + \frac{1}{2}(L^{\mathbf{s}}_{\gamma})^2$ at $(\mathbf{s}, 0)$ is

$$\mathcal{W}_{\mathbf{s}} + \begin{pmatrix} L^{\mathbf{s}}_{\gamma;x} \\ L^{\mathbf{s}}_{\gamma;v} \end{pmatrix} \begin{pmatrix} L^{\mathbf{s}}_{\gamma;x} \\ L^{\mathbf{s}}_{\gamma;v} \end{pmatrix}^{\mathsf{T}}$$

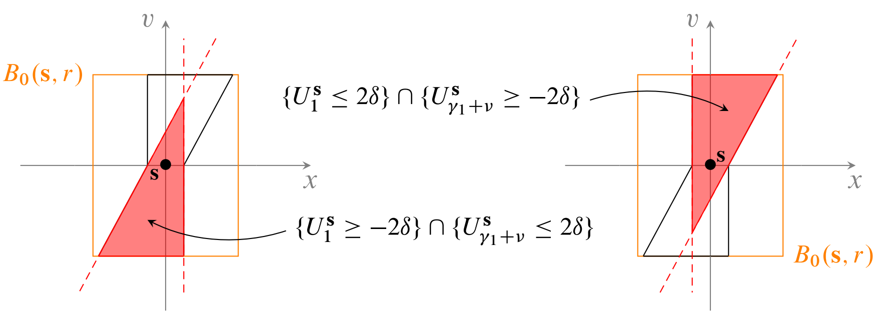
and apply Lemma B.1. ∎

Proposition 3.2. *Up to changing the sign of* $\ell^{\mathbf{s}}$, *for all* $\nu \in (0, |\gamma_1|)$, *we can choose* $\varepsilon > 0$ *and* $\delta > 0$ *small enough so that the function* $\theta^{\mathbf{m}}_{\nu,h}$ *is smooth on the neighborhood of the support of* $\chi_{\mathbf{m}}$ *given by* $\Omega + B(0, \varepsilon)$.

Proof. By Hypothesis 2.8 (b), each $\ell^{\mathbf{s}}$ corresponds to a unique $\mathbf{m} \in \mathsf{U}^{(0)} \backslash \{\underline{\mathbf{m}}\}$. Let us first show that in $B_0(\mathbf{s}, r)$ we have

$$\bigcup_{\gamma \in [\gamma_1+\nu, 1]} \{|U^{\mathbf{s}}_{\gamma}| \leq 2\delta\} = \big(\{U^{\mathbf{s}}_1 \geq -2\delta\} \cap \{U^{\mathbf{s}}_{\gamma_1+\nu} \leq 2\delta\}\big) \cup \big(\{U^{\mathbf{s}}_1 \leq 2\delta\} \cap \{U^{\mathbf{s}}_{\gamma_1+\nu} \geq -2\delta\}\big) \tag{3.13}$$

(so in particular, this set is closed).



Let $(x,v) \in \{U_1^{\mathbf{s}} \geq -2\delta\} \cap \{U_{\gamma_1+\nu}^{\mathbf{s}} \leq 2\delta\}$. If $U_1^{\mathbf{s}}(x,v) \leq 2\delta$, then $(x,v) \in \{|U_1^{\mathbf{s}}| \leq 2\delta\}$ and similarly, if $U_{\gamma_1+\nu}^{\mathbf{s}}(x,v) \geq -2\delta$, then $(x,v) \in \{|U_{\gamma_1+\nu}^{\mathbf{s}}| \leq 2\delta\}$. Now, if $U_1^{\mathbf{s}}(x,v) > 2\delta$ and $U_{\gamma_1+\nu}^{\mathbf{s}}(x,v) < -2\delta$, by the intermediate value theorem, there exists $\gamma \in [\gamma_1 + \nu, 1]$ such that $U_\gamma^{\mathbf{s}}(x,v) = 0$; so, in particular, $(x,v) \in \{|U_\gamma^{\mathbf{s}}| \leq 2\delta\}$. Thus, we have shown that

$$\{U_1^{\mathbf{s}} \geq -2\delta\} \cap \{U_{\gamma_1+\nu}^{\mathbf{s}} \leq 2\delta\} \subseteq \bigcup_{\gamma \in [\gamma_1+\nu,1]} \{|U_\gamma^{\mathbf{s}}| \leq 2\delta\} \tag{3.14}$$

and clearly the same strategy of proof enables to show that

$$\{U_1^{\mathbf{s}} \leq 2\delta\} \cap \{U_{\gamma_1+\nu}^{\mathbf{s}} \geq -2\delta\} \subseteq \bigcup_{\gamma \in [\gamma_1+\nu,1]} \{|U_\gamma^{\mathbf{s}}| \leq 2\delta\}. \tag{3.15}$$

Conversely, let

$$(x,v) \notin (\{U_1^{\mathbf{s}} \geq -2\delta\} \cap \{U_{\gamma_1+\nu}^{\mathbf{s}} \leq 2\delta\}) \cup (\{U_1^{\mathbf{s}} \leq 2\delta\} \cap \{U_{\gamma_1+\nu}^{\mathbf{s}} \geq -2\delta\}).$$

Since $\{U_1^{\mathbf{s}} < -2\delta\} \cap \{U_1^{\mathbf{s}} > 2\delta\}$ and $\{U_{\gamma_1+\nu}^{\mathbf{s}} < -2\delta\} \cap \{U_{\gamma_1+\nu}^{\mathbf{s}} > 2\delta\}$ are empty, we have

$$(x,v) \in \{U_1^{\mathbf{s}} < -2\delta\} \cap \{U_{\gamma_1+\nu}^{\mathbf{s}} < -2\delta\} \quad \text{or} \quad (x,v) \in \{U_{\gamma_1+\nu}^{\mathbf{s}} > 2\delta\} \cap \{U_1^{\mathbf{s}} > 2\delta\}. \tag{3.16}$$

Besides, using (2.8) and (3.2), one can check that the sign of $\partial_\gamma U_\gamma^{\mathbf{s}}(x,v)$ is given by

$$\ell_x^{\mathbf{s}} \cdot \tilde{x}_{\mathbf{s}} - |\ell_x^{\mathbf{s}}|^2 \ell_v^{\mathbf{s}} \cdot v - (\ell_x^{\mathbf{s}} \cdot \tilde{x}_{\mathbf{s}} + |\ell_x^{\mathbf{s}}|^2 \ell_v^{\mathbf{s}} \cdot v)\gamma \tag{3.17}$$

which vanishes at most once in $(\gamma_1 + \nu, 1)$. If it does not vanish in $(\gamma_1 + \nu, 1)$, then, by monotonicity, (3.16) implies that, for any $\gamma \in [\gamma_1 + \nu, 1]$, we have $(x,v) \notin \{|U_\gamma^{\mathbf{s}}| \leq 2\delta\}$. Now, in the case where the expression from (3.17) vanishes at some point in $(\gamma_1 + \nu, 1)$, its values at $\gamma_1 + \nu$ and 1 have opposite signs, i.e.,

$$|\ell_x^{\mathbf{s}}|^2 \ell_v^{\mathbf{s}} \cdot v((1 - \gamma_1 - \nu)\ell_x^{\mathbf{s}} \cdot \tilde{x}_{\mathbf{s}} - |\ell_x^{\mathbf{s}}|^2(1 + \gamma_1 + \nu)\ell_v^{\mathbf{s}} \cdot v) > 0. \tag{3.18}$$

When both factors from (3.18) are positive, we have $\ell_x^{\mathbf{s}} \cdot \tilde{x}_{\mathbf{s}} > 0$, so $U_1^{\mathbf{s}}(x,v) > 0$ and it follows that $(x,v) \in \{U_{\gamma_1+\nu}^{\mathbf{s}} > 2\delta\} \cap \{U_1^{\mathbf{s}} > 2\delta\}$. Moreover, in this case, we also have that the minimum of $\gamma \mapsto U_\gamma^{\mathbf{s}}(x,v)$ on $[\gamma_1 + \nu, 1]$ is attained on the boundary of the interval since $\partial_\gamma U_\gamma^{\mathbf{s}}(x,v)|_{\gamma=1} < 0$, so for any $\gamma \in [\gamma_1 + \nu, 1]$ it holds $(x,v) \in \{U_\gamma^{\mathbf{s}} > 2\delta\}$. Here again, the same strategy of proof enables to show that if both factors from (3.18) are negative, then for any $\gamma \in [\gamma_1 + \nu, 1]$, it holds $(x,v) \in \{U_\gamma^{\mathbf{s}} < -2\delta\}$. Combined with (3.14) and (3.15), this proves (3.13).

From (3.9), (3.10), (3.11), and (3.13), we see that the only parts on which it is not clear that $\theta^{\mathbf{m}}_{\nu,h}$ is smooth are

$$F_1 = \bigsqcup_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\big(\{U_1^{\mathbf{s}} = 2\delta\} \cap \{U^{\mathbf{s}}_{\gamma_1+\nu} \geq 2\delta\} \cap B_0(\mathbf{s}, r)\big),$$

$$F_2 = \bigsqcup_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\big(\{U_1^{\mathbf{s}} \geq 2\delta\} \cap \{U^{\mathbf{s}}_{\gamma_1+\nu} = 2\delta\} \cap B_0(\mathbf{s}, r)\big),$$

$$F_3 = \bigsqcup_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\big(\{U_1^{\mathbf{s}} = -2\delta\} \cap \{U^{\mathbf{s}}_{\gamma_1+\nu} \leq -2\delta\} \cap B_0(\mathbf{s}, r)\big),$$

$$F_4 = \bigsqcup_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\big(\{U_1^{\mathbf{s}} \leq -2\delta\} \cap \{U^{\mathbf{s}}_{\gamma_1+\nu} = -2\delta\} \cap B_0(\mathbf{s}, r)\big),$$

$$F_5 = \bigsqcup_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\Big(\bigcup_{\gamma\in[\gamma_1+\nu,1]}\{|U_\gamma^{\mathbf{s}}| \leq 2\delta\} \cap \partial B_0(\mathbf{s}, r)\Big),$$

and

$$F_6 = \partial(E(\mathbf{m}) + B(0,\varepsilon))\backslash\Big(\bigsqcup_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\Big(\bigcup_{\gamma\in[\gamma_1+\nu,1]}\{|U_\gamma^{\mathbf{s}}| \leq 2\delta\} \cap B_0(\mathbf{s}, r)\Big)\Big).$$

Note that (3.13) suggested to put $\{U_1^{\mathbf{s}} = 2\delta\} \cap \{U^{\mathbf{s}}_{\gamma_1+\nu} \geq -2\delta\} \cap B_0(\mathbf{s}, r)$ in the definition of F_1, but we allowed ourselves to discard the part $\{U_1^{\mathbf{s}} = 2\delta\} \cap \{U^{\mathbf{s}}_{\gamma_1+\nu} \in [-2\delta, 2\delta)\} \cap B_0(\mathbf{s}, r)$ since it is included in the interior of $\{U_1^{\mathbf{s}} \geq -2\delta\} \cap \{U^{\mathbf{s}}_{\gamma_1+\nu} \leq 2\delta\} \cap B_0(\mathbf{s}, r)$ (and we did similarly for F_2, F_3 and F_4).

Now, let $\mathbf{s} \in \mathbf{j}(\mathbf{m})$ and $(\gamma, x, v) \in [\gamma_1 + \nu, 1] \times \overline{B_0(\mathbf{s}, r)}\backslash\{(\mathbf{s}, 0)\}$ such that

$$U_\gamma^{\mathbf{s}}(x, v) = L_\gamma^{\mathbf{s}}(x, v) = 0.$$

Using Lemma 3.1, we see that if $r > 0$ is small enough,

$$W(x, v) = W(x, v) + \frac{1}{2}L_\gamma^{\mathbf{s}}(x, v)^2 > W(\mathbf{s}, 0). \tag{3.19}$$

Hence, for all $\gamma \in [\gamma_1 + \nu, 1]$, the set $\{U_\gamma^{\mathbf{s}} = 0\} \cap B_0(\mathbf{s}, r)$ is contained in $\{W \geq \boldsymbol{\sigma}(\mathbf{m})\}$. Assume by contradiction that for any $r > 0$, the function $U_\gamma^{\mathbf{s}}$ takes both positive and negative values on $E(\mathbf{m}) \cap B_0(\mathbf{s}, r)$. Then, according to Lemma 2.5, the two CCs of $\mathcal{O}_r \cap \{W < \boldsymbol{\sigma}(\mathbf{m})\}$ are both included in $E(\mathbf{m})$ (the one on which $U_\gamma^{\mathbf{s}} > 0$ and the one where $U_\gamma^{\mathbf{s}} < 0$). This is a contradiction with the fact that $\mathbf{s} \in \mathtt{V}^{(1)}$. Therefore, $U_\gamma^{\mathbf{s}}$ has a sign on $E(\mathbf{m}) \cap B_0(\mathbf{s}, r)$ and since it depends smoothly on γ and cannot vanish on $E(\mathbf{m}) \cap B_0(\mathbf{s}, r)$, this sign does not depend on γ. In particular, it is given by the sign of $U_0^{\mathbf{s}}$ on $E(\mathbf{m}) \cap B_0(\mathbf{s}, r)$ so taking $\ell^{\mathbf{s}}$ such that

$$\ell^{\mathbf{s}} \cdot (\phi_{\mathbf{s}}(x_0), v_0) > 0 \tag{3.20}$$

for some $(x_0, v_0) \in E(\mathbf{m}) \cap B_0(\mathbf{s}, r)$, we get that for each $\gamma \in [\gamma_1 + \nu, 1]$, the function $U_\gamma^{\mathbf{s}}$ is positive on $E(\mathbf{m}) \cap B_0(\mathbf{s}, r)$. We can then choose $\varepsilon(\delta) > 0$ small enough so that

$$((E(\mathbf{m}) + B(0, \varepsilon)) \cap B_0(\mathbf{s}, r)) \subseteq \{U_1^{\mathbf{s}} \geq -\delta\} \cap \{U_{\gamma_1+\nu}^{\mathbf{s}} \geq -\delta\}. \tag{3.21}$$

Similarly, if we denote $\Omega_{\mathbf{s}}$ the other CC of $\{W < \boldsymbol{\sigma}(\mathbf{m})\}$ which contains $(\mathbf{s}, 0)$ on its boundary, one can check that $(\phi_{\mathbf{s}}^{-1}(-\phi_{\mathbf{s}}(x_0)), -v_0) \in \Omega_{\mathbf{s}} \cap B_0(\mathbf{s}, r) \cap \{U_0^{\mathbf{s}} < 0\}$ where (x_0, v_0) was introduced in (3.20) so $U_\gamma^{\mathbf{s}}$ is negative on $\Omega_{\mathbf{s}} \cap B_0(\mathbf{s}, r)$ and

$$\big((\Omega_{\mathbf{s}} + B(0, \varepsilon)) \cap B_0(\mathbf{s}, r)\big) \subseteq \{U_1^{\mathbf{s}} \leq \delta\} \cap \{U_{\gamma_1+\nu}^{\mathbf{s}} \leq \delta\}. \tag{3.22}$$

Choosing once again $\varepsilon(r)$ small enough, we can even assume that

$$\big(\overline{E(\mathbf{m}) + B(0, \varepsilon)} \cap \overline{\Omega_{\mathbf{s}} + B(0, \varepsilon)}\big) \subseteq \mathbf{j}^W(\mathbf{m}) + B_0(0, r) \tag{3.23}$$

(see [11, Lemma 3.2] for more details). We first prove that $\theta_{\nu,h}^{\mathbf{m}}$ is smooth on $F_1 \cap (\Omega + B(0, \varepsilon))$: let $\mathbf{s} \in \mathbf{j}(\mathbf{m})$ and $(x, v) \in B_0(\mathbf{s}, r) \cap \{U_1^{\mathbf{s}} = 2\delta\} \cap \{U_{\gamma_1+\nu}^{\mathbf{s}} \geq 2\delta\} \cap (\Omega + B(0, \varepsilon))$. According to (3.22), there exists a small ball B centered in (x, v) such that

$$B \subset (B_0(\mathbf{s}, r) \cap \{U_1^{\mathbf{s}} > \delta\} \cap \{U_{\gamma_1+\nu}^{\mathbf{s}} > \delta\} \cap (E(\mathbf{m}) + B(0, \varepsilon))).$$

Thus, according to (3.9), (3.10), and (3.13), with δ instead of 2δ, we have $\theta_{\nu,h}^{\mathbf{m}} = 1$ on B, so $\theta_{\nu,h}^{\mathbf{m}}$ is smooth at (x, v). Obviously, the same goes for $F_2 \cap (\Omega + B(0, \varepsilon))$, and, similarly, for $(x, v) \in (F_3 \cup F_4) \cap (\Omega + B(0, \varepsilon))$, we can show that $\theta_{\nu,h}^{\mathbf{m}} = 0$ in a neighborhood of (x, v).

