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Metastabilité de processus non locaux

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Chapitre 1

Introduction : Petites valeurs propres d'opérateurs semiclassiques

1.1 Motivations

Les principaux objets d'étude de cette thèse sont des équations de Boltzmann

$$\begin{cases} \partial_t u + X_0(u) + Q(u) = 0 \\ u_{|t=0} = u_0 \end{cases} \tag{1.1.1}$$

qui permettent de modéliser l'évolution d'un système de particules formant un gaz. L'inconnue est ici la fonction $u : \mathbb{R}_+ \to L^1(\mathbb{R}^{2d})$ et donne la densité de probabilité du système de particules au temps $t \in \mathbb{R}_+$, position $x \in \mathbb{R}^d$ et vitesse $v \in \mathbb{R}^d$. L'opérateur Q est appelé *opérateur de collisions* et rend compte de l'interaction entre les particules (ou entre une particule et le milieu). Il peut donc être quadratique ou linéaire et n'agit que dans la variable de vitesse ; tandis que X_0 est un opérateur de transport agissant dans les variables (x, v).

Les équations que l'on va considérer ici seront bien posées, admettront un état stable (aussi appelé Maxwellienne) et un de nos objectifs sera d'étudier le *retour à l'équilibre* des solutions (i.e leur convergence en temps long vers cette Maxwellienne). Dans le cas où Q est quadratique, il s'agit d'un problème qui remonte aux travaux de Boltzmann et pour lequel des réponses ont été apportées dans [11, 40, 44] avec $X_0 = v \cdot \partial_x$. Pour ce type d'opérateurs de collisions, il n'existe à notre connaissance pas de résultat similaire en présence d'une force extérieure dérivant d'un potentiel V, c'est-à-dire lorsque

$$X_0 = v \cdot \partial_x - \partial_x V \cdot \partial_v. \tag{1.1.2}$$

Ici V est une fonction ne dépendant que de la variable spatiale et à valeurs dans $\mathbb{R}$, tandis que les notations ∂_x et ∂_v désignent les gradients partiels par rapport à x et v. Ce nouveau terme dans l'opérateur de transport vient compliquer l'étude de l'équation (1.1.1) notamment en introduisant des coefficients non explicites.

Dans le cas spatialement non homogène (i.e où $X_0 \neq 0$) donné par (1.1.2), il semble donc raisonnable de commencer par traiter des opérateurs de collisions linéaires. On se placera dans un cadre Hilbertien que l'on détaille à la fin de cettre section et on notera $Q_{\mathcal{H}}$ ces opérateurs. Cela permet de mettre en place des méthodes spectrales pour l'étude de l'opérateur

$$X_0 + Q_{\mathcal{H}} \tag{1.1.3}$$

associé à (1.1.1). Une façon d'établir le retour à l'équilibre est alors de montrer que l'opérateur (1.1.3) présente un trou spectral. C'est par exemple l'approche adoptée dans [1, 32] où est traité le cas spatialement homogène. Dans notre cas non homogène, il faut prendre en compte l'interaction entre les opérateurs de transport et de collisions, ce qui rend plus difficile de montrer l'existence d'un trou spectral. On a alors généralement recours à des méthodes d'*hypocoercivité hilbertienne*, voir par exemple [7, 13, 21, 39, 45], ou encore [2, 5] dans le cas sans potentiel ; mais aussi [14, 22, 37] pour d'autres équations hypocoercives.

Lorsque l'on prend également en compte la température du système de particules que l'on considère (on note h un paramètre qui lui est proportionnel), l'équation (1.1.1) devient

$$\begin{cases} h\partial_t u + v\cdot h\partial_x u - \partial_x V\cdot h\partial_v u + Q_{\mathcal{H}}(h,u) = 0 \\ u_{|t=0} = u_0. \end{cases} \tag{1.1.4}$$

On va effectuer ici une étude de (1.1.4) à basse température, c'est-à-dire dans la limite *semiclassique* $h\to 0$. Robbe a obtenu dans ce cadre des premiers résultats d'hypocoercivité pour des potentiels confinants et des choix spécifiques d'opérateurs de collisions [38,39]. On se propose ici de poursuivre et d'étendre ses travaux en cherchant notamment à établir une formule d'Eyring-Kramers (i.e une asymptotique précise des petites valeurs propres). Il va nous falloir faire face à des difficultés propres à l'analyse semiclassique, comme la nécessité de fournir des estimations uniformes en h ou encore la présence de plusieurs petites valeurs propres presque indiscernables car toutes exponentiellement petites par rapport à $1/h$. Commençons par spécifier le cadre fonctionnel dans lequel on va se placer.