Now, we show that F_5 does not meet $\Omega + B(0, \varepsilon)$. Recall that Ω denotes the CC of $\{W \leq \boldsymbol{\sigma}(\mathbf{m})\}$ containing $\mathbf{m}$. For $\mathbf{s} \in \mathbf{j}(\mathbf{m})$, we can deduce from (3.19) that if $(\gamma, x, v) \in [\gamma_1 + \nu, 1] \times \partial B_0(\mathbf{s}, r)$ is such that $U_\gamma^{\mathbf{s}}(x, v) = 0$, then $(x, v) \notin \Omega$. Hence, $(\gamma, x, v) \mapsto |U_\gamma^{\mathbf{s}}(x, v)|$ must attain a positive minimum on $[\gamma_1 + \nu, 1] \times (\partial B_0(\mathbf{s}, r) \cap \Omega)$, so we can choose $\delta(r, \nu) > 0$ independent of γ such that for all $\gamma \in [\gamma_1 + \nu, 1]$, the set $\partial B_0(\mathbf{s}, r) \cap \{|U_\gamma^{\mathbf{s}}| \leq 2\delta\}$ does not intersect Ω. It follows that we can choose $\varepsilon(\delta) > 0$ such that

$$F_5 \subseteq \big(\mathbb{R}^{2d} \backslash \overline{\Omega + B(0, \varepsilon)}\big).$$

It only remains to prove that, as for F_5, the set F_6 does not meet $\Omega + B(0, \varepsilon)$. If $(x, v) \in F_6 \cap B_0(\mathbf{s}, r)$, (3.21) and (3.13) imply that $(x, v) \in \{U_1^{\mathbf{s}} \geq 2\delta\} \cap \{U_{\gamma_1+\nu}^{\mathbf{s}} \geq 2\delta\}$ so using (3.22), we see that (x, v) is outside $\Omega_{\mathbf{s}} + B(0, \varepsilon)$. Since it is not in $(E(\mathbf{m}) + B(0, \varepsilon))$ either, it is outside $\Omega + B(0, \varepsilon)$. Now, if $(x, v) \in F_6 \backslash (\mathbf{j}^W(\mathbf{m}) + B_0(0, r))$, (3.23) implies that (x, v) is outside $\cup_{\mathbf{j}(\mathbf{m})}(\Omega_{\mathbf{s}} + B(0, \varepsilon))$ so it is also outside $\Omega + B(0, \varepsilon)$ for ε small enough and the proof is complete. ∎

From now on, we fix the sign of $\ell^{\mathbf{s}}$ as well as $\varepsilon > 0$ and $\delta > 0$ such that the conclusion of Proposition 3.2 holds true. In particular, even though we do not make it appear in the notations, the functions $\chi_{\mathbf{m}}$ and ζ now depend on ν. Finally, we denote

$$W^{\mathbf{m}}(x, v) = W(x, v) - \frac{1}{2} V(\mathbf{m}) \tag{3.24}$$

and it is clear from (3.12) that

$$W^{\mathbf{m}} \geq S(\mathbf{m}) + \tilde{\varepsilon} \quad \text{on supp } \nabla \chi_{\mathbf{m}}. \tag{3.25}$$

Our quasimodes will be the L^2-renormalizations of the functions

$$f^{\mathbf{m}}_{\nu,h}(x, v) = \chi_{\mathbf{m}}(x, v) \theta^{\mathbf{m}}_{\nu,h}(x, v)\, \mathrm{e}^{-W^{\mathbf{m}}(x,v)/h}, \quad \mathbf{m} \in \mathrm{U}^{(0)} \backslash \{\underline{\mathbf{m}}\}, \tag{3.26}$$

and, for $\mathbf{m} = \underline{\mathbf{m}}$,

$$f_{\underline{\mathbf{m}},h}(x, v) = \mathrm{e}^{-W^{\underline{\mathbf{m}}}(x,v)/h} \in \operatorname{Ker} P_h.$$

Note that for $\mathbf{m} \neq \underline{\mathbf{m}}$, we have $f^{\mathbf{m}}_{\nu,h} \in \mathscr{C}^{\infty}_c(\mathbb{R}^{2d})$ thanks to Proposition 3.2 and

$$\text{supp } f^{\mathbf{m}}_{\nu,h} \subseteq E(\mathbf{m}) + B(0, \varepsilon') \tag{3.27}$$

where $\varepsilon' = \max(\varepsilon, r)$.

3.2. Action of the operator P_h

Let us fix $\mathbf{m} \in \mathrm{U}^{(0)} \backslash \{\underline{\mathbf{m}}\}$. For $\gamma \in (\gamma_1, 1]$, we will denote

$$\widetilde{W}^{\mathbf{m}}_{\gamma}(x, v) = W^{\mathbf{m}}(x, v) + \frac{1}{2} \sum_{\mathbf{s} \in \mathbf{j}(\mathbf{m})} L^{\mathbf{s}}_{\gamma}(x, v)^2. \tag{3.28}$$

For $\mathbf{s} \in \mathbf{j}(\mathbf{m})$ and $x \in B(\mathbf{s}, \tilde{r})$, we also denote

$$\tilde{\theta}^{\mathbf{s}}_{\gamma,h}(x, v) = \int_0^{L^{\mathbf{s}}_{\gamma}(x,v)} \mathrm{e}^{-\frac{s^2}{2h}}\, \mathrm{d}s. \tag{3.29}$$

We now have to compute $P_h f^{\mathbf{m}}_{\nu,h}$. We will see fairly easily thanks to (3.35) that X^h_0 applied to $f^{\mathbf{m}}_{\nu,h}$ will yield a superposition of the exponentials

$$(\mathrm{e}^{-\widetilde{W}^{\mathbf{m}}_{\gamma}/h})_{\gamma \in [\gamma_1 + \nu, 1]}. \tag{3.30}$$

In view of (3.9), we see that the computation of $Q_h f^{\mathbf{m}}_{\nu,h}$ will essentially boil down to the one of $Q_h(\tilde{\theta}^{\mathbf{s}}_{\gamma,h}\,\mathrm{e}^{-W^{\mathbf{m}}/h})$ which we are already able to do thanks to Lemma 2.4:

$$Q_h(\tilde{\theta}^{\mathbf{s}}_{\gamma,h}\,\mathrm{e}^{-W^{\mathbf{m}}/h}) = -h\int_0^1 \partial_y \mathcal{L}^{\mathbf{s}}(\gamma,y)\exp\Big[-\frac{1}{h}\Big(W^{\mathbf{m}}(x,v)+\frac{1}{2}[\mathcal{L}^{\mathbf{s}}(\gamma,y)\cdot(\tilde{x}_{\mathbf{s}},v)]^2\Big)\Big]\,\mathrm{d}y\cdot\begin{pmatrix}\tilde{x}_{\mathbf{s}}\\ v\end{pmatrix},$$

where $\mathcal{L}^{\mathbf{s}}(\gamma,y)$ stands for the vector

$$\Big(\frac{1+y}{(4|L^{\mathbf{s}}_{\gamma,v}|^2y+(y+1)^2)^{1/2}}\,L^{\mathbf{s}}_{\gamma,x},\ \frac{1-y}{(4|L^{\mathbf{s}}_{\gamma,v}|^2y+(y+1)^2)^{1/2}}\,L^{\mathbf{s}}_{\gamma,v}\Big). \tag{3.31}$$

Here we disregarded the fact that the linear form L_γ is twisted in x as Q_h only acts in v. Our concern is now to see whether the functions

$$\Big(\exp\Big[-\frac{1}{h}\Big(W^{\mathbf{m}}(x,v)+\frac{1}{2}[\mathcal{L}^{\mathbf{s}}(\gamma,y)\cdot(\tilde{x}_{\mathbf{s}},v)]^2\Big)\Big]\Big)_{\gamma\in[\gamma_1+\nu,1],y\in[0,1]}$$

belong to the family (3.30) as we hoped for some compensations between $X^h_0 f^{\mathbf{m}}_{\nu,h}$ and $Q_h f^{\mathbf{m}}_{\nu,h}$. It appears to be the case as, denoting for $\gamma\in(\gamma_1,1]$ and $y\in[0,1]$

$$\Gamma_\gamma(y)=\frac{y+\gamma}{1+y\gamma}, \tag{3.32}$$

an easy computation shows that

$$\mathcal{L}^{\mathbf{s}}(\gamma,y)=(L^{\mathbf{s}}_{\Gamma_\gamma(y),x};L^{\mathbf{s}}_{\Gamma_\gamma(y),v}). \tag{3.33}$$

We sum up the above discussion in the following updated version of Lemma 2.4.

Lemma 3.3. *With the notations* (3.29), (3.32), *and* (3.28), *we have*

$$\begin{aligned} h\,\mathrm{Op}_h(g)(\mathrm{e}^{-\frac{\widetilde{W}^{\mathbf{m}}_\gamma}{h}}L^{\mathbf{s}}_{\gamma,v}) &= Q_h(\tilde{\theta}^{\mathbf{s}}_{\gamma,h}\,\mathrm{e}^{-W^{\mathbf{m}}/h})(x,v)\\ &= -h\int_0^1 \partial_y(L_{\Gamma_\gamma(y)})\,\mathrm{e}^{-\frac{\widetilde{W}^{\mathbf{m}}_{\Gamma_\gamma(y)}}{h}}\,\mathrm{d}y\cdot\begin{pmatrix}\tilde{x}_{\mathbf{s}}\\ v\end{pmatrix}.\end{aligned}$$

Moreover,

$$\begin{aligned}&\mathrm{Op}_h(m_{y,h}\,\mathrm{Id})\circ b_h(\tilde{\theta}^{\mathbf{s}}_{\gamma,h}\,\mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}})\\ &\quad= 2h(2\pi h)^{-d/2}\,\mathrm{e}^{-\frac{V(x)-V(\mathbf{m})}{2h}}\,\frac{(y+1)^{d-2}}{(4y)^{\frac{d}{2}}}\\ &\qquad\times\int_{v'\in\mathbb{R}^d}\mathrm{e}^{-\frac{1}{h}(\frac{v'^2}{4}+\frac{y}{8}(v+v')^2+\frac{(v-v')^2}{8y}+\frac{1}{2}L^{\mathbf{s}}_\gamma(x,v')^2)}\,\mathrm{d}v'\,L^{\mathbf{s}}_{\gamma,v}.\end{aligned} \tag{3.34}$$

We are now in position to give a precise computation of $P_h f^{\mathbf{m}}_{\nu,h}$.

Proposition 3.4. *Let $f^{\mathbf{m}}_{\nu,h}$ be the quasimode defined in (3.26) and recall the notations (3.7) and (3.28). There exist some functions $R^{\mathbf{m}}_{\nu,h}$ and $(\omega^{\mathbf{m}}_{\nu,z})_{z\in[\gamma_1+\nu,1]}$ in $L^2(\mathbb{R}^{2d})$ such that*

(1) *the function $P_h f^{\mathbf{m}}_{\nu,h} - R^{\mathbf{m}}_{\nu,h}$ is supported in $\mathbf{j}^W(\mathbf{m}) + B_0(0,r)$;*

(2) *the function $R^{\mathbf{m}}_{\nu,h}$ is $O_{\nu,L^2}(h^{\frac{3+d}{2}}\,\mathrm{e}^{-\frac{S(\mathbf{m})}{h}})$;*

(3) *for $(x,v) \in \mathbf{j}^W(\mathbf{m}) + B_0(0,r)$, one has*

$$(P_h f^{\mathbf{m}}_{\nu,h} - R^{\mathbf{m}}_{\nu,h})(x,v) = \Big(\frac{h}{2\pi}\Big)^{1/2}\int_{\gamma_1+\nu}^{1} \omega^{\mathbf{m}}_{\nu,z}(x,v)\exp\Big[-\frac{1}{h}\widetilde{W}^{\mathbf{m}}_z(x,v)\Big]\,\mathrm{d}z$$

where, using the notation (1.11), we have the expression

$$\omega^{\mathbf{m}}_{\nu,z}(x,v) = \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\Bigg[k^{\mathbf{s}}_\nu(z)\begin{pmatrix}0 & -\mathcal{V}_{\mathbf{s}}\\ \mathrm{Id} & 0\end{pmatrix}\begin{pmatrix}L^{\mathbf{s}}_{z;x}\\ L^{\mathbf{s}}_{z;v}\end{pmatrix} - \int_{\gamma_1+\nu}^{z} k^{\mathbf{s}}_\nu(\gamma)\,\mathrm{d}\gamma\begin{pmatrix}\partial_z L^{\mathbf{s}}_{z;x}\\ \partial_z L^{\mathbf{s}}_{z;v}\end{pmatrix}\Bigg]\cdot\begin{pmatrix}\tilde{x}_{\mathbf{s}}\\ v\end{pmatrix}.$$

Proof. In order to lighten the notations, we will drop some of the exponents and indexes $\mathbf{m}$, $\mathbf{s}$, ν and h in the proof. We know that θ is smooth on the support of χ and since θ is constant outside of $\mathbf{j}^W(\mathbf{m}) + B_0(0,r)$, we have

$$\nabla\theta = \frac{1}{2}\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}(A^{\mathbf{s}}_h)^{-1}\int_{\gamma_1+\nu}^{1} k^{\mathbf{s}}(\gamma)\zeta(U^{\mathbf{s}}_\gamma)\,\mathrm{e}^{-(L^{\mathbf{s}}_\gamma)^2/2h}\,\nabla L^{\mathbf{s}}_\gamma\,\mathbf{1}_{B_0(\mathbf{s},r)}\,\mathrm{d}\gamma. \tag{3.35}$$