Cadre L^2 pour les équations de Boltzmann linéaires

Dans l'optique de notre étude de (1.1.4), on définit les racines carrées des distributions Maxwelliennes usuelles

$$\mu_h(v) = \frac{\mathrm{e}^{-\frac{v^2}{4h}}}{(2\pi h)^{d/4}} \qquad \text{et} \qquad \mathcal{M}_h = \mathrm{e}^{-\frac{V}{2h}}\mu_h. \tag{1.1.5}$$

On considérera des opérateurs de collisions tels que, en notant $Q^*_{\mathcal{H}}(h,\cdot)$ l'adjoint formel de $Q_{\mathcal{H}}(h,\cdot)$, on a

$$Q_{\mathcal{H}}(h,\mathcal{M}_h^2) = 0 \qquad \text{et} \qquad Q^*_{\mathcal{H}}(h,1) = 0. \tag{1.1.6}$$

En particulier, $\mathcal{M}_h^2$ est un état stable de (1.1.4) et $Q_{\mathcal{H}}$ conserve localement la masse. Un exemple de tel opérateur qu'on étudiera dans cette thèse est celui du modèle linéaire de BGK

$$Q_{\mathcal{H}}(h,u) = h\Big(u - \int_{v'\in\mathbb{R}^d} u(x,v')\,\mathrm{d}v'\mu_h^2\Big) \tag{1.1.7}$$

qui correspond à une simple relaxation vers la Maxwellienne.

Dans le but d'effectuer une étude perturbative de l'opérateur associé à (1.1.4) près de $\mathcal{M}_h^2$ et dans l'esprit de [21,39], on définit l'espace de Hilbert naturel

$$\mathcal{H} = \big\{u\in\mathcal{D}'\,;\,\mathcal{M}_h^{-1}u\in L^2(\mathbb{R}^{2d})\big\}.$$

Il est clair par l'inégalité de Cauchy Schwarz que $\mathcal{H}$ est bien un sous espace de $L^1(\mathbb{R}^{2d})$ pourvu que $\mathrm{e}^{-\frac{V}{2h}}\in L^2(\mathbb{R}^d_x)$. Au vu de (1.1.6) et de la définition de $\mathcal{H}$, il apparait plus commode de travailler avec la nouvelle inconnue

$$f = \mathcal{M}_h^{-1}u\,:\mathbb{R}_+\to L^2(\mathbb{R}^{2d})$$

pour laquelle la nouvelle équation devient

$$\begin{cases} h\partial_t f + v\cdot h\partial_x f - \partial_x V\cdot h\partial_v f + Q_h(f) = 0 \\ f_{|t=0} = f_0 \end{cases} \tag{1.1.8}$$

où

$$Q_h = \mathcal{M}_h^{-1}\circ Q_{\mathcal{H}}(h,\cdot)\circ\mathcal{M}_h \tag{1.1.9}$$

est un opérateur borné et auto-adjoint. Dans le cas de l'opérateur de BGK (1.1.7), en définissant avec la notation (1.1.5),

$$\Pi_h : L^2(\mathbb{R}^{2d})\to L^2(\mathbb{R}^{2d}) \tag{1.1.10}$$

le projecteur orthogonal sur $\mu_h\, L^2(\mathbb{R}^d_x)$, on obtient par (1.1.9)

$$Q_h = h(\mathrm{Id} - \Pi_h). \tag{1.1.11}$$

Notre étude sera concentrée sur l'opérateur indépendant du temps associé à (1.1.8)

$$P_h = X_0^h + Q_h \tag{1.1.12}$$

où la notation X_0^h désignera l'opérateur $v \cdot h\partial_x - \partial_x V \cdot h\partial_v$, mais aussi le champ de vecteurs $(x,v) \mapsto h(v, -\partial_x V(x))$. Muni d'un domaine convenable, ce dernier est anti-adjoint et par conséquent (voir par exemple [43, Proposition 3.1.11]) P_h est maximal-accrétif. On sera plus particulièrement intéressé par les propriétés spectrales de P_h. On verra notament à la section 1.4 que la connaissance de ces dernières nous permettra de donner des informations précises sur le comportement en temps long des solutions de (1.1.8).