Using Corollary 2.3, we can then begin by computing

$$\begin{aligned}Q_h(f) &= h\,\mathrm{Op}_h(g)((\partial_v\theta)\chi\,\mathrm{e}^{-W^{\mathbf{m}}/h} + (\partial_v\chi)\theta\,\mathrm{e}^{-W^{\mathbf{m}}/h})\\ &= \frac{h}{2}\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}(A^{\mathbf{s}}_h)^{-1}\int_{\gamma_1+\nu}^{1} k^{\mathbf{s}}(\gamma)\,\mathrm{Op}_h(g)(\chi\zeta(U^{\mathbf{s}}_\gamma)\,\mathrm{e}^{-\frac{\widetilde{W}^{\mathbf{m}}_\gamma}{h}}\,\mathbf{1}_{B_0(\mathbf{s},r)}L^{\mathbf{s}}_{\gamma,v})\,\mathrm{d}\gamma\\ &\quad + O_\nu(h\,\mathrm{e}^{-\frac{S(\mathbf{m})+\tilde{\varepsilon}}{h}}),\end{aligned} \tag{3.36}$$

as χ now depends on ν, where we used (3.25) as well as the fact that $\mathrm{Op}_h(g)$ is bounded uniformly in h since $g \in S^{1/2}(\langle(v,\eta)\rangle^{-1})$. Now, since $\chi\zeta(U^{\mathbf{s}}_\gamma) - 1 = O_\nu((x-\mathbf{s},v)^2)$, we have thanks to Lemma 3.1 and by a standard Laplace method (see Proposition D.1) that

$$(\chi\zeta(U_\gamma^{\mathbf{s}}) - 1)\, e^{-\frac{\widetilde{W}_\gamma^{\mathbf{m}}}{h}}\, \mathbf{1}_{B_0(\mathbf{s},r)} \nabla L_\gamma^{\mathbf{s}} = O_\nu(h^{1+\frac{d}{2}}\, e^{-\frac{S(\mathbf{m})}{h}}). \tag{3.37}$$

Hence, still by the boundedness of $\mathrm{Op}_h(g)$, we get that

$$\begin{aligned}\mathrm{Op}_h(g)&(\chi\zeta(U_\gamma^{\mathbf{s}})\, e^{-\frac{\widetilde{W}_\gamma^{\mathbf{m}}}{h}}\, \mathbf{1}_{B_0(\mathbf{s},r)} L_{\gamma,v}^{\mathbf{s}})\\ &= \mathrm{Op}_h(g)(e^{-\frac{\widetilde{W}_\gamma^{\mathbf{m}}}{h}}\, \mathbf{1}_{B_0(\mathbf{s},r)} L_{\gamma,v}^{\mathbf{s}})d + O_\nu(h^{1+\frac{d}{2}}\, e^{-\frac{S(\mathbf{m})}{h}}).\end{aligned} \tag{3.38}$$

In the same spirit, we can write

$$\mathbf{1}_{B_0(\mathbf{s},r)} L_{\gamma,v}^{\mathbf{s}} = \mathbf{1}_{|x-\mathbf{s}|<\tilde{r}}(\mathbf{1}_{|v|<r} - 1 + 1) L_{\gamma,v}^{\mathbf{s}} = \mathbf{1}_{|x-\mathbf{s}|<\tilde{r}}\, L_{\gamma,v}^{\mathbf{s}} + \rho_\gamma$$

with ρ_γ supported in $\{(x,v);\ |x-\mathbf{s}| < \tilde{r} \text{ and } |v| \geq r\}$ and such that $\|\rho_\gamma\|_\infty \leq C_\nu$; so, using the boudedness of $\mathrm{Op}_h(g)$ again and the fact that it is local in the variable x, as well as (3.1), we get

$$\begin{aligned}\mathrm{Op}_h(g)&(e^{-\frac{\widetilde{W}_\gamma^{\mathbf{m}}}{h}}\, \mathbf{1}_{B_0(\mathbf{s},r)} L_{\gamma,v}^{\mathbf{s}})\\ &= \mathrm{Op}_h(g)(e^{-\frac{\widetilde{W}_\gamma^{\mathbf{m}}}{h}}\, L_{\gamma,v}^{\mathbf{s}})\mathbf{1}_{|x-\mathbf{s}|<\tilde{r}} + O_\nu(h^{1+\frac{d}{2}}\, e^{-\frac{S(\mathbf{m})}{h}}).\end{aligned} \tag{3.39}$$

Hence, putting (3.38) and (3.39) together and using (3.8), we get that (3.36) becomes

$$\begin{aligned}Q_h(f) = \Big(\frac{h}{2\pi}\Big)^{1/2} \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} \int_{\gamma_1+\nu}^{1} k^{\mathbf{s}}(\gamma)\, \mathrm{Op}_h(g)(e^{-\frac{\widetilde{W}_\gamma^{\mathbf{m}}}{h}}\, L_{\gamma,v}^{\mathbf{s}})\, \mathrm{d}\gamma\, \mathbf{1}_{|x-\mathbf{s}|<\tilde{r}}\\ + O_\nu(h^{\frac{3+d}{2}}\, e^{-\frac{S(\mathbf{m})}{h}}),\end{aligned} \tag{3.40}$$

which further gives

$$\begin{aligned}Q_h(f) + O_\nu(h^{\frac{3+d}{2}}\, e^{-\frac{S(\mathbf{m})}{h}}) = -\Big(\frac{h}{2\pi}\Big)^{1/2} \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} \int_{\gamma_1+\nu}^{1} k^{\mathbf{s}}(\gamma) \int_0^1 \partial_y(L_{\Gamma_\gamma(y)})\\ \times \exp\Big[-\frac{1}{h}\widetilde{W}_{\Gamma_\gamma(y)}^{\mathbf{m}}\Big]\, \mathrm{d}y \cdot \begin{pmatrix}\tilde{x}_{\mathbf{s}}\\ v\end{pmatrix} \mathrm{d}\gamma\, \mathbf{1}_{|x-\mathbf{s}|<\tilde{r}}\end{aligned} \tag{3.41}$$

thanks to Lemma 3.3. By the change of variable $z = \Gamma_\gamma(y)$, the integral in y from (3.41) becomes

$$\int_\gamma^1 \partial_z(L_z^{\mathbf{s}}) \exp\Big[-\frac{1}{h}\widetilde{W}_z^{\mathbf{m}}(x,v)\Big]\, \mathrm{d}z.$$

Therefore, switching the order of integration and using (3.1) again, (3.41) yields that the function $Q_h(f)$ satisfies

$$Q_h(f)(x,v) + O_\nu(h^{\frac{3+d}{2}} e^{-\frac{S(\mathbf{m})}{h}})$$
$$= -\Big(\frac{h}{2\pi}\Big)^{1/2} \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} \int_{\gamma_1+\nu}^{1} \int_{\gamma_1+\nu}^{z} k^{\mathbf{s}}(\gamma)\, d\gamma\, \partial_z(L_z^{\mathbf{s}}) \cdot \begin{pmatrix} \tilde{x}_{\mathbf{s}} \\ v \end{pmatrix} e^{-\frac{\widetilde{W}_z^{\mathbf{m}}(x,v)}{h}}\, dz\, \mathbf{1}_{B_0(\mathbf{s},r)}(x,v). \tag{3.42}$$

Now, the computation for the transport term is easier: according to (3.35), we have

$$\begin{aligned} X_0^h f &= h\begin{pmatrix} v \\ -\partial_x V \end{pmatrix} \cdot \nabla f \\ &= h\begin{pmatrix} v \\ -\partial_x V \end{pmatrix} \cdot \nabla\theta\, \chi\, e^{-W^{\mathbf{m}}/h} + h\begin{pmatrix} v \\ -\partial_x V \end{pmatrix} \cdot \nabla\chi\, \theta\, e^{-W^{\mathbf{m}}/h} \\ &= \frac{h}{2}\chi \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} (A_h^{\mathbf{s}})^{-1} \int_{\gamma_1+\nu}^{1} k^{\mathbf{s}}(z)\zeta(U_z^{\mathbf{s}}) \begin{pmatrix} v \\ -\partial_x V \end{pmatrix} \cdot \nabla L_z^{\mathbf{s}}\, e^{-\frac{\widetilde{W}_z^{\mathbf{m}}}{h}}\, \mathbf{1}_{B_0(\mathbf{s},r)}\, dz \\ &\quad + O_\nu(h\, e^{-\frac{S(\mathbf{m})+\tilde{\varepsilon}}{h}}) \end{aligned}$$

thanks to (3.25). Here again, we can use (3.8) and (3.37) to get

$$\begin{aligned} X_0^h f &= \Big(\frac{h}{2\pi}\Big)^{1/2} \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} \int_{\gamma_1+\nu}^{1} k^{\mathbf{s}}(z) \begin{pmatrix} v \\ -\partial_x V \end{pmatrix} \cdot \nabla L_z^{\mathbf{s}}\, e^{-\frac{\widetilde{W}_z^{\mathbf{m}}}{h}}\, \mathbf{1}_{B_0(\mathbf{s},r)}\, dz \\ &\quad + O_\nu(h^{\frac{3+d}{2}} e^{-\frac{S(\mathbf{m})}{h}}). \end{aligned}$$

Recalling that the differential of $\phi_{\mathbf{s}}$ at $\mathbf{s}$ is the identity, the last step consists in using (3.6) to write

$$\begin{aligned} \begin{pmatrix} v \\ -\partial_x V \end{pmatrix} \cdot \nabla L_z^{\mathbf{s}} &= \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathcal{V}_{\mathbf{s}} & 0 \end{pmatrix} \begin{pmatrix} \tilde{x}_{\mathbf{s}} \\ v \end{pmatrix} \cdot \begin{pmatrix} L_{z,x}^{\mathbf{s}} \\ L_{z,v}^{\mathbf{s}} \end{pmatrix} + O_\nu((\tilde{x}_{\mathbf{s}}, v)^2) \\ &= \begin{pmatrix} 0 & -\mathcal{V}_{\mathbf{s}} \\ \mathrm{Id} & 0 \end{pmatrix} \begin{pmatrix} L_{z,x}^{\mathbf{s}} \\ L_{z,v}^{\mathbf{s}} \end{pmatrix} \cdot \begin{pmatrix} \tilde{x}_{\mathbf{s}} \\ v \end{pmatrix} + O_\nu((\tilde{x}_{\mathbf{s}}, v)^2) \end{aligned}$$

and the same argument that we used to establish (3.37) yields that the function $X_0^h f$ satisfies

$$X_0^h f(x,v) + O_{\nu,L^2}(h^{\frac{3+d}{2}} e^{-S(\mathbf{m})/h})$$
$$= \Big(\frac{h}{2\pi}\Big)^{1/2} \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} \int_{\gamma_1+\nu}^{1} k^{\mathbf{s}}(z) \begin{pmatrix} 0 & -\mathcal{V}_{\mathbf{s}} \\ \mathrm{Id} & 0 \end{pmatrix} \begin{pmatrix} L_{z,x}^{\mathbf{s}} \\ L_{z,v}^{\mathbf{s}} \end{pmatrix} \cdot \begin{pmatrix} \tilde{x}_{\mathbf{s}} \\ v \end{pmatrix} e^{-\frac{\widetilde{W}_z^{\mathbf{m}}(x,v)}{h}}\, dz\, \mathbf{1}_{B_0(\mathbf{s},r)}(x,v). \tag{3.43}$$

The conclusion follows from (3.42) and (3.43). ■

Remark 3.5. Since $P_h^* = -X_0^h + Q_h$, it is clear from (3.42) and (3.43) that

$$P_h^* f_{\nu,h}^{\mathbf{m}} = \Big(\frac{h}{2\pi}\Big)^{1/2} \int_{\gamma_1+\nu}^{1} \overset{*}{\omega}{}_{\nu,z}^{\mathbf{m}}(x,v)\, \exp\Big[-\frac{1}{h}\widetilde{W}_z^{\mathbf{m}}(x,v)\Big]\, \mathrm{d}z + O_{\nu,L^2}(h^{\frac{3+d}{2}}\, \mathrm{e}^{-\frac{S(\mathbf{m})}{h}})$$

with

$$\overset{*}{\omega}{}_{\nu,z}^{\mathbf{m}}(x,v) = -\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} \Bigg[k_\nu^{\mathbf{s}}(z)\begin{pmatrix}0 & -\mathcal{V}_{\mathbf{s}}\\ \mathrm{Id} & 0\end{pmatrix}\begin{pmatrix}L_{z;x}^{\mathbf{s}}\\ L_{z;v}^{\mathbf{s}}\end{pmatrix} + \int_{\gamma_1+\nu}^{z} k_\nu^{\mathbf{s}}(\gamma)\,\mathrm{d}\gamma \begin{pmatrix}\partial_z L_{z;x}^{\mathbf{s}}\\ \partial_z L_{z;v}^{\mathbf{s}}\end{pmatrix}\Bigg]\cdot\begin{pmatrix}\tilde{x}_{\mathbf{s}}\\ v\end{pmatrix}\mathbf{1}_{\mathbf{j}^W(\mathbf{m})+B_0(0,r)}(x,v).$$

3.3. Choices of ℓ and k

Following the steps from [1, 11], we would like in view of Proposition 3.4 to find $(\ell^{\mathbf{s}})_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} \subset \mathbb{R}^{2d}$ satisfying (3.2) and (3.3) as well as some probability densities $(k_\nu^{\mathbf{s}})_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}$ on $[\gamma_1+\nu, 1]$ for which the leading term of $P_h f_{\nu,h}^{\mathbf{m}}$ vanishes, i.e., such that

$$k_\nu^{\mathbf{s}}(z)\begin{pmatrix}0 & -\mathcal{V}_{\mathbf{s}}\\ \mathrm{Id} & 0\end{pmatrix}\begin{pmatrix}L_{z;x}^{\mathbf{s}}\\ L_{z;v}^{\mathbf{s}}\end{pmatrix} - \int_{\gamma_1+\nu}^{z} k_\nu^{\mathbf{s}}(\gamma)\,\mathrm{d}\gamma \begin{pmatrix}\partial_z L_{z;x}^{\mathbf{s}}\\ \partial_z L_{z;v}^{\mathbf{s}}\end{pmatrix} = 0, \tag{3.44}$$

for all $\mathbf{s}\in\mathbf{j}(\mathbf{m})$, $z\in[\gamma_1+\nu,1]$.

As it will be more convenient to handle than the function $k_\nu^{\mathbf{s}}$, let us introduce the cumulative distribution function (CDF) on $[\gamma_1+\nu,1]$ associated to $k_\nu^{\mathbf{s}}$:

$$K_\nu^{\mathbf{s}}(z) = \int_{\gamma_1+\nu}^{z} k_\nu^{\mathbf{s}}(\gamma)\,\mathrm{d}\gamma. \tag{3.45}$$

Lemma 3.6. *Recall the notations (1.11)–(1.12). If $(\ell^{\mathbf{s}})_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}$ is a family of vectors satisfying (3.2) and $(k_\nu^{\mathbf{s}})_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}$ is a family of probability densities on $[\gamma_1+\nu,1]$ for which (3.44) holds true, then*

$$\mathcal{V}_{\mathbf{s}}\ell_v^{\mathbf{s}} = \tau_{\mathbf{s}}\ell_v^{\mathbf{s}}, \quad \ell_x^{\mathbf{s}} = -\sqrt{2|\tau_{\mathbf{s}}|}\,\ell_v^{\mathbf{s}} \tag{3.46}$$

(in particular, $\ell_x^{\mathbf{s}}$ satisfies (3.3)) and the function $K_\nu^{\mathbf{s}}$ defined in (3.45) is a CDF on $[\gamma_1+\nu,1]$ satisfying the ODE

$$(K_\nu^{\mathbf{s}})'(z) - \frac{2\sqrt{2}}{\sqrt{|\tau_{\mathbf{s}}|}P(z)}K_\nu^{\mathbf{s}}(z) = 0. \tag{3.47}$$

Proof. Let $(\ell^{\mathbf{s}})_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}$ and $(k_\nu^{\mathbf{s}})_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}$ satisfying the hypotheses of the lemma. According to (2.8), (3.2), and (3.44), we have

$$-k_\nu^{\mathbf{s}}(z)\mathcal{V}_{\mathbf{s}}\ell_v^{\mathbf{s}} + 2\frac{K_\nu^{\mathbf{s}}(z)}{P(z)}\ell_x^{\mathbf{s}} = 0 \quad \text{and} \quad k_\nu^{\mathbf{s}}(z)\ell_x^{\mathbf{s}} + 4\frac{K_\nu^{\mathbf{s}}(z)}{P(z)}\ell_v^{\mathbf{s}} = 0, \tag{3.48}$$

from which we deduce that there exists $\sigma_{\mathbf{s}} < 0$ such that $\ell_x^{\mathbf{s}} = \sigma_{\mathbf{s}}\ell_v^{\mathbf{s}}$ and consequently, that $\ell_v^{\mathbf{s}}$ is an eigenvector of $\mathcal{V}_{\mathbf{s}}$ associated to its negative eigenvalue $\tau_{\mathbf{s}}$. Plugging these informations in (3.48), we obtain

$$|\tau_{\mathbf{s}}|k_\nu^{\mathbf{s}}(z) + 2\sigma_{\mathbf{s}}\frac{K_\nu^{\mathbf{s}}(z)}{P(z)} = 0 \quad \text{and} \quad \sigma_{\mathbf{s}}k_\nu^{\mathbf{s}}(z) + 4\frac{K_\nu^{\mathbf{s}}(z)}{P(z)} = 0$$

which yield $\sigma_{\mathbf{s}} = -\sqrt{2|\tau_{\mathbf{s}}|}$ and (3.47). ■

Since the sign of $\ell^{\mathbf{s}}$ was fixed by Proposition 3.2 and $|\ell_v^{\mathbf{s}}|^2 = 1$, the choice of $\ell^{\mathbf{s}}$ is entirely determined by (3.46). Unfortunately, there is no CDF on $[\gamma_1 + \nu, 1]$ satisfying (3.47). However, there exists a CDF on the whole segment $(\gamma_1, 1]$ solving (3.47), which up to renormalization is given by

$$K_0^{\mathbf{s}}(z) = \Big(\frac{z-\gamma_1}{z-\gamma_2}\Big)^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}}, \quad \text{i.e.,} \quad k_0^{\mathbf{s}}(z) = \frac{\gamma_1-\gamma_2}{2\sqrt{|\tau_{\mathbf{s}}|}(z-\gamma_2)^2}\Big(\frac{z-\gamma_1}{z-\gamma_2}\Big)^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}-1}. \tag{3.49}$$

This leads to the introduction of the following CDF on $[\gamma_1 + \nu, 1]$ which will be an approximate solution of (3.47):

$$K_\nu^{\mathbf{s}}(z) = \frac{K_0^{\mathbf{s}}(z) - K_0^{\mathbf{s}}(\gamma_1+\nu)}{B_\nu^{\mathbf{s}}} \quad \text{and} \quad k_\nu^{\mathbf{s}}(z) = (K_\nu^{\mathbf{s}})'(z) = \frac{k_0^{\mathbf{s}}(z)}{B_\nu^{\mathbf{s}}} \tag{3.50}$$

where

$$B_\nu^{\mathbf{s}} = K_0^{\mathbf{s}}(1) - K_0^{\mathbf{s}}(\gamma_1+\nu) = K_0^{\mathbf{s}}(1) + O(\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}}). \tag{3.51}$$

Lemma 3.7. *Recall the notation* (1.12) *and let* $(\ell^{\mathbf{s}})_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}$ *a family of vectors satisfying* (3.2) *and* (3.46) *and whose signs are fixed by Proposition* 3.2. *Let also* $(k_\nu^{\mathbf{s}})_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}$ *be the probability densities on* $[\gamma_1+\nu, 1]$ *defined in* (3.50)*. Then, for all* $\mathbf{s}\in\mathbf{j}(\mathbf{m})$ *and* $(x,v)\in B_0(\mathbf{s},r)$*, the prefactor from Proposition* 3.4 *satisfies*

$$\omega_{\nu,z}^{\mathbf{m}}(x,v) = O(\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}})\begin{pmatrix}\partial_z L_{z;x}^{\mathbf{s}}\\ \partial_z L_{z;v}^{\mathbf{s}}\end{pmatrix}\cdot\begin{pmatrix}\tilde{x}_{\mathbf{s}}\\ v\end{pmatrix}.$$

Proof. By some computations similar to the ones we made in the proof of Lemma 3.6, we get that the choice of $(\ell^{\mathbf{s}})_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}$ implies that

$$\omega_{\nu,z}^{\mathbf{m}}(x,v) = \frac{\sqrt{|\tau_{\mathbf{s}}|}P(z)}{2\sqrt{2}}\Big[k_\nu^{\mathbf{s}}(z) - \frac{2\sqrt{2}}{\sqrt{|\tau_{\mathbf{s}}|}P(z)}K_\nu(z)\Big]\begin{pmatrix}\partial_z L_{z;x}^{\mathbf{s}}\\ \partial_z L_{z;v}^{\mathbf{s}}\end{pmatrix}\cdot\begin{pmatrix}\tilde{x}_{\mathbf{s}}\\ v\end{pmatrix}.$$

The term between brackets is exactly the one appearing in (3.47); so, using (3.50) and the fact that K_0 is a solution of (3.47), we get

$$\omega^{\mathbf{m}}_{\nu,z}(x,v) = \frac{K^{\mathbf{s}}_0(\gamma_1+\nu)}{B^{\mathbf{s}}_\nu}\begin{pmatrix}\partial_z L^{\mathbf{s}}_{z;x}\\ \partial_z L^{\mathbf{s}}_{z;v}\end{pmatrix}\cdot\begin{pmatrix}\tilde{x}_{\mathbf{s}}\\ v\end{pmatrix} = O(\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}})\begin{pmatrix}\partial_z L^{\mathbf{s}}_{z;x}\\ \partial_z L^{\mathbf{s}}_{z;v}\end{pmatrix}\cdot\begin{pmatrix}\tilde{x}_{\mathbf{s}}\\ v\end{pmatrix}$$

by (3.51) and the definition of $K^{\mathbf{s}}_0$. ■

Proposition 3.8. *Recall the notation (1.12) and let $f^{\mathbf{m}}_{\nu,h}$ be the quasimode defined in (3.26) with $(\ell^{\mathbf{s}})_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}$ and $(k^{\mathbf{s}}_\nu)_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}$ satisfying the hypotheses from Lemma 3.7. Then*

$$\|P_h f^{\mathbf{m}}_{\nu,h}\| = h\,\mathrm{e}^{-\frac{S(\mathbf{m})}{h}}\,\|f^{\mathbf{m}}_{\nu,h}\|\big(O_\nu(h^{\frac{1}{2}}) + O(\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}}|\ln(\nu)|)\big).$$

Proof. First notice that, thanks Hypothesis 2.8 (a), one can apply a standard Laplace method (see Proposition D.1) to obtain with the notation (1.11)

$$\|f^{\mathbf{m}}_{\nu,h}\|^2 = \frac{(2\pi h)^d}{\det(\mathcal{V}_{\mathbf{m}})^{1/2}}(1+O(h)). \tag{3.52}$$

Hence, according to Proposition 3.4, it is sufficient to show that

$$\big\|P_h f^{\mathbf{m}}_{\nu,h} - R^{\mathbf{m}}_{\nu,h}\big\| = h^{1+\frac{d}{2}}\,\mathrm{e}^{-\frac{S(\mathbf{m})}{h}}\,O(\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}}|\ln(\nu)|). \tag{3.53}$$

Still using Proposition 3.4 as well as Minkowski's integral inequality and Lemma 3.7, we have

$$\begin{aligned}
&\|P_h f^{\mathbf{m}}_{\nu,h} - R^{\mathbf{m}}_{\nu,h}\|\\
&\leq Ch^{1/2}\int_{\gamma_1+\nu}^{1}\Big(\int_{\mathbf{j}^W(\mathbf{m})+B_0(0,r)}\omega^{\mathbf{m}}_{\nu,z}(x,v)^2\exp\Big[-\frac{2}{h}\widetilde{W}^{\mathbf{m}}_z(x,v)\Big]\,\mathrm{d}(x,v)\Big)^{1/2}\mathrm{d}z\\
&\leq Ch^{1/2}\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}}\int_{\gamma_1+\nu}^{1}\Big(\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\int_{B_0(\mathbf{s},r)}\begin{pmatrix}\partial_z L^{\mathbf{s}}_{z;x}\\ \partial_z L^{\mathbf{s}}_{z;v}\end{pmatrix}\begin{pmatrix}\partial_z L^{\mathbf{s}}_{z;x}\\ \partial_z L^{\mathbf{s}}_{z;v}\end{pmatrix}^{\intercal}\begin{pmatrix}\tilde{x}_{\mathbf{s}}\\ v\end{pmatrix}\cdot\begin{pmatrix}\tilde{x}_{\mathbf{s}}\\ v\end{pmatrix}\\
&\qquad\qquad\times\exp\Big[-\frac{2}{h}\widetilde{W}^{\mathbf{m}}_z(x,v)\Big]\,\mathrm{d}(x,v)\Big)^{1/2}\mathrm{d}z.
\end{aligned}$$

With the notation (1.11), the change of variables

$$(y,w) = \Big(\frac{2}{h}\Big)^{1/2}\Big[\mathcal{W}_{\mathbf{s}} + \begin{pmatrix}L^{\mathbf{s}}_{z;x}\\ L^{\mathbf{s}}_{z;v}\end{pmatrix}\begin{pmatrix}L^{\mathbf{s}}_{z;x}\\ L^{\mathbf{s}}_{z;v}\end{pmatrix}^{\intercal}\Big]^{1/2}(\tilde{x}_{\mathbf{s}},v)$$

then yields according to Lemma 3.1

$$
\begin{aligned}
&\| P_h f^{\mathbf{m}}_{\nu,h} - R^{\mathbf{m}}_{\nu,h} \| \\
&\leq C h^{1+\frac{d}{2}} \, \mathrm{e}^{-\frac{S(\mathbf{m})}{h}} \, \nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}} \int_{\gamma_1+\nu}^{1} \Big(\sum_{\mathbf{s}\in \mathbf{j}(\mathbf{m})} \int_{\mathbb{R}^{2d}} a_z a_z^{\intercal} \begin{pmatrix} y \\ w \end{pmatrix} \cdot \begin{pmatrix} y \\ w \end{pmatrix} \mathrm{e}^{-\frac{(y,w)^2}{2}} \mathrm{d}(y,w) \Big)^{1/2} \mathrm{d}z,
\end{aligned} \tag{3.54}
$$

where

$$
a_z = \left[\mathcal{W}_{\mathbf{s}} + \begin{pmatrix} L^{\mathbf{s}}_{z;x} \\ L^{\mathbf{s}}_{z;v} \end{pmatrix} \begin{pmatrix} L^{\mathbf{s}}_{z;x} \\ L^{\mathbf{s}}_{z;v} \end{pmatrix}^{\intercal} \right]^{-1/2} \begin{pmatrix} \partial_z L^{\mathbf{s}}_{z;x} \\ \partial_z L^{\mathbf{s}}_{z;v} \end{pmatrix}.
$$

Thanks to Proposition C.1, we know that

$$
\begin{aligned}
&(2\pi)^{-d} \int_{\mathbb{R}^{2d}} a_z a_z^{\intercal} \begin{pmatrix} y \\ w \end{pmatrix} \cdot \begin{pmatrix} y \\ w \end{pmatrix} \mathrm{e}^{-\frac{(y,w)^2}{2}} \mathrm{d}(y,w) \\
&\quad = |a_z|^2 = \left[\mathcal{W}_{\mathbf{s}} + \begin{pmatrix} L^{\mathbf{s}}_{z;x} \\ L^{\mathbf{s}}_{z;v} \end{pmatrix} \begin{pmatrix} L^{\mathbf{s}}_{z;x} \\ L^{\mathbf{s}}_{z;v} \end{pmatrix}^{\intercal} \right]^{-1} \begin{pmatrix} \partial_z L^{\mathbf{s}}_{z;x} \\ \partial_z L^{\mathbf{s}}_{z;v} \end{pmatrix} \cdot \begin{pmatrix} \partial_z L^{\mathbf{s}}_{z;x} \\ \partial_z L^{\mathbf{s}}_{z;v} \end{pmatrix}.
\end{aligned} \tag{3.55}
$$

Since by (2.8)

$$
\left[\mathcal{W}_{\mathbf{s}} + \begin{pmatrix} L^{\mathbf{s}}_{z;x} \\ L^{\mathbf{s}}_{z;v} \end{pmatrix} \begin{pmatrix} L^{\mathbf{s}}_{z;x} \\ L^{\mathbf{s}}_{z;v} \end{pmatrix}^{\intercal} \right]^{-1} \begin{pmatrix} \partial_z L^{\mathbf{s}}_{z;x} \\ \partial_z L^{\mathbf{s}}_{z;v} \end{pmatrix} = \frac{-8}{P(z)^{3/2}} \begin{pmatrix} (2|\tau_{\mathbf{s}}|)^{-1/2}(1-z)\,\ell^{\mathbf{s}}_v \\ (1+z)\ell^{\mathbf{s}}_v \end{pmatrix},
$$

we get

$$
|a_z|^2 = \frac{16}{P(z)^3}(2(1+z)^2 - (1-z)^2) = \frac{16}{P(z)^2}. \tag{3.56}
$$

Putting together (3.54), (3.55), (3.56), and computing the integral in z, we obtain (3.53), so the proof is complete. ■

4. Computation of the approximated small eigenvalues

Let us denote

$$
\tilde{f}^{\mathbf{m}}_{\nu,h} = \frac{f^{\mathbf{m}}_{\nu,h}}{\| f^{\mathbf{m}}_{\nu,h} \|} \tag{4.1}
$$

the renormalization of the quasimodes defined in (3.26) and satisfying the hypotheses of Proposition 3.8. The goal of this section is to compute the *approximated eigenvalues*

$$
\tilde{\lambda}^{\mathbf{m}}_{\nu,h} := \langle P_h \tilde{f}^{\mathbf{m}}_{\nu,h}, \tilde{f}^{\mathbf{m}}_{\nu,h} \rangle = \langle Q_h \tilde{f}^{\mathbf{m}}_{\nu,h}, \tilde{f}^{\mathbf{m}}_{\nu,h} \rangle \tag{4.2}
$$

as X_0^h is a skew-adjoint differential operator and $\tilde{f}^{\mathbf{m}}_{\nu,h}$ is real valued.

This will require to study the matrix

$$H_\gamma^{\mathbf{s}} = \begin{pmatrix} \mathcal{V}_{\mathbf{s}} + 2L_{\gamma,x}L_{\gamma,x}^\intercal & L_{\gamma,x}L_{\gamma,v}^\intercal & L_{\gamma,x}L_{\gamma,v}^\intercal \\ L_{\gamma,v}L_{\gamma,x}^\intercal & \frac{1}{2} + L_{\gamma,v}L_{\gamma,v}^\intercal & 0 \\ L_{\gamma,v}L_{\gamma,x}^\intercal & 0 & \frac{1}{2} + L_{\gamma,v}L_{\gamma,v}^\intercal \end{pmatrix}, \tag{4.3}$$

where we used the notation (1.11) and for shortness, we wrote $L_{\gamma,x}$ and $L_{\gamma,v}$ instead of $L_{\gamma,x}^{\mathbf{s}}$ and $L_{\gamma,v}^{\mathbf{s}}$.