1.2 Contexte et état de l'art

On s'intéresse tout au long de cette thèse à étudier des opérateurs semiclassiques, c'est à dire dépendant d'un paramètre $h > 0$ qu'on fera tendre vers 0 ; et faisant intervenir un potentiel qu'on notera V ou parfois W. Des exemples typiques de tels opérateurs sont fournis par l'opérateur de Schrödinger

$$-h^2\Delta + V; \tag{1.2.1}$$

des opérateurs issus de l'étude de processus stochastiques de diffusion comme le Laplacien de Witten

$$\Delta_V = -h^2\Delta + |\nabla V|^2 - h\Delta V; \tag{1.2.2}$$

ou encore des opérateurs associés à des équations cinétiques comme l'opérateur de Fokker-Planck ou des opérateurs de Boltzmann inhomogènes linéaires, ces derniers étant tous de la forme

$$v \cdot h\partial_x - \partial_x V \cdot h\partial_v + Q_h$$

pour différents choix de l'opérateur Q_h (local pour l'opérateur de Fokker-Planck, non local dans le cas de l'équation de Boltzmann linéaire). On s'intéressera plus particulièrement ici au spectre près de 0 de ces opérateurs, ce qui nous permettra de discuter la stabilité et la *metastabilité* (qu'on présentera plus loin) des solutions des équations associées. Les premiers résultats mathématiques sur ces problématiques spectrales bien connues en physique théorique ainsi qu'en chimie quantique ont été établis dans les années 1980 pour l'opérateur de Schrödinger (1.2.1) (voir par exemple [8]). Grâce à des approximations harmoniques près de chaque minimum du potentiel, Helffer-Sjöstrand [15–17] et Simon [41, 42] ont notamment obtenu des équivalents de ses valeurs propres, menant par exemple à l'étude de l'effet tunnel pour cet opérateur.

Cela a permis dans la foulée de fournir une première description du bas du spectre du Laplacien de Witten Δ_V par Helffer et Sjöstrand dans [19]. Ces derniers ont montré que, pour un potentiel V de Morse (c'est à dire dont les points critiques sont non dégénérés), satisfaisant des hypothèses de confinement et possédant $n_0 \in \mathbb{N}_{\geq 2}$ minima locaux, il existe $c > 0$ tel que le spectre de Δ_V dans $[0, ch[$ est constitué d'exactement n_0 valeurs propres (comptées avec multiplicité) qui sont $O(\mathrm{e}^{-c/h})$. Ce résultat peut se démontrer en remarquant que Δ_V est une perturbation de l'opérateur de Schrödinger pour le potentiel $|\nabla V|^2$ dont les minima sont les points critiques de V. En appliquant une approximation harmonique près de ces derniers, on s'aperçoit que le terme $-h\Delta V$ contribue à produire soit une valeur propre dans $[0, ch[$ dans le cas d'un minimum, soit une valeur propre hors de $[0, ch[$ dans le cas d'un autre point critique. Après avoir remarqué que

$$\Delta_V \mathrm{e}^{-V/h} = 0,$$

l'idée pour montrer que les valeurs propres dans $[0, ch[$ sont exponentiellement petites est de construire des *quasimodes* (fonctions propres approchées) de la forme

$$\chi_j \mathrm{e}^{-V/h} \tag{1.2.3}$$

où χ_j est une fonction plateau localisant près du j-ème minimum de V. Ces fonctions vérifient bien

$$\Delta_V\big(\chi_j \mathrm{e}^{-V/h}\big) = O\big(\mathrm{e}^{-c/h}\|\chi_j \mathrm{e}^{-V/h}\|\big)$$

et on peut donc conclure en appliquant le principe du min-max à la famille obtenue. Ce résultat permet notamment de fournir une preuve analytique des inégalités de Morse [9, 46], ce qui était la principale motivation de Witten pour l'étude de l'opérateur Δ_V.

Il est alors assez naturel de chercher à aller plus loin dans la description de ces petites valeurs propres en essayant par exemple d'établir une formule d'Eyring-Kramers (voir par exemple [25] pour un premier calcul formel). La taille exponentiellement petite de ces valeurs propres pose alors de sérieuses difficultés. On ne peut par exemple pas espérer utiliser les méthodes BKW classiques qui fournissent des restes d'ordre h^∞. Pire encore, on sait aujourd'hui que la petite valeur propre associée au j-ème minimum de V est de taille $O(\mathrm{e}^{-2S_j/h})$ où S_j est la hauteur d'un certain puits contenant ce minimum. L'idée d'en obtenir une localisation en réutilisant les quasimodes (1.2.3) semble donc compromise. En effet, même en poussant le support de χ_j jusqu'au bord du puits, on n'obtient à priori pas mieux que

$$\|\Delta_V\big(\chi_j \mathrm{e}^{-V/h}\big)\|^2 = O\Big(\mathrm{e}^{-2(S_j-\varepsilon)/h}\|\chi_j \mathrm{e}^{-V/h}\|^2\Big).$$

Ce n'est donc pas cette approche qui a permis d'obtenir la première description précise des petites valeurs propres de Δ_V (on verra cependant à la section 1.3 que l'on sait aujourd'hui construire des troncatures finement choisies de $\mathrm{e}^{-V/h}$ qui fournissent le résultat espéré).