Lemma 4.1. *For $\gamma \in [\gamma_1 + \nu, 1]$, the matrix $H_\gamma^{\mathbf{s}}$ is positive definite.*

Proof. It suffices to notice that

$$\begin{aligned} H_\gamma^{\mathbf{s}}\begin{pmatrix} x \\ v \\ v' \end{pmatrix} \cdot \begin{pmatrix} x \\ v \\ v' \end{pmatrix} &= \left[\mathcal{W}_{\mathbf{s}} + \begin{pmatrix} L_{\gamma;x}^{\mathbf{s}} \\ L_{\gamma;v}^{\mathbf{s}} \end{pmatrix}\begin{pmatrix} L_{\gamma;x}^{\mathbf{s}} \\ L_{\gamma;v}^{\mathbf{s}} \end{pmatrix}^\intercal\right]\begin{pmatrix} x \\ v \end{pmatrix} \cdot \begin{pmatrix} x \\ v \end{pmatrix} \\ &\quad + \left[\mathcal{W}_{\mathbf{s}} + \begin{pmatrix} L_{\gamma;x}^{\mathbf{s}} \\ L_{\gamma;v}^{\mathbf{s}} \end{pmatrix}\begin{pmatrix} L_{\gamma;x}^{\mathbf{s}} \\ L_{\gamma;v}^{\mathbf{s}} \end{pmatrix}^\intercal\right]\begin{pmatrix} x \\ v' \end{pmatrix} \cdot \begin{pmatrix} x \\ v' \end{pmatrix} \end{aligned}$$

and apply Lemma 3.1. ∎

In the spirit of Proposition 2.2 and with the notation (2.3), let us denote

$$Q_{y,h} = b_h^* \circ \mathrm{Op}_h(m_{y,h}\,\mathrm{Id}) \circ b_h. \tag{4.4}$$

For $\mathbf{m} \in \mathrm{U}^{(0)} \backslash \{\underline{\mathbf{m}}\}$, $\mathbf{s} \in \mathbf{j}(\mathbf{m})$, we also denote $\langle \cdot, \cdot \rangle_{\tilde{r}}$ the inner product on $L^2(B(\mathbf{s}, \tilde{r}) \times \mathbb{R}_v^d)$.

Lemma 4.2. *Let $\mathbf{s} \in \mathbf{j}(\mathbf{m})$ for some $\mathbf{m} \in \mathrm{U}^{(0)} \backslash \{\underline{\mathbf{m}}\}$ and recall the notations (1.11), (3.24), and (3.29). Then, for all $\gamma \in [\gamma_1 + \nu, 1]$ and $y \in (0, 1)$,*

$$\begin{aligned} &\langle Q_{y,h}(\tilde{\theta}_{\gamma,h}^{\mathbf{s}}\, \mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}}),\ \tilde{\theta}_{\gamma,h}^{\mathbf{s}}\, \mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}} \rangle_{\tilde{r}} \\ &\quad = 2h^2\, \mathrm{e}^{\frac{-2S(\mathbf{m})}{h}} (2\pi h)^d\, |\det \mathcal{V}_{\mathbf{s}}|^{-1/2} \frac{1 + O_\nu(h)}{(1+y)(1+(1+2|L_{\gamma,v}^{\mathbf{s}}|^2)y)} |L_{\gamma,v}^{\mathbf{s}}|^2. \end{aligned}$$

Proof. First, let us use the definition of $Q_{y,h}$ to write

$$\begin{aligned} &\langle Q_{y,h}(\tilde{\theta}_{\gamma,h}^{\mathbf{s}}\, \mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}}),\ \tilde{\theta}_{\gamma,h}^{\mathbf{s}}\, \mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}} \rangle_{\tilde{r}} \\ &\quad = \langle \mathrm{Op}_h(m_{y,h}\,\mathrm{Id}) \circ b_h(\tilde{\theta}_{\gamma,h}^{\mathbf{s}}\, \mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}}),\ b_h(\tilde{\theta}_{\gamma,h}^{\mathbf{s}}\, \mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}}) \rangle_{\tilde{r}}. \end{aligned}$$

Using (3.34), we get

$$\begin{aligned}
&\langle Q_{y,h}(\tilde{\theta}^{\mathbf{s}}_{\gamma,h}\, \mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}}),\ \tilde{\theta}^{\mathbf{s}}_{\gamma,h}\, \mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}}\rangle_{\tilde{r}} \\
&\quad = 2h^2(2\pi h)^{-d/2}\frac{(y+1)^{d-2}}{(4y)^{\frac{d}{2}}}\,\mathrm{e}^{\frac{V(\mathbf{m})}{h}}\,|L^{\mathbf{s}}_{\gamma,v}|^2 \\
&\qquad \times \int_{\substack{|x-\mathbf{s}|<\tilde{r}\\ v,v'\in\mathbb{R}^d}} \exp\Big[-\frac{1}{h}\Big(V(x)+\frac{v^2+v'^2}{4}+\frac{y}{8}(v+v')^2+\frac{(v-v')^2}{8y} \\
&\qquad\qquad + \frac{L^{\mathbf{s}}_{\gamma}(x,v)^2+L^{\mathbf{s}}_{\gamma}(x,v')^2}{2}\Big)\Big]\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}v'. \qquad (4.5)
\end{aligned}$$

By the change of variables $x'=\phi_{\mathbf{s}}(x)$ and with the notation $\boldsymbol{\sigma}(\mathbf{m})$ from Definition 2.7, the last integral becomes

$$\mathrm{e}^{\frac{-2\boldsymbol{\sigma}(\mathbf{m})}{h}}\int_{\substack{|\phi_{\mathbf{s}}^{-1}(x')-\mathbf{s}|<\tilde{r},\\ v,\,v'\in\mathbb{R}^d}} \exp\left[-\frac{1}{2h}H^{\mathbf{s}}_{\gamma,y}\begin{pmatrix}x'\\ v\\ v'\end{pmatrix}\cdot\begin{pmatrix}x'\\ v\\ v'\end{pmatrix}\right]|\det D_{x'}\phi_{\mathbf{s}}^{-1}|\,\mathrm{d}x'\,\mathrm{d}v\,\mathrm{d}v' \qquad (4.6)$$

where, using the notation (4.3),

$$\begin{aligned}
H^{\mathbf{s}}_{\gamma,y} &= \begin{pmatrix} \mathcal{V}_{\mathbf{s}}+2L_{\gamma,x}L_{\gamma,x}^{\intercal} & L_{\gamma,x}L_{\gamma,v}^{\intercal} & L_{\gamma,x}L_{\gamma,v}^{\intercal} \\ L_{\gamma,v}L_{\gamma,x}^{\intercal} & \frac{(y+1)^2}{4y}+L_{\gamma,v}L_{\gamma,v}^{\intercal} & \frac{y^2-1}{4y} \\ L_{\gamma,v}L_{\gamma,x}^{\intercal} & \frac{y^2-1}{4y} & \frac{(y+1)^2}{4y}+L_{\gamma,v}L_{\gamma,v}^{\intercal} \end{pmatrix} \\
&= H^{\mathbf{s}}_{\gamma} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{y^2+1}{4y} & \frac{y^2-1}{4y} \\ 0 & \frac{y^2-1}{4y} & \frac{y^2+1}{4y} \end{pmatrix} \qquad (4.7)
\end{aligned}$$

is a positive-definite matrix uniformly in $(\gamma, y)\in[\gamma_1+\nu, 1]\times(0,1)$, thanks to Lemma 4.1. Hence, $(H^{\mathbf{s}}_{\gamma,y})^{-1/2}$ exists and is $O_\nu(1)$ so by a standard Laplace method (see Proposition D.1),

$$\begin{aligned}
&\int_{\substack{|\phi_{\mathbf{s}}^{-1}(x')-\mathbf{s}|<\tilde{r},\\ v,\,v'\in\mathbb{R}^d}} \exp\left[-\frac{1}{2h}H^{\mathbf{s}}_{\gamma,y}\begin{pmatrix}x'\\ v\\ v'\end{pmatrix}\cdot\begin{pmatrix}x'\\ v\\ v'\end{pmatrix}\right]|\det D_{x'}\phi_{\mathbf{s}}^{-1}|\,\mathrm{d}x'\,\mathrm{d}v\,\mathrm{d}v' \\
&\quad = (2\pi h)^{3d/2}\det(H^{\mathbf{s}}_{\gamma,y})^{-1/2}(1+O_\nu(h)) \\
&\quad = (2\pi h)^{3d/2}|\det\mathcal{V}_{\mathbf{s}}|^{-1/2}\frac{(4y)^{d/2}}{(1+y)^{d-1}(1+(1+2|L^{\mathbf{s}}_{\gamma,v}|^2)y)}(1+O_\nu(h)), \qquad (4.8)
\end{aligned}$$

where we also used Lemma B.2. The conclusion then follows from (4.5), (4.6), and (4.8). ∎

Lemma 4.3. *Recall the notation* (3.32) *and let* $\gamma_1 + \nu \leq z \leq \gamma < 1$. *For* $y \in [0, 1)$, *we have*

$$\Gamma_z^{-1} \circ \Gamma_\gamma(y) \in [0, 1)$$

and

$$Q_{y,h}(\tilde{\theta}^{\mathbf{s}}_{\gamma,h}\, \mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}}) = (\Gamma_z^{-1} \circ \Gamma_\gamma)'(y)\, Q_{\Gamma_z^{-1} \circ \Gamma_\gamma(y),h}(\tilde{\theta}^{\mathbf{s}}_{z,h}\, \mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}})$$

on $B(\mathbf{s}, \tilde{r}) \times \mathbb{R}^d_v$.

Proof. First, notice that for all $\gamma \in [\gamma_1 + \nu, 1)$, the function $\Gamma_\gamma : [0, 1) \to [\gamma, 1)$ is an increasing bijection whose inverse is given by

$$\Gamma_\gamma^{-1}(y) = \frac{y - \gamma}{1 - y\gamma}, \tag{4.9}$$

so the first assertion follows from the hypothesis on z and γ. Now, by Lemma 3.3 applied with $Q_{y,h}$ instead of Q_h, we get, using the notation (3.31) as well as (3.33), that, on $B(\mathbf{s}, \tilde{r}) \times \mathbb{R}^d_v$,

$$Q_{y,h}(\tilde{\theta}^{\mathbf{s}}_{\gamma,h}\, \mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}}) = -h\, \partial_y \mathcal{L}(\gamma, y)\, \mathrm{e}^{-\frac{\tilde{W}_{\Gamma_\gamma(y)}(x,v)}{h}} \cdot \begin{pmatrix} \tilde{x}_{\mathbf{s}} \\ v \end{pmatrix} \tag{4.10}$$

(here we once again disregarded the fact that the linear form L_γ is twisted in x as $Q_{y,h}$ only acts in v). Thus, denoting $\partial_2 \mathcal{L}(\gamma, \cdot)$ the derivative of $\mathcal{L}$ with respect toits second argument and still using (3.33), we also have

$$\begin{aligned} &Q_{y,h}(\tilde{\theta}^{\mathbf{s}}_{\gamma,h}\, \mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}}) \\ &\quad = -h\, \partial_y (L_{\Gamma_\gamma(y)})\, \mathrm{e}^{-\frac{\tilde{W}_{\Gamma_\gamma(y)}(x,v)}{h}} \cdot \begin{pmatrix} \tilde{x}_{\mathbf{s}} \\ v \end{pmatrix} \\ &\quad = -h\, \partial_y (\mathcal{L}(z, \Gamma_z^{-1} \circ \Gamma_\gamma(y)))\, \mathrm{e}^{-\frac{\tilde{W}_{\Gamma_\gamma(y)}(x,v)}{h}} \cdot \begin{pmatrix} \tilde{x}_{\mathbf{s}} \\ v \end{pmatrix} \\ &\quad = -h\, (\Gamma_z^{-1} \circ \Gamma_\gamma)'(y)\, \partial_2 \mathcal{L}(z, \Gamma_z^{-1} \circ \Gamma_\gamma(y))\, \mathrm{e}^{-\frac{\tilde{W}_{\Gamma_\gamma(y)}(x,v)}{h}} \cdot \begin{pmatrix} \tilde{x}_{\mathbf{s}} \\ v \end{pmatrix} \end{aligned}$$

so (4.10) with $Q_{\Gamma_z^{-1} \circ \Gamma_\gamma(y),h}$ and $\tilde{\theta}^{\mathbf{s}}_{z,h}$ yields the last statement. ∎

Proposition 4.4. *With the notations* (1.11), (1.12), (3.49), *and* (4.2), *we have for* $\mathbf{m} \in \mathrm{U}^{(0)} \backslash \{\underline{\mathbf{m}}\}$

$$\tilde{\lambda}^{\mathbf{m}}_{\nu,h} = h\, \tilde{\varrho}_{\nu,h}(\mathbf{m})\, \mathrm{e}^{\frac{-2S(\mathbf{m})}{h}}$$

with

$$\tilde{\varrho}_{\nu,h}(\mathbf{m})+O_\nu(h)+O(\nu^{\frac{1}{2\sqrt{|\tau_\mathbf{s}|}}}) = \frac{1}{\pi}\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\Big(\frac{2+\sqrt{2}}{2-\sqrt{2}}\Big)^{\frac{1}{\sqrt{|\tau_\mathbf{s}|}}}\Big(\frac{\det\mathcal{V}_\mathbf{m}}{|\det\mathcal{V}_\mathbf{s}|}\Big)^{1/2} \times\int_{\gamma_1\le z\le\gamma<1} k_0^\mathbf{s}(\gamma)k_0^\mathbf{s}(z)\ln\Big(2\frac{(1+z)(1+\gamma)}{1+3z+3\gamma+z\gamma}\Big)\,\mathrm{d}z\,\mathrm{d}\gamma.$$

Proof. As we mentioned at the beginning of the section, since X_0^h is a skew-adjoint differential operator and $\tilde{f}_{\nu,h}^\mathbf{m}$ is real valued, we have

$$\langle P_h\tilde{f}_{\nu,h}^\mathbf{m},\tilde{f}_{\nu,h}^\mathbf{m}\rangle = \langle Q_h\tilde{f}_{\nu,h}^\mathbf{m},\tilde{f}_{\nu,h}^\mathbf{m}\rangle.$$

Now, by Proposition 2.2, we get

$$\langle Q_h f_{\nu,h}^\mathbf{m}, f_{\nu,h}^\mathbf{m}\rangle = \langle \mathrm{Op}_h(m_h\,\mathrm{Id})(b_h f_{\nu,h}^\mathbf{m}), b_h f_{\nu,h}^\mathbf{m}\rangle \tag{4.11}$$

and we saw through (3.36)–(3.40) that

$$b_h f_{\nu,h}^\mathbf{m} = \Big(\frac{h}{2\pi}\Big)^{1/2}\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\int_{\gamma_1+\nu}^1 k_\nu^\mathbf{s}(\gamma)\,\mathrm{e}^{-\frac{\widetilde{W}_\gamma^\mathbf{m}}{h}}\,L_{\gamma,\nu}^\mathbf{s}\,\mathrm{d}\gamma\,\mathbf{1}_{|x-\mathbf{s}|<\tilde{r}} + O_\nu(h^{\frac{3+d}{2}}\,\mathrm{e}^{-\frac{S(\mathbf{m})}{h}}) \tag{4.12}$$

$$= (2\pi h)^{-1/2}\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\int_{\gamma_1+\nu}^1 k_\nu^\mathbf{s}(\gamma)b_h(\tilde{\theta}_{\gamma,h}\,\mathrm{e}^{-\frac{W^\mathbf{m}(x,v)}{h}})\,\mathrm{d}\gamma\,\mathbf{1}_{|x-\mathbf{s}|<\tilde{r}} + O_\nu(h^{\frac{3+d}{2}}\,\mathrm{e}^{-\frac{S(\mathbf{m})}{h}}) \tag{4.13}$$

Note that (4.12) also implies

$$b_h f_{\nu,h}^\mathbf{m} = O_\nu(h^{\frac{1+d}{2}}\,\mathrm{e}^{-\frac{S(\mathbf{m})}{h}}). \tag{4.14}$$

Combining the boundedness of $\mathrm{Op}_h(m_h\,\mathrm{Id})$ with (4.13)–(4.14) and using the notation (4.4), (4.11) becomes

$$\langle Q_h f_{\nu,h}^\mathbf{m}, f_{\nu,h}^\mathbf{m}\rangle = \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\int_{[\gamma_1+\nu,1]^2} k_\nu^\mathbf{s}(\gamma)k_\nu^\mathbf{s}(z)\langle Q_h(\tilde{\theta}_{\gamma,h}\,\mathrm{e}^{-\frac{W^\mathbf{m}(x,v)}{h}}),\,\tilde{\theta}_{z,h}\,\mathrm{e}^{-\frac{W^\mathbf{m}(x,v)}{h}}\rangle_{\tilde{r}}\,\mathrm{d}\gamma\,\mathrm{d}z\,(2\pi h)^{-1} + O_\nu(h^{d+2}\,\mathrm{e}^{-\frac{2S(\mathbf{m})}{h}})$$

$$= \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} \int_0^1 \int_{[\gamma_1+\nu,1]^2} k_\nu^{\mathbf{s}}(\gamma) k_\nu^{\mathbf{s}}(z) \langle Q_{y,h}(\tilde{\theta}_{\gamma,h} \, \mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}}), \tilde{\theta}_{z,h} \, \mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}} \rangle_{\tilde{r}} \, \mathrm{d}\gamma \, \mathrm{d}z \, \mathrm{d}y$$

$$\times (2\pi h)^{-1} + O_\nu(h^{d+2} \, \mathrm{e}^{-\frac{2S(\mathbf{m})}{h}})$$

$$= \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} \int_0^1 \int_{\gamma_1+\nu\le z\le\gamma<1} k_\nu^{\mathbf{s}}(\gamma) k_\nu^{\mathbf{s}}(z) \langle Q_{y,h}(\tilde{\theta}_{\gamma,h} \, \mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}}), \tilde{\theta}_{z,h} \, \mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}} \rangle_{\tilde{r}} \, \mathrm{d}z \, \mathrm{d}\gamma \, \mathrm{d}y$$

$$\times 2(2\pi h)^{-1} + O_\nu(h^{d+2} \, \mathrm{e}^{-\frac{2S(\mathbf{m})}{h}}), \tag{4.15}$$

where for the last equation we used the fact that $Q_{y,h}$ is self-adjoint. Applying Lemma 4.3 together with the change of variables $\tilde{y} = \Gamma_z^{-1} \circ \Gamma_\gamma(y)$, we get that (4.15) yields

$$\langle Q_h f_{\nu,h}^{\mathbf{m}}, f_{\nu,h}^{\mathbf{m}} \rangle + O_\nu(h^{d+2} \, \mathrm{e}^{-\frac{2S(\mathbf{m})}{h}})$$
$$= 2(2\pi h)^{-1} \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} \int_{\gamma_1+\nu\le z\le\gamma<1} \int_{\Gamma_z^{-1}(\gamma)}^1 k_\nu^{\mathbf{s}}(\gamma) k_\nu^{\mathbf{s}}(z) \langle Q_{\tilde{y},h}(\tilde{\theta}_{z,h} \, \mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}}), \tilde{\theta}_{z,h} \, \mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}} \rangle_{\tilde{r}} \, \mathrm{d}\tilde{y} \, \mathrm{d}z \, \mathrm{d}\gamma$$

which by Lemma 4.2 is further equal to

$$\frac{2}{\pi} h (2\pi h)^d \, \mathrm{e}^{\frac{-2S(\mathbf{m})}{h}} \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} |\det \mathcal{V}_{\mathbf{s}}|^{-1/2}$$
$$\times \int_{\gamma_1+\nu\le z\le\gamma<1} \int_{\Gamma_z^{-1}(\gamma)}^1 \frac{k_\nu^{\mathbf{s}}(\gamma) k_\nu^{\mathbf{s}}(z) |L_{z,v}^{\mathbf{s}}|^2}{(1+\tilde{y})(1+(1+2|L_{z,v}^{\mathbf{s}}|^2)\tilde{y})} \, \mathrm{d}\tilde{y} \, \mathrm{d}z \, \mathrm{d}\gamma. \tag{4.16}$$

By partial fraction decomposition, the $\tilde{y}$-integral becomes

$$\int_{\Gamma_z^{-1}(\gamma)}^1 \frac{1}{(1+\tilde{y})(1+(1+2|L_{z,v}^{\mathbf{s}}|^2)\tilde{y})} \, \mathrm{d}\tilde{y}$$
$$= \frac{1}{2|L_{z,v}^{\mathbf{s}}|^2} \int_{\Gamma_z^{-1}(\gamma)}^1 \frac{1+2|L_{z,v}^{\mathbf{s}}|^2}{1+(1+2|L_{z,v}^{\mathbf{s}}|^2)\tilde{y}} - \frac{1}{1+\tilde{y}} \, \mathrm{d}\tilde{y}$$
$$= \frac{1}{2|L_{z,v}^{\mathbf{s}}|^2} \ln\Big(\frac{(1+|L_{z,v}^{\mathbf{s}}|^2)(1+\Gamma_z^{-1}(\gamma))}{1+(1+2|L_{z,v}^{\mathbf{s}}|^2)\Gamma_z^{-1}(\gamma)} \Big) \tag{4.17}$$

and, using (3.4)–(3.5) as well as (4.9), the quantity in the logarithm from (4.17) simplifies as follows:

$$\begin{aligned}\frac{(1+|L^{\mathbf{s}}_{z,\nu}|^2)(1+\Gamma_z^{-1}(\gamma))}{1+(1+2|L^{\mathbf{s}}_{z,\nu}|^2)\Gamma_z^{-1}(\gamma)} &= \frac{(P(z)+(1-z)^2)(1-z)(1+\gamma)}{P(z)(1-\gamma z)+(3z^2+2z+3)(\gamma-z)}\\ &= 2\,\frac{(1+z)^2(1-z)(1+\gamma)}{(1-z^2)(1+3z+3\gamma+z\gamma)}\\ &= 2\,\frac{(1+z)(1+\gamma)}{1+3z+3\gamma+z\gamma}. \qquad (4.18)\end{aligned}$$

Putting together (4.16), (4.17), (4.18) and using (3.52), we get

$$\begin{aligned}&\langle P_h \tilde f^{\mathbf{m}}_{\nu,h}, \tilde f^{\mathbf{m}}_{\nu,h}\rangle + O_\nu(h^2\,\mathrm{e}^{-\frac{2S(\mathbf{m})}{h}})\\ &\quad = \frac{h}{\pi}\,\mathrm{e}^{\frac{-2S(\mathbf{m})}{h}} \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\Big(\frac{\det \mathcal{V}_{\mathbf{m}}}{|\det \mathcal{V}_{\mathbf{s}}|}\Big)^{1/2} \int_{\gamma_1+\nu\le z\le\gamma<1} k^{\mathbf{s}}_\nu(\gamma)k^{\mathbf{s}}_\nu(z)\ln\Big(2\frac{(1+z)(1+\gamma)}{1+3z+3\gamma+z\gamma}\Big)\,\mathrm{d}z\,\mathrm{d}\gamma. \qquad (4.19)\end{aligned}$$

Now, the function $1+3z+3\gamma+z\gamma$ is non-negative on $[\gamma_1,1]^2$ and vanishes only at (γ_1,γ_1). Moreover, we have by Taylor expansion that

$$1+3z+3\gamma+z\gamma \ge \frac{|(\gamma,z)-(\gamma_1,\gamma_1)|}{C} \ge \max\Big(\frac{z-\gamma_1}{C}, \frac{\gamma-\gamma_1}{C}\Big)$$

for $(\gamma,z)\in[\gamma_1,1]^2$ close enough to (γ_1,γ_1) and thus

$$\ln\Big(2\,\frac{(1+z)(1+\gamma)}{1+3z+3\gamma+z\gamma}\Big) = O(|\ln(z-\gamma_1)|)$$

holds as well as

$$\ln\Big(2\,\frac{(1+z)(1+\gamma)}{1+3z+3\gamma+z\gamma}\Big) = O(|\ln(\gamma-\gamma_1)|).$$

Besides, by (3.50) and (3.51), we have

$$k^{\mathbf{s}}_\nu(z) = \Big(\frac{2-\sqrt{2}}{2+\sqrt{2}}\Big)^{\frac{-1}{2\sqrt{|\tau_{\mathbf{s}}|}}} k^{\mathbf{s}}_0(z)(1+O(\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}})), \qquad (4.20)$$

with

$$k^{\mathbf{s}}_0(z) = O(|z-\gamma_1|^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}-1})$$

on $[\gamma_1,1]$. Consequently, the integral

$$\int_{\gamma_1\le z\le\gamma<1} k^{\mathbf{s}}_0(\gamma)k^{\mathbf{s}}_0(z)\ln\Big(2\frac{(1+z)(1+\gamma)}{1+3z+3\gamma+z\gamma}\Big)\,\mathrm{d}z\,\mathrm{d}\gamma$$

exists and we have

$$\int\limits_{\gamma_1+\nu\le z\le\gamma<1} k_0^{\mathbf{s}}(\gamma)k_0^{\mathbf{s}}(z)\ln\Big(2\frac{(1+z)(1+\gamma)}{1+3z+3\gamma+z\gamma}\Big)\,\mathrm{d}z\,\mathrm{d}\gamma+O(\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}})$$
$$=\int\limits_{\gamma_1\le z\le\gamma<1} k_0^{\mathbf{s}}(\gamma)k_0^{\mathbf{s}}(z)\ln\Big(2\frac{(1+z)(1+\gamma)}{1+3z+3\gamma+z\gamma}\Big)\,\mathrm{d}z\,\mathrm{d}\gamma. \tag{4.21}$$

Combining (4.19), (4.20), and (4.21), we get the announced result. ■

5. Proof of the main results

We now introduce a series of results which will enable us to go from the approximated eigenvalues of P_h to the actual ones.

Lemma 5.1. *Let* $\mathbf{m}\in\mathrm{U}^{(0)}\setminus\{\underline{\mathbf{m}}\}$. *Using the notations* (1.12), (4.1)*, and* (4.2)*, we have*

(i) $\|P_h\tilde f^{\mathbf{m}}_{\nu,h}\|=\sqrt{h\tilde\lambda^{\mathbf{m}}_{\nu,h}}(O_\nu(h^{\frac12})+O(\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}}|\ln(\nu)|))$;

(ii) $\|P_h^*\tilde f^{\mathbf{m}}_{\nu,h}\|=\sqrt{h\tilde\lambda^{\mathbf{m}}_{\nu,h}}(O_\nu(h^{\frac12})+O(|\ln(\nu)|))$.

Proof. The first item is an immediate consequence of Propositions 3.8 and 4.4. The second one can be obtained similarly using Remark 3.5 and mimicking the proof of Proposition 3.8 after noticing that

$$\overset{*}{\omega}{}^{\mathbf{m}}_{\nu,z}(x,v)=O(1)\begin{pmatrix}\partial_z L^{\mathbf{s}}_{z;x}\\ \partial_z L^{\mathbf{s}}_{z;v}\end{pmatrix}\cdot\begin{pmatrix}\tilde x_{\mathbf{s}}\\ v\end{pmatrix}.$$ ■

Lemma 5.2. *For* $\mathbf{m}$ *and* $\mathbf{m}'$ *two distinct elements of* $\mathrm{U}^{(0)}$,

(i) $\langle P_h\tilde f^{\mathbf{m}}_{\nu,h},\tilde f^{\mathbf{m}'}_{\nu,h}\rangle=O_\nu(h^\infty\sqrt{\tilde\lambda^{\mathbf{m}}_{\nu,h}\tilde\lambda^{\mathbf{m}'}_{\nu,h}})$;

(ii) *there exists* $c>0$ *such that* $\langle\tilde f^{\mathbf{m}}_{\nu,h},\tilde f^{\mathbf{m}'}_{\nu,h}\rangle=O(\mathrm{e}^{-c/h})$.

Proof. The proof is a straightforward adaptation of the one in [11, Lemma 5.5], even though the operator P_h and the quasimodes $(\tilde f^{\mathbf{m}}_{\nu,h})_{\mathbf{m}}$ differ from the ones of this reference. We recall the main steps for the reader's convenience.

(i) The idea is to use (3.27), the fact that P_h is local in x, Hypothesis 2.8 and the support properties of $\nabla\theta^{\mathbf{m}}_{\nu,h}$ and $\nabla\chi_{\mathbf{m}}$ to show that

$$|\langle P_h f^{\mathbf{m}}_{\nu,h},f^{\mathbf{m}'}_{\nu,h}\rangle|\le\langle\mathrm{Op}_h(m_h\,\mathrm{Id})(\theta^{\mathbf{m}}_{\nu,h}(\partial_v\chi_{\mathbf{m}})\,\mathrm{e}^{-W^{\mathbf{m}}/h}),\,b_h f_{\mathbf{m}',h}\rangle$$
$$=O_\nu(h^\infty\,\mathrm{e}^{-\frac{S(\mathbf{m})+S(\mathbf{m}')}{h}})$$

by (4.14). We can then conclude with (3.52).

(ii) It is shown in [11, proof of Lemma 5.5] that, when $V(\mathbf{m}) = V(\mathbf{m}')$, the supports of $f^{\mathbf{m}}_{\nu,h}$ and $f^{\mathbf{m}'}_{\nu,h}$ do not meet. Thus, we can suppose that $V(\mathbf{m}) > V(\mathbf{m}')$ and, in that case, using once again (3.27) and Hypothesis 2.8, we show that

$$\langle f^{\mathbf{m}}_{\nu,h}, f^{\mathbf{m}'}_{\nu,h}\rangle = \int_{E(\mathbf{m})+B(0,\varepsilon')} \theta^{\mathbf{m}}_{\nu,h}\theta^{\mathbf{m}'}_{\nu,h}\chi_{\mathbf{m}}\chi_{\mathbf{m}'}\,\mathrm{e}^{-\frac{2V-V(\mathbf{m})-V(\mathbf{m}')+v^2}{2h}}\,\mathrm{d}(x,v) = O(\mathrm{e}^{-\frac{V(\mathbf{m})-V(\mathbf{m}')}{2h}}),$$

so the conclusion immediately follows from (3.52). ∎

In order to go from quasimodes to functions that actually belong to the generalized eigenspace associated to the small eigenvalues of P_h, let us now consider the operator

$$\Pi_0 = \frac{1}{2i\pi}\int_{|z|=ch}(z-P_h)^{-1}\,\mathrm{d}z$$

introduced in [13]. Using the resolvent estimates from Theorem 1.2, the following is established in [13].