Il faudra ainsi attendre les années 2000 pour qu'une description précise (en l'occurence un développement asymptotique) des petites valeurs propres du Laplacien de Witten soit établie dans [6] par des méthodes probabilistes et dans [18] par des méthodes spectrales. Cette dernière approche consiste à exploiter la structure *supersymétrique* du Laplacien de Witten :

$$\Delta_V = d_V^* d_V \qquad \text{avec} \qquad d_V = h\nabla + \nabla V(x)$$

et d'étudier la dérivée d_V de l'espace des 0-formes vers l'espace des 1-formes ainsi que son adjoint.

C'est également ce type de méthodes qui a permis à Hérau et al. de fournir une description similaire du spectre près de 0 de l'opérateur de Fokker-Planck dans [22, 23]. Contrairement au Laplacien de Witten, cet opérateur n'est pas auto-adjoint et une étape préliminaire présentant déjà des difficultés conséquentes est alors d'obtenir des estimations de résolvante près de 0 qui permettront ensuite de définir des projecteurs spectraux (cf. (1.3.2)).

De telles estimations de résolvante ont pu être obtenues autour de 2015 dans le cas d'opérateurs de Boltzmann linéaires par Robbe [38, 39] grâce à l'emploi de techniques hypocoercives que l'on présentera dans la section 2.2 (voir aussi [21] dans le cas $h = 1$). Ces dernières permettent d'obtenir une première description du spectre près de 0 de ces opérateurs similaire à celle obtenue dans les années 1980 par Helffer et Sjöstrand pour le Laplacien de Witten. Cependant, les méthodes supersymétriques ayant permis de passer à une description précise pour le Laplacien de Witten ou l'opérateur de Fokker-Planck échouent à traiter les opérateurs de Boltzmann linéaires car la structure supersymétrique de ces derniers fait intervenir une modification du produit scalaire ne vérifiant pas de bonnes estimées par rapport à h.

Les principales contributions de cette thèse consistent à établir des asymptotiques précises des petites valeurs propres des opérateurs de Boltzmann de relaxation douce et de relaxation linéaire (voir chapitres 2, 4, 5 et 7). Cela fournit donc non seulement le trou spectral de ces opérateurs et les résultats sur la vitesse exponentielle de convergence vers l'équilibre qui en découlent, mais le calcul des valeurs propres suivantes permet également de mettre en évidence le caractère metastable des solutions (voir chapitre 1.4). Les méthodes employées consistent à adapter aux cadres non locaux de ces opérateurs des constructions de quasimodes précis appelés *quasimodes gaussiens* (QG) récemment développées par Le Peutrec-Michel [26] et Bony - Le Peutrec - Michel [4] pour traiter les petites valeurs propres d'opérateurs différentiels de type Fokker-Planck sans hypothèse de supersymétrie.

On commence donc dans la section suivante par décrire ces constructions et expliquer comment elles fournissent une asymptotique des petites valeurs propres. La section 1.4 est dédiée à l'obtention de résultats de retour à l'équilibre et de metastabilité pour les solutions des équations d'évolution associées. Enfin, on étudie dans les sections 1.5 et 1.6 un modèle jouet d'opérateur non local pour lequel la méthode présentée à la section 1.3 échoue et nécessite une adaptation. Les difficultés qui y sont présentées ainsi que la résolution proposée rendent compte de façon pertinente de celles que l'on retrouve pour l'étude de l'équation de Boltzmann de relaxation linéaire.

1.3 La méthode des QG usuels

On présente dans cette section les constructions de quasimodes gaussiens inspirées de [6] et développées dans [4,26] pour l'étude précise du spectre près de 0 d'opérateurs différentiels de type Fokker-Planck. Cette méthode s'applique à des opérateurs pour lesquels on dispose déjà d'une première description du spectre près de 0, ainsi que d'estimations de résolvante qui serviront notamment à introduire des projecteurs spectraux adaptés.
Plus précisément, on va considérer un opérateur semiclassique P_h agissant sur $L^2(\mathbb{R}^d)$ et tel que :

a) P_h est maximal accrétif.
b) Il existe un potentiel W à valeurs réelles, confinant, de Morse (c'est à dire dont les points critiques sont non dégénérés), qui possède $n_0 \in \mathbb{N}_{\geq 2}$ minima locaux et tel que
$$\mathrm{e}^{-W/h} \in \operatorname{Ker} P_h \cap \operatorname{Ker} P_h^*.$$
c) 0 est valeur propre simple de P_h.
d) Il existe $c > 0$, et $h_0 > 0$ tels que pour $0 < h \leq h_0$, le spectre de P_h dans le demi-plan $\{\operatorname{Re} z \leq ch\}$ est constitué d'exactement n_0 valeurs propres (comptées avec multiplicité algébrique) qui sont $O(\mathrm{e}^{-c/h})$.
e) Pour tout $0 < \tilde{c} \leq c$, on a l'estimation de résolvante
$$(P_h - z)^{-1} = O(h^{-1})$$
uniformément sur $\{\operatorname{Re} z \leq ch\} \backslash B(0, \tilde{c}h)$.

Le but de cette méthode est de construire une famille de n_0 quasimodes (fonctions propres approchées) associés aux petites valeurs propres de P_h que l'on souhaitera être les plus précis possibles. Par quasimode précis, on entend une fonction f_h telle que

$$\|P_h f_h\|^2 << \langle P_h f_h, f_h\rangle. \tag{1.3.1}$$

Cela peut se comprendre de la façon suivante : si f_h était une véritable fonction propre associée à une petite valeur propre λ_h de P_h, on aurait
$$\frac{\|P_h f_h\|^2}{\langle P_h f_h, f_h\rangle} = \lambda_h = O(\mathrm{e}^{-c/h}).$$
Dans la pratique, (1.3.1) servira lorsque l'on considérera le projecteur spectral (qui n'est pas nécessairement orthogonal)

$$\Pi_0 = \frac{1}{2i\pi} \int_{|z|=ch} (z - P_h)^{-1} \mathrm{d}z. \tag{1.3.2}$$

Ce dernier vérifie
$$\operatorname{Spec}(P_h) \cap B(0, ch) = \operatorname{Spec}(P_h|_{\operatorname{Ran} \Pi_0})$$
et commute avec P_h (voir par exemple [24, chapitre III, Théorème 6.17]). On montre également qu'il est borné uniformément en h grâce aux estimées de résolvante dont on dispose sur P_h et la petitesse de $\|P_h f_h\|$ induite par (1.3.1) permet en écrivant

$$(1 - \Pi_0) f_h = \frac{-1}{2i\pi} \int_{|z|=ch} z^{-1}(z - P_h)^{-1} P_h f_h \mathrm{d}z. \tag{1.3.3}$$

de montrer une estimation du type

$$\|(1 - \Pi_0) f_h\|^2 = O\big(\langle P_h f_h, f_h\rangle\big). \tag{1.3.4}$$

Le membre de gauche correspond au coût à payer pour passer du quasimode f_h à la fonction $\Pi_0 f_h$ qui appartient effectivement au sous-espace généralisé associé aux petites valeurs propres de P_h. C'est cette estimation qui permet ensuite de contrôler les termes d'erreur apparaissant lorsque l'on cherche à obtenir des informations sur les valeurs propres à partir d'informations obtenues sur les quasimodes.

L'idée de cette méthode est de construire, pour chaque minimum $\mathbf{m}$ de W, un *quasimode gaussien* $f_{\mathbf{m},h}$. Afin d'alléger la présentation, on va se placer ici dans un cadre simplifié mais qui permet néanmoins de présenter toutes les idées fondamentales qui apparaissent.

Hypothèse “double puits” 1.3.1. *Le potentiel* W *a exactement 2 minima locaux* $\underline{\mathbf{m}}$ *et* $\mathbf{m}$ *et ces derniers vérifient*

$$W(\underline{\mathbf{m}}) < W(\mathbf{m}) = 0.$$

Enfin, 0 *est l'unique point selle de* W*.*

Sous cette hypothèse, P_h a exactement 2 petites valeurs propres et on a donc à construire seulement 2 quasimodes $f_{\underline{\mathbf{m}},h}$ et $f_{\mathbf{m},h}$ associés respectivement à $\underline{\mathbf{m}}$ et $\mathbf{m}$. Pour le premier, on va simplement prendre la fonction propre connue de notre opérateur et donc poser

$$f_{\underline{\mathbf{m}},h} = \mathrm{e}^{-W/h}.$$

Tout le travail consiste donc maintenant à construire le second quasimode qui nous permettra de décrire précisément l'unique petite valeur propre non nulle de P_h que l'on note λ_h. On va prendre une fonction de la forme

$$f_{\mathbf{m},h}(x) = \theta(x)\mathrm{e}^{-W(x)/h} \tag{1.3.5}$$

avec θ une *troncature gaussienne*; c'est à dire une fonction plateau localisant à échelle $\sqrt{h}$ près de la composante connexe (CC) de $\{W < W(0)\}$ contenant $\mathbf{m}$ et dont le profil est essentiellement donné près de 0 (le point selle de W, qui se trouve être sur le bord de cette CC, voir Lemme C.0.6) par

$$\theta(x) = \frac{1}{\sqrt{h}}\int_0^{\ell(x)} \mathrm{e}^{-s^2/2h}\mathrm{d}s.$$