Proposition 5.3. *The operator Π_0 is a projector on the generalized eigenspace associated to the small eigenvalues of P_h and satisfies $\|\Pi_0\| = O(1)$.*

Lemma 5.4. *Using the notations (1.12), (4.1), and (4.2), for any $\mathbf{m} \in \mathrm{U}^{(0)}$, we have*

$$\|(1-\Pi_0)\tilde f^{\mathbf{m}}_{\nu,h}\| = \sqrt{\tilde\lambda_{\mathbf{m},h}}\big(O_\nu(1) + O(h^{-1/2}\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}}|\ln(\nu)|)\big).$$

Proof. We simply recall the proof from [8]. We write

$$\begin{aligned}(1-\Pi_0)\tilde f^{\mathbf{m}}_{\nu,h} &= \frac{1}{2i\pi}\int_{|z|=ch}(z^{-1}-(z-P_h)^{-1})\tilde f^{\mathbf{m}}_{\nu,h}\,\mathrm{d}z\\ &= \frac{-1}{2i\pi}\int_{|z|=ch} z^{-1}(z-P_h)^{-1}P_h\tilde f^{\mathbf{m}}_{\nu,h}\,\mathrm{d}z.\end{aligned}$$

We can then conclude using Lemma 5.1 and the resolvent estimate from Theorem 1.2. ∎

Lemma 5.5. *Recall the notations (1.12), (4.1), and (4.2). The family $(\Pi_0\tilde f^{\mathbf{m}}_{\nu,h})_{\mathbf{m}\in\mathrm{U}^{(0)}}$ is almost orthonormal: there exists $c > 0$ such that*

$$\langle \Pi_0\tilde f^{\mathbf{m}}_{\nu,h}, \Pi_0\tilde f^{\mathbf{m}'}_{\nu,h}\rangle = \delta_{\mathbf{m},\mathbf{m}'} + O_\nu(\mathrm{e}^{-c/h}).$$

In particular, it is a basis of the space $\operatorname{Ran}\Pi_0$.

Moreover, we have

$$\langle P_h\Pi_0\tilde f^{\mathbf{m}}_{\nu,h}, \Pi_0\tilde f^{\mathbf{m}'}_{\nu,h}\rangle = \delta_{\mathbf{m},\mathbf{m}'}\tilde\lambda^{\mathbf{m}}_{\nu,h} + \sqrt{\tilde\lambda^{\mathbf{m}}_{\nu,h}\tilde\lambda^{\mathbf{m}'}_{\nu,h}}\big(O_\nu(\sqrt{h}) + O(\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}}|\ln(\nu)|^2)\big).$$

Proof. The proof is the same as the one of [8, Proposition 4.10]. It suffices to write

$$\langle \Pi_0 \tilde{f}^{\mathbf{m}}_{\nu,h}, \Pi_0 \tilde{f}^{\mathbf{m}'}_{\nu,h} \rangle = \langle \tilde{f}^{\mathbf{m}}_{\nu,h}, \tilde{f}^{\mathbf{m}'}_{\nu,h} \rangle + \langle \tilde{f}^{\mathbf{m}}_{\nu,h}, (\Pi_0 - 1) \tilde{f}^{\mathbf{m}'}_{\nu,h} \rangle + \langle (\Pi_0 - 1) \tilde{f}^{\mathbf{m}}_{\nu,h}, \Pi_0 \tilde{f}^{\mathbf{m}'}_{\nu,h} \rangle$$

as well as

$$\langle P_h \Pi_0 \tilde{f}^{\mathbf{m}}_{\nu,h}, \Pi_0 \tilde{f}^{\mathbf{m}'}_{\nu,h} \rangle = \langle P_h \tilde{f}^{\mathbf{m}}_{\nu,h}, \tilde{f}^{\mathbf{m}'}_{\nu,h} \rangle + \langle (\Pi_0 - 1) \tilde{f}^{\mathbf{m}}_{\nu,h}, P_h^* \tilde{f}^{\mathbf{m}'}_{\nu,h} \rangle + \langle \Pi_0 P_h \tilde{f}^{\mathbf{m}}_{\nu,h}, (\Pi_0 - 1) \tilde{f}^{\mathbf{m}'}_{\nu,h} \rangle.$$

and use all the previous results of this section together with Proposition 4.4. ■

Let us re-label the local minima $\mathbf{m}_1, \dots, \mathbf{m}_{n_0}$ so that $(S(\mathbf{m}_j))_{j=1,\dots,n_0}$ is non-increasing in j. For shortness, we will now denote

$$\tilde{f}_j = \tilde{f}^{\mathbf{m}_j}_{\nu,h} \quad \text{and} \quad \tilde{\lambda}_j = \tilde{\lambda}^{\mathbf{m}_j}_{\nu,h}$$

which still depend on ν and h. Note in particular that, according to Proposition 4.4, $\tilde{\lambda}_j = O_\nu(\tilde{\lambda}_k)$ whenever $1 \leq j \leq k \leq n_0$. We also denote $(\tilde{u}_j)_{j=1,\dots,n_0}$ the orthogonalization by the Gram–Schmidt procedure of the family $(\Pi_0 \tilde{f}_j)_{j=1,\dots,n_0}$ and

$$u_j = \frac{\tilde{u}_j}{\|\tilde{u}_j\|}.$$

In this setting and with our previous results, we get the following (see [8, Proposition 4.12] for a proof).

Lemma 5.6. *With the notations* (1.12), (4.1), *and* (4.2), *for all* $1 \leq j, k \leq n_0$, *it holds*

$$\langle P_h u_j, u_k \rangle = \delta_{j,k} \tilde{\lambda}_j + \sqrt{\tilde{\lambda}_j \tilde{\lambda}_k} \big(O_\nu(\sqrt{h}) + O(\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}} |\ln(\nu)|^2) \big).$$

In order to compute the small eigenvalues of P_h, let us now consider the restriction $P_h|_{\operatorname{Ran} \Pi_0} \colon \operatorname{Ran} \Pi_0 \to \operatorname{Ran} \Pi_0$. We denote by $\hat{u}_j = u_{n_0 - j + 1}$, $\hat{\lambda}_j = \tilde{\lambda}_{n_0 - j + 1}$ and $\mathcal{M}$ the matrix of $P_h|_{\operatorname{Ran} \Pi_0}$ in the orthonormal basis $(\hat{u}_1, \dots, \hat{u}_{n_0})$. Since $\hat{u}_{n_0} = u_1 = \tilde{f}_1$, we have

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}' & 0 \\ 0 & 0 \end{pmatrix} \quad \text{where } \mathcal{M}' = (\langle P_h \hat{u}_j, \hat{u}_k \rangle)_{1 \leq j,k \leq n_0 - 1}$$

and it is sufficient to study the spectrum of $\mathcal{M}'$. We will also denote by $\{\widehat{S}_1 < \dots < \widehat{S}_p\}$ the set $\{S(\mathbf{m}_j) : 2 \leq j \leq n_0\}$ and' for $1 \leq k \leq p$, by E_k the subspace of $L^2(\mathbb{R}^{2d})$ generated by $\{\hat{u}_r : S(\mathbf{m}_r) = \widehat{S}_k\}$. Finally, we set

$$\varpi_k = \mathrm{e}^{-(\widehat{S}_k - \widehat{S}_{k-1})/h} \quad \text{for } 2 \leq k \leq p$$

and

$$\varepsilon_j(\varpi) = \prod_{k=2}^{j} \varpi_k = \mathrm{e}^{-(\widehat{S}_j - \widehat{S}_1)/h} \quad \text{for } 2 \leq j \leq p$$

(with the convention $\varepsilon_1(\varpi) = 1$). In view of Proposition 4.4, let us also denote

$$\tilde{\varrho}_0(\mathbf{m}) = \frac{1}{\pi} \sum_{\mathbf{s}\in \mathbf{j}(\mathbf{m})} \Big(\frac{2+\sqrt{2}}{2-\sqrt{2}}\Big)^g r \frac{1}{\sqrt{|\tau_{\mathbf{s}}|}} \Big(\frac{\det \mathcal{V}_{\mathbf{m}}}{|\det \mathcal{V}_{\mathbf{s}}|}\Big)^{1/2} \times \int_{\gamma_1\le z\le \gamma<1} k_0^{\mathbf{s}}(\gamma) k_0^{\mathbf{s}}(z) \ln\Big(2\frac{(1+z)(1+\gamma)}{1+3z+3\gamma+z\gamma}\Big)\, \mathrm{d}z\, \mathrm{d}\gamma$$

and

$$\hat{\lambda}_j^0 = h\, \tilde{\varrho}_0(\mathbf{m}_{n_0-j+1})\, \mathrm{e}^{\frac{-2S(\mathbf{m}_{n_0-j+1})}{h}}.$$

Lemma 5.7. *With the above notations, the matrix $\mathcal{M}'$ satisfies*

$$h^{-1}\, \mathrm{e}^{2\widehat{S}_1/h}\, \mathcal{M}' = \Omega(\varpi)\big(M_0^{\#} + O_\nu(\sqrt{h}) + O(\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}} |\ln(\nu)|^2)\big)\Omega(\varpi)$$

with

$$M_0^{\#} = \mathrm{diag}(\tilde{\varrho}_0(\mathbf{m}_{n_0-j+1}) : 1 \le j \le n_0 - 1)$$

and

$$\Omega(\varpi) = \mathrm{diag}(\varepsilon_1(\varpi)\,\mathrm{Id}_{E_1}, \dots, \varepsilon_p(\varpi)\,\mathrm{Id}_{E_p}).$$

In particular, for all $\nu > 0$, there exists $h_0 > 0$ such that, for all $0 < h < h_0$,

$$h^{-1}\, \mathrm{e}^{2\widehat{S}_1/h}\, \mathcal{M}' = \Omega(\varpi)\big(M_0^{\#} + O(\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}} |\ln(\nu)|^2)\big)\Omega(\varpi).$$

Remark 5.8. In the words of [8, Definition A.1], the last lemma implies that for all $\nu > 0$, there exists $h_0 > 0$ such that for all $0 < h < h_0$,

$$h^{-1}\, \mathrm{e}^{2\widehat{S}_1/h}\, \mathcal{M}' \text{ is a } ((E_k)_k, \varpi, \nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}} |\ln(\nu)|^2)\text{-graded matrix.}$$

Proof. According to Lemma 5.6 and Proposition 4.4, we can decompose

$$\mathcal{M}' = \mathcal{M}_1' + \mathcal{M}_2'$$

with

$$\mathcal{M}_1' = \mathrm{diag}(\hat{\lambda}_j^0 : 1 \le j \le n_0 - 1)$$

and

$$\mathcal{M}_2' = \Big(\sqrt{\hat{\lambda}_j \hat{\lambda}_k}\,[O_\nu(\sqrt{h}) + O(\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}} |\ln(\nu)|^2)]\Big)_{1\le j,k\le n_0-1}.$$

It then suffices to notice that $M_0^{\#} = h^{-1}\, \mathrm{e}^{2\widehat{S}_1/h}\, \Omega(\varpi)^{-1} \mathcal{M}_1' \Omega(\varpi)^{-1}$ and that

$$h^{-1}\, \mathrm{e}^{2\widehat{S}_1/h}\, \Omega(\varpi)^{-1} \mathcal{M}_2' \Omega(\varpi)^{-1} = O_\nu(\sqrt{h}) + O(\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}} |\ln(\nu)|^2),$$

where we still used Proposition 4.4. ∎

Proof of Theorem 1.3. According to Remark 5.8, it now suffices to combine the result of Lemma 5.7 with [8, Theorem A.4] which gives a description of the spectrum of graded matrices. We get that for all $\nu > 0$, there exists $h_0 > 0$ such that for all $0 < h < h_0$,

$$h^{-1}\,\mathrm{e}^{2S(\mathbf{m})/h}\,\lambda(\mathbf{m},h) - \tilde{\varrho}_0(\mathbf{m}) = O(\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}}\,|\ln(\nu)|^2)$$

and the result is proven. ∎

Proof of Corollaries 1.4 *and* 1.5. With the notations from Theorem 1.3, it is shown in [13, Section 4], with the use of PT-Symmetry arguments and a quantitative version of the Gearhart–Prüss theorem, that there exist $c > 0$ and some projectors $(\Pi_j)_{1\leq j\leq n_0}$ which are $O(1)$ and such that

- $\Pi_1 = \mathbb{P}_1$,
- $\Pi_j \Pi_k = \delta_{j,k}\Pi_j$,
- $\mathbb{P}_k = \sum_{\{j: S(\mathbf{m}_j)\geq S(\mathbf{m}_k)\}} \Pi_j$,
- $\mathrm{e}^{-tP_h/h} = \sum_{j=1}^{n_0} \mathrm{e}^{-t\lambda(\mathbf{m}_j,h)/h}\,\Pi_j + O(\mathrm{e}^{-ct})$ for $t \geq 0$ and h small enough.

Corollary 1.4 directly follows, while the proof of Corollary 1.5 is then an easy adaptation of the one of [1, Corollary 1.6]. (Note that our notations t_k^- and t_k^+ differ from that in [1].) ∎

A. Structure of the collision operator

The aim of this section is to show Proposition 2.2 and Corollary 2.3. For a, b two symbols, we denote by $a \# b$ the symbol of $\mathrm{Op}_h(a) \circ \mathrm{Op}_h(b)$. We start by showing that Q_h defined in (1.7) is a pseudo-differential operator.