Ici ℓ est une forme linéaire que l'on choisira de sorte à minimiser $\|P_h f_{\mathbf{m},h}\|$, au vu de (1.3.1). Intuitivement, on peut voir le vecteur représentant ℓ comme la direction dans laquelle notre plateau doit être profilé. La méthode peut être raffinée en considérant ℓ^h une fonction lisse admettant un développement classique, cf [4, 36]; cela permet notamment de traiter des cas où le demi-plan $\{\mathrm{Re}\, z \leq ch\}$ est remplacé par $\{\mathrm{Re}\, z \leq ch^N\}$ et les estimées de résolvante sont de la forme $(P_h - z)^{-1} = O(h^{-N})$. On dira que la fonction $f_{\mathbf{m},h}$ est un *quasimode gaussien usuel*, par opposition avec les *superpositions de quasimodes gaussiens* que l'on présentera à la section 1.6 ainsi qu'aux chapitres 4 et 7.

On peut remarquer par une méthode de Laplace (cf Appendice B) que la norme de $f_{\mathbf{m},h}$ se concentre près de $\mathbf{m}$ et ainsi, grâce à l'Hypothèse “double puits” 1.3.1, on a

$$\frac{\langle f_{\underline{\mathbf{m}},h}, f_{\mathbf{m},h}\rangle}{\|f_{\underline{\mathbf{m}},h}\|\|f_{\mathbf{m},h}\|} = O(\mathrm{e}^{-c/h}). \tag{1.3.6}$$

Il s'agit maintenant de calculer $P_h f_{\mathbf{m},h}$. Ce calcul, relativement aisé dans le cas d'un opérateur différentiel, s'avère nettement plus compliqué lorsque P_h est non local. Pour les opérateurs différentiels que cette méthode a déjà permis de traiter, on obtient à des termes négligeables près un résultat localisé près de 0 et de la forme

$$P_h f_{\mathbf{m},h}(x) = \sqrt{h}\Big(a_\ell \cdot x + O\big(x^2\big)\Big)\mathrm{e}^{-\widetilde{W}_\ell(x)/h} \tag{1.3.7}$$

où $a_\ell \in \mathbb{R}^d$ et

$$\widetilde{W}_\ell = W + \frac{1}{2}\ell(x)^2. \tag{1.3.8}$$

On verra aux chapitres 2 et 5 que (1.3.7) est encore vérifiée pour des opérateurs pseudo-différentiels dont le symbole est dans la classe $S^0(1)$ définie en Appendice A. Au vu de (1.3.8) et en se rappelant que notre but est de minimiser la norme de $P_h f_{\mathbf{m},h}$, on dira que

(1.3.9) $\qquad \ell$ est *convenable* si le point selle 0 est un minimum non dégénéré de $\widetilde{W}_\ell$.

On va d'abord chercher à choisir ℓ de sorte à annuler le préfacteur a_ℓ présent dans (1.3.7). Cela détermine en général entièrement le choix de ℓ (au signe près, que l'on choisira de façon à ce que $f_{\mathbf{m},h}$ localise près du bon minimum) et on démontre ensuite que la forme linéaire obtenue est convenable. En appliquant la méthode de Laplace (voir Appendice B), (1.3.7) fournit alors pour ce choix de ℓ

$$\|P_h f_{\mathbf{m},h}\|^2 = O\big(h^3 \mathrm{e}^{-2W(0)/h}\big)\|f_{\mathbf{m},h}\|^2. \tag{1.3.10}$$

Dans l'esprit de (1.3.1), on introduit alors la valeur propre approchée

$$\tilde{\lambda}_h = \frac{\langle P_h f_{\mathbf{m},h}, f_{\mathbf{m},h}\rangle}{\|f_{\mathbf{m},h}\|^2}. \tag{1.3.11}$$

Toujours à l'aide de la méthode de Laplace (voir Appendice B), on montre que cette dernière vérifie

$$\tilde{\lambda}_h = h\big(\varrho + O(h)\big)\mathrm{e}^{-2W(0)/h} \tag{1.3.12}$$

avec $\varrho > 0$ que l'on calcule explicitement. Comme indiqué plus haut, on va maintenant pouvoir montrer qu'en remplaçant nos quasimodes par des fonctions appartenant effectivement au sous-espace généralisé Ran Π_0 associé aux petites valeurs propres de P_h, les termes d'erreur apparaissant sont négligeables. En effet, (1.3.10) et (1.3.12) fournissent en suivant (1.3.3) une estimation du type (1.3.4). Cette dernière nous permet, en définissant la nouvelle notion de valeur propre approchée

$$\hat{\lambda}_h = \frac{\langle P_h \Pi_0 f_{\mathbf{m},h}, \Pi_0 f_{\mathbf{m},h}\rangle}{\|\Pi_0 f_{\mathbf{m},h}\|^2}$$

et en écrivant

$$\big\langle P_h \Pi_0 f_{\mathbf{m},h}, \Pi_0 f_{\mathbf{m},h}\big\rangle = \big\langle P_h f_{\mathbf{m},h}, f_{\mathbf{m},h}\big\rangle + \big\langle (\Pi_0 - 1) f_{\mathbf{m},h}, P_h^* f_{\mathbf{m},h}\big\rangle + \big\langle \Pi_0 P_h f_{\mathbf{m},h}, (\Pi_0 - 1) f_{\mathbf{m},h}\big\rangle.$$

de montrer que

$$\hat{\lambda}_h = \tilde{\lambda}_h\Big(1 + O\big(h^{1/2}\big)\Big). \tag{1.3.13}$$

Dans le cas où P_h n'est pas auto-adjoint, on a utilisé ici que (1.3.10) reste vraie en remplaçant P_h par P_h^* et h^3 par h^2.
On peut maintenant conclure en orthonormalisant la famille $(f_{\underline{\mathbf{m}},h}, \Pi_0 f_{\mathbf{m},h})$ par une procédure de Gram-Schmidt pour obtenir une base orthonormée $(u_{\underline{\mathbf{m}},h}, u_{\mathbf{m},h})$ de Ran Π_0. Là encore les termes d'erreur apparaissant sont négligeables en vue de (1.3.6). Dans cette nouvelle base, la matrice de $P_h|_{\mathrm{Ran}\,\Pi_0}$ est donnée par

$$\begin{pmatrix} 0 & 0 \\ 0 & \langle P_h u_{\mathbf{m},h}, u_{\mathbf{m},h}\rangle \end{pmatrix}; \tag{1.3.14}$$

ainsi, on a

$$\lambda_h = \langle P_h u_{\mathbf{m},h}, u_{\mathbf{m},h}\rangle$$

et par des arguments similaires à ceux utilisée pour établir (1.3.13), on obtient finalement

$$\lambda_h = \hat{\lambda}_h\Big(1 + O\big(h^{1/2}\big)\Big) = \tilde{\lambda}_h\Big(1 + O\big(h^{1/2}\big)\Big) = h\Big(\varrho + O\big(h^{1/2}\big)\Big)\mathrm{e}^{-2W(0)/h}, \tag{1.3.15}$$

soit la description précise de la petite valeur propre non nulle de P_h que l'on attendait.

Lorsque l'Hypothèse "double puits" 1.3.1 n'est plus vérifiée et en particulier lorsque $n_0 \geq 3$, la matrice apparaissant à la place de (1.3.14) n'est plus aussi simple. Établir une formule du type (1.3.15) nécessite alors là encore de travailler avec des restes exponentiellement petits. On y parvient en utilisant un résultat d'algèbre linéaire de [4] dont la preuve repose en partie sur l'utilisation des compléments de Schur.

1.4 Retour à l'équilibre et metastabilité

On va voir dans cette section comment la connaissance précise des petites valeurs propres d'un opérateur semiclassique peut nous permettre de récupérer des informations sur le comportement en temps long des solutions de l'équation associée

$$\begin{cases} \partial_t f + P_h f = 0 \\ f_{|t=0} = f_0. \end{cases} \tag{1.4.1}$$

On considère donc un opérateur P_h satisfaisant les points a) - e) présentés au début de la section 1.3 et pour lequel on dispose d'une asymptotique (disons d'un équivalent) de chacune de ses petites valeurs propres de la forme

$$\lambda_j \sim h\, a_j\, \mathrm{e}^{-2S_j/h}; \qquad 1 \leq j \leq n_0,$$

avec a_j, $S_j > 0$ et pour simplifier la présentation

$$S_{n_0} < \ldots < S_1 = +\infty.$$

Afin d'isoler les difficultés, on va également commencer par supposer que P_h est auto-adjoint. On verra à la fin de la section les principales idées qui permettent de s'affranchir de cette hypothèse (et en particulier de traiter le cas d'opérateurs de Boltzmann linéaires).