Lemma A.1. *One has* $\Pi_h = \mathrm{Op}_h(\varpi_h)$ *with* $\varpi_h \in S^{1/2}(1)$ *given by*

$$\varpi_h(v,\eta) = 2^d\,\mathrm{e}^{-\frac{v^2+4\eta^2}{2h}}\,.$$

Proof of Lemma A.1. First, notice that the distributional kernel of Π_h is $\mu_h(v)\mu_h(v')$. Using the formula (1.8) to compute the symbol of a pseudo-differential operator from its distributional kernel, we get

$$\mathcal{F}_{h,v'}\Big(\mu_h\Big(v+\frac{v'}{2}\Big)\mu_h\Big(v-\frac{v'}{2}\Big)\Big)(v,\eta) = 2^d\,\mathrm{e}^{-\frac{v^2+4\eta^2}{2h}}$$

which is clearly in $S^{1/2}(1)$ as $\mathrm{e}^{-\frac{v^2+4\eta^2}{2}} \in S^0(1)$. ∎

Proof of Proposition 2.2. Let us first check that $m_h \in S^{1/2}(\langle v, \eta\rangle^{-2})$. We have

$$m_h(v,\eta) = \tilde{m}(h^{-1/2}v, h^{-1/2}\eta) \quad \text{and} \quad \tilde{m}(v,\eta) = \check{m}\Big(\frac{v^2}{2} + 2\eta^2\Big) \tag{A.1}$$

with

$$\tilde{m}(v,\eta) = 2\int_0^1 (y+1)^{d-2}\, \mathrm{e}^{-y(\frac{v^2}{2}+2\eta^2)}\, \mathrm{d}\, y$$

and

$$\check{m}(\mu) = 2\int_0^1 (y+1)^{d-2}\, \mathrm{e}^{-y\mu}\, \mathrm{d}\, y.$$

One can then check, using integration by parts that, for all $k \in \mathbb{N}$, there exists C_k such that $|\partial_\mu^k \check{m}(\mu)| \leq C_k\langle\mu\rangle^{-k-1}$, from which we deduce, using (A.1), that $\tilde{m} \in S^0(\langle v,\eta\rangle^{-2})$. Thus, still using (A.1), for $\alpha \in \mathbb{N}^{2d}$, there exists C_α such that

$$\begin{aligned}|\partial^\alpha m_h(v,\eta)| = h^{-|\alpha|/2}|\partial^\alpha \tilde{m}(h^{-1/2}v, h^{-1/2}\eta)| &\leq C_\alpha h^{-|\alpha|/2}\langle h^{-1/2}v, h^{-1/2}\eta\rangle^{-2}\\ &\leq C_\alpha h^{-|\alpha|/2}\langle v,\eta\rangle^{-2},\end{aligned}$$

so m_h indeed belongs to $S^{1/2}(\langle v,\eta\rangle^{-2})$. Using symbolic calculus and Lemma A.1, one could then simply check that

$$\Big(-i\eta^\mathsf{T} + \frac{v^\mathsf{T}}{2}\Big) \# (m_h \operatorname{Id}) \# \Big(i\eta + \frac{v}{2}\Big) = h(1-\varpi_h) \tag{A.2}$$

but let us explain how the suitable m_h (i.e., the one solving (A.2)) was found. Since $\big(i\eta + \frac{v}{2}\big)$ and its conjugate are both polynomials of degree 1, we compute

$$\begin{aligned}&\Big(-i\eta^\mathsf{T} + \frac{v^\mathsf{T}}{2}\Big) \# (m_h \operatorname{Id}) \# \Big(i\eta + \frac{v}{2}\Big)\\ &\quad = \Big(\eta^2 + \frac{v^2}{4}\Big)m_h - \frac{h}{2}(d m_h + v\cdot\partial_v m_h + \eta\cdot\partial_\eta m_h) + \frac{h^2}{4}\Big(\Delta_v + \frac{1}{4}\Delta_\eta\Big)m_h.\end{aligned} \tag{A.3}$$

Let us look for solutions under the form $m_h(v,\eta) = u_h(v,\eta)\,\mathrm{e}^{\frac{v^2+4\eta^2}{2h}}$. In that case,

$$\partial_v m_h = \mathrm{e}^{\frac{v^2+4\eta^2}{2h}}\Big(\partial_v u_h + \frac{u_h}{h}v\Big)$$

and

$$\Delta_v m_h = \Big(\Delta_v u_h + \frac{2v}{h}\cdot\partial_v u_h + \frac{d}{h}u_h + \frac{v^2}{h^2}u_h\Big)\,\mathrm{e}^{\frac{v^2+4\eta^2}{2h}},$$

so

$$\frac{h^2}{4}\Delta_v m_h - \frac{h}{2} v\cdot\partial_v m_h = \Big(\frac{h^2}{4}\Delta_v u_h + \frac{hd}{4}u_h - \frac{v^2}{4}u_h\Big)\,\mathrm{e}^{\frac{v^2+4\eta^2}{2h}}.$$

Similarly, we compute

$$\frac{h^2}{16}\Delta_\eta m_h - \frac{h}{2}\eta\cdot\partial_\eta m_h = \Big(\frac{h^2}{16}\Delta_\eta u_h + \frac{hd}{4}u_h - \eta^2 u_h\Big),\mathrm{e}^{\frac{v^2+4\eta^2}{2h}};$$

so, according to (A.3), (A.2) becomes

$$\frac{h^2}{4}\Big(\Delta_v u_h + \frac{1}{4}\Delta_\eta u_h\Big) = h(\mathrm{e}^{-\frac{v^2+4\eta^2}{2h}} - 2^d\,\mathrm{e}^{-\frac{v^2+4\eta^2}{h}}).$$

Applying the semiclassical Fourier transform on $\mathbb{R}^{2d}$, this yields

$$\begin{aligned}-\frac{1}{4}\Big(v^{*2} + \frac{\eta^{*2}}{4}\Big)\mathcal{F}_h u_h &= h(\pi h)^d(\mathrm{e}^{-\frac{4v^{*2}+\eta^{*2}}{8h}} - \mathrm{e}^{-\frac{4v^{*2}+\eta^{*2}}{16h}})\\ &= -\frac{(\pi h)^d}{4}\Big(v^{*2}+\frac{\eta^{*2}}{4}\Big)\int_1^2 \mathrm{e}^{-s\frac{4v^{*2}+\eta^{*2}}{16h}}\,\mathrm{d}s,\end{aligned}$$

where (v^*,η^*) denotes the dual variable of (v,η). Hence,

$$\mathcal{F}_h u_h(v^*,\eta^*) = (\pi h)^d\int_1^2 \mathrm{e}^{-s\frac{4v^{*2}+\eta^{*2}}{16h}}\,\mathrm{d}s$$

and, applying the inverse semiclassical Fourier transform, we get

$$u_h(v,\eta) = 2^d\int_1^2 s^{-d}\,\mathrm{e}^{-\frac{v^2+4\eta^2}{sh}}\,\mathrm{d}s.$$

Consequently,

$$m_h(v,\eta) = 2^d\int_1^2 s^{-d}\,\mathrm{e}^{-\frac{v^2+4\eta^2}{2h}(\frac{2}{s}-1)}\,\mathrm{d}s$$

and we find the final expression of m_h by substituting $y = \frac{2}{s}-1$. ∎

Proof of Corollary 2.3. By symbolic calculus, we just have to check that

$$g_h = \Big(-i\eta^\intercal + \frac{v^\intercal}{2}\Big)\,\#\,(m_h\,\mathrm{Id}).$$

Since the symbol on the left-hand side is a polynomial of degree 1, we have

$$\Big(-i\eta^\intercal + \frac{v^\intercal}{2}\Big)\,\#\,(m_h\,\mathrm{Id}) = m_h\Big(-i\eta^\intercal + \frac{v^\intercal}{2}\Big) - \frac{h}{2}\Big(\partial_v^\intercal - \frac{i}{2}\partial_\eta^\intercal\Big)m_h.$$

Now,

$$-\frac{h}{2}\partial_v^{\mathsf{T}} m_h(v,\eta) = \int_0^1 y(y+1)^{d-2}\, e^{-\frac{y}{h}(\frac{v^2}{2}+2\eta^2)}\, \mathrm{d}y\, v^{\mathsf{T}}$$

so we easily get

$$m_h(v,\eta)\,\frac{v^{\mathsf{T}}}{2} - \frac{h}{2}\partial_v^{\mathsf{T}} m_h(v,\eta) = \int_0^1 (y+1)^{d-1}\, e^{-\frac{y}{h}(\frac{v^2}{2}+2\eta^2)}\, \mathrm{d}y\, v^{\mathsf{T}}.$$

One can show similarly that

$$-i m_h(v,\eta)\,\eta^{\mathsf{T}} + \frac{ih}{4}\partial_\eta^{\mathsf{T}} m_h(v,\eta) = -2i\int_0^1 (y+1)^{d-1}\, e^{-\frac{y}{h}(\frac{v^2}{2}+2\eta^2)}\, \mathrm{d}y\, \eta^{\mathsf{T}}$$

which is enough to conclude. ■

B. Bilinear algebra

Lemma B.1. *Let $L(x,v) = L_x\cdot x + L_v\cdot v$ a linear form on $\mathbb{R}^{2d}$ and recall the notation (1.11). Then for any $\mathbf{s}\in \mathrm{U}^{(1)}$, the matrix $\mathcal{W}_{\mathbf{s}} + \nabla L\,\nabla L^{\mathsf{T}}$ is positive definite if and only if*

$$-\mathcal{V}_{\mathbf{s}}^{-1} L_x\cdot L_x - L_v^2 > \frac{1}{2}. \tag{B.1}$$

Moreover, its determinant is

$$2^{-2d}\det \mathcal{V}_{\mathbf{s}}\,(1 + 2\mathcal{V}_{\mathbf{s}}^{-1}L_x\cdot L_x + 2L_v^2).$$

Proof. First notice that since $\mathbf{s}\in\mathrm{U}^{(1)}$ and $\mathcal{W}_{\mathbf{s}} + \nabla L\,\nabla L^{\mathsf{T}} \geq \mathcal{W}_{\mathbf{s}}$, the matrix $\mathcal{W}_{\mathbf{s}} + \nabla L\,\nabla L^{\mathsf{T}}$ has at most one negative eigenvalue, so it is sufficient to show that its determinant is positive if and only if (B.1) holds. The rest of the proof is inspired by [1, Lemma 3.3]. We have

$$\begin{aligned}\det(\mathcal{W}_{\mathbf{s}} + \nabla L\,\nabla L^{\mathsf{T}}) &= \det \mathcal{W}_{\mathbf{s}}\det(\mathrm{Id} + \mathcal{W}_{\mathbf{s}}^{-1}\nabla L\,\nabla L^{\mathsf{T}})\\ &= 2^{-2d}\det\mathcal{V}_{\mathbf{s}}\det(\mathrm{Id} + \mathcal{W}_{\mathbf{s}}^{-1}\nabla L\,\nabla L^{\mathsf{T}})\end{aligned}$$

and since $\det\mathcal{V}_{\mathbf{s}} < 0$, it only remains to show that

$$(\mathrm{B.1}) \iff \det(\mathrm{Id} + \mathcal{W}_{\mathbf{s}}^{-1}\nabla L\,\nabla L^{\mathsf{T}}) < 0.$$

Now, it is easy to see that

$$(\mathrm{Id} + \mathcal{W}_{\mathbf{s}}^{-1} \nabla L \, \nabla L^{\mathsf{T}})|_{\nabla L^{\perp}} = \mathrm{Id}$$

and

$$(\mathrm{Id} + \mathcal{W}_{\mathbf{s}}^{-1} \nabla L \, \nabla L^{\mathsf{T}}) \nabla L \cdot \nabla L = (1 + 2\mathcal{V}_{\mathbf{s}}^{-1} L_x \cdot L_x + 2L_v^2) |\nabla L|^2.$$

Hence, $\det(\mathrm{Id} + \mathcal{W}_{\mathbf{s}}^{-1} \nabla L \, \nabla L^{\mathsf{T}}) = 1 + 2\mathcal{V}_{\mathbf{s}}^{-1} L_x \cdot L_x + 2L_v^2$ which is negative if and only if (B.1) holds true. ∎

Lemma B.2. *Recall the notations* (1.11) *and* (4.7). *For* $\gamma \in [\gamma_1 + \nu, 1]$ *and* $y \in (0, 1)$, *we have*

$$\det H_{\gamma,y}^{\mathbf{s}} = \frac{(1+y)^{2d-2}}{(4y)^d} (1 + (1 + 2|L_{\gamma,v}^{\mathbf{s}}|^2) y)^2 \, |\det \mathcal{V}|. \tag{B.2}$$

Proof. We drop some exponents and indexes $\mathbf{s}$ in the notations for shortness. Let us begin by writing

$$H_{\gamma,y} = \begin{pmatrix} \mathcal{V} & 0 & 0 \\ 0 & \frac{(y+1)^2}{4y} & \frac{y^2-1}{4y} \\ 0 & \frac{y^2-1}{4y} & \frac{(y+1)^2}{4y} \end{pmatrix} \left[\mathrm{Id} + \begin{pmatrix} \mathcal{V}^{-1} & 0 & 0 \\ 0 & 1 & \frac{1-y}{1+y} \\ 0 & \frac{1-y}{1+y} & 1 \end{pmatrix} \begin{pmatrix} L_{\gamma,x} & L_{\gamma,x} \\ L_{\gamma,v} & 0 \\ 0 & L_{\gamma,v} \end{pmatrix} \times \begin{pmatrix} L_{\gamma,x}^{\mathsf{T}} & L_{\gamma,v}^{\mathsf{T}} & 0 \\ L_{\gamma,x}^{\mathsf{T}} & 0 & L_{\gamma,v}^{\mathsf{T}} \end{pmatrix} \right]. \tag{B.3}$$

Clearly, the determinant of the first factor is $(4y)^{-d} (y+1)^{2d} \det \mathcal{V}$. Denoting

$$\widetilde{H}_{\gamma,y} = \begin{pmatrix} \mathcal{V}^{-1} & 0 & 0 \\ 0 & 1 & \frac{1-y}{1+y} \\ 0 & \frac{1-y}{1+y} & 1 \end{pmatrix} \begin{pmatrix} L_{\gamma,x} & L_{\gamma,x} \\ L_{\gamma,v} & 0 \\ 0 & L_{\gamma,v} \end{pmatrix} \begin{pmatrix} L_{\gamma,x}^{\mathsf{T}} & L_{\gamma,v}^{\mathsf{T}} & 0 \\ L_{\gamma,x}^{\mathsf{T}} & 0 & L_{\gamma,v}^{\mathsf{T}} \end{pmatrix},$$

it is also clear that $\widetilde{H}_{\gamma,y}$ has rank 2, so it has at most 2 non-zero eigenvalues. Besides, using Lemma 3.1, one can easily check that

$$\widetilde{H}_{\gamma,y} \begin{pmatrix} (1+y)\mathcal{V}^{-1} L_{\gamma,x} \\ L_{\gamma,v} \\ L_{\gamma,v} \end{pmatrix} = \frac{-2}{1+y} (1 + (1 + |L_{\gamma,v}^{\mathbf{s}}|^2) y) \begin{pmatrix} (1+y)\mathcal{V}^{-1} L_{\gamma,x} \\ L_{\gamma,v} \\ L_{\gamma,v} \end{pmatrix}$$

and

$$\widetilde{H}_{\gamma,y} \begin{pmatrix} 0 \\ L_{\gamma,v} \\ -L_{\gamma,v} \end{pmatrix} = \frac{2y|L_{\gamma,v}|^2}{1+y} \begin{pmatrix} 0 \\ L_{\gamma,v} \\ -L_{\gamma,v} \end{pmatrix}.$$

Hence, the determinant of the second factor from (B.3) is

$$-(1+y)^{-2}(1+(1+2|L^{\mathbf{s}}_{\gamma,v}|^2)y)^2$$

and we get (B.2). ∎

C. Multivariate Gaussian moment

Using the formulas of the first moments of the one-dimensional Gaussian, we easily establish the following.

Proposition C.1. *If A is a real symmetric matrix, then*

$$\int_{\mathbb{R}^{d'}} Ax\cdot x\, \mathrm{e}^{-\frac{x^2}{2}}\, \mathrm{d}x = (2\pi)^{d'/2}\,\mathrm{Tr}(A).$$

D. Laplace's method

Here we give a precise statement of Laplace's method that we use to approximate h-dependent integrals.

Proposition D.1. *Let $x_0 \in \mathbb{R}^{d'}$, K a compact neighborhood of x_0, and $\varphi \in \mathscr{C}^\infty(K)$ such that x_0 is a non-degenerate minimum of φ and its only global minimum on K. Denote $H \in \mathcal{M}_{d'}(\mathbb{R})$ the Hessian of φ at x_0.*

- *If a_h is a function bounded uniformly in h on K such that*

$$a_h = O((x-x_0)^{2n}),$$

  then

$$h^{-d'/2}\int_K a_h(x)\,\mathrm{e}^{-\frac{\varphi(x)-\varphi(x_0)}{h}}\,\mathrm{d}x = O(h^n).$$

- *If $a_h \sim \sum_{j\geq 0} h^j a_j$ in $\mathscr{C}^\infty(K)$, then the integral*

$$\frac{\det(H)^{1/2}}{(2\pi h)^{d'/2}}\int_K a_h(x)\,\mathrm{e}^{-\frac{\varphi(x)-\varphi(x_0)}{h}}\,\mathrm{d}x$$

  admits a classical expansion whose first term is given by $a_0(x_0)$.

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