Puisque P_h est supposé auto-adjoint, en notant $\mathbb{P}_j$ le projecteur spectral associé à λ_j et

$$\Pi_0 = \sum_{j=1}^{n_0} \mathbb{P}_j,$$

on a grâce aux points c) et d) supposés au début de la section que

$$\mathrm{e}^{-tP_h} = \mathrm{e}^{-tP_h}\Pi_0 + O(\mathrm{e}^{-cht}) \tag{1.4.2}$$

$$= \sum_{j=1}^{n_0} \mathrm{e}^{-t\lambda_j}\mathbb{P}_j + O(\mathrm{e}^{-cht}) \tag{1.4.3}$$

$$= \mathbb{P}_1 + O(\mathrm{e}^{-t\lambda_2}). \tag{1.4.4}$$

On peut déjà conclure quant au retour à l'équilibre des solutions de (1.4.1) grâce à (1.4.4) : pour toute condition initiale f_0, la solution associée vérifie

$$f_t = \mathbb{P}_1 f_0 + O(\mathrm{e}^{-t\lambda_2}\|f_0\|)$$

avec $\mathbb{P}_1 f_0 \in \mathrm{Ker}\ P_h$. La solution f_t converge donc bien vers l'équilibre $\mathbb{P}_1 f_0$ et la vitesse de convergence dépend explicitement de λ_2.
Par ailleurs, (1.4.3) permet également d'établir le caractère metastable des solutions, c'est à dire l'existence d'intervalles de temps pendant lesquels la solution peut sembler avoir atteint une forme d'équilibre et ne plus dépendre du temps. En effet, (1.4.3) se réécrit pour $1 \leq k < n_0$

$$\begin{aligned}
\mathrm{e}^{-tP_h} + O(\mathrm{e}^{-cht}) &= \sum_{j=1}^{k} \mathbb{P}_j + \sum_{j=1}^{k} \left(\mathrm{e}^{-t\lambda_j} - 1\right)\mathbb{P}_j + \sum_{j=k+1}^{n_0} \mathrm{e}^{-t\lambda_j}\mathbb{P}_j \\
&= \sum_{j=1}^{k} \mathbb{P}_j + \sum_{j=1}^{k} \left(\mathrm{e}^{-t\lambda_j} - 1\right)\mathbb{P}_j + O(h^\infty) \quad \text{dès lors que } t \geq |\ln(h^\infty)|\lambda_{k+1}^{-1} \\
&= \sum_{j=1}^{k} \mathbb{P}_j + O(h^\infty) \qquad \text{si de plus } t = O\big(h^\infty \lambda_k^{-1}\big).
\end{aligned}$$

Ainsi, en prenant

$$t_k^- \geq h^{-1}|\ln(h^\infty)|\mathrm{e}^{2S_{k+1}/h}; \qquad t_k^+ = O\big(h^\infty \mathrm{e}^{2S_k/h}\big)$$

(avec la convention $t_1^+ = +\infty$), on obtient que pour $t \in [t_k^-, t_k^+]$,

$$f_t = \sum_{j=1}^{k} \mathbb{P}_j f_0 + O\big(h^\infty \|f_0\|\big).$$

En d'autres termes, on observe l'existence d'échelles de temps au cours desquelles, durant sa convergence vers l'équilibre, la solution de (1.4.1) va essentiellement visiter les états metastables $\sum_{j=1}^{k} \mathbb{P}_j f_0$.

Lorsque P_h n'est pas auto-adjoint, une première adaptation consiste à utiliser un Théorème de Gearhart-Prüss (dans sa version quantitative dûe à Helffer et Sjöstrand [20] afin d'avoir des estimations uniformes en h) pour obtenir (1.4.2) :

Théorème 1.4.1. *Soit $S(t)$ un semigroupe continu, engendré par A et pour lequel*

- $\|S(t)\| \leq \widehat{M}\,\mathrm{e}^{\hat{\omega}t}$
- *Il existe $\omega < \hat{\omega}$ tel que $B(\omega) := \sup_{\operatorname{Re} z > \omega} \|(z-A)^{-1}\| < +\infty$*

Alors

$$\|S(t)\| \leq \widehat{M}\Big(1 + 2\widehat{M}(\hat{\omega}-\omega)B(\omega)\Big)\,\mathrm{e}^{\omega t}.$$

En appliquant ce résultat avec $A = -P_h(1-\Pi_0)$, $\widehat{M} = 1$, $\hat{\omega} = 0$, $\omega = -ch$ et $B(\omega) = Ch^{-1}$, on obtient bien (1.4.2).

La deuxième adaptation réside dans le fait de définir les projecteurs spectraux comme

$$\mathbb{P}_j = \frac{1}{2i\pi}\int_{\partial D_j} (z - P_h|_{\operatorname{Ran}\Pi_0})^{-1}\mathrm{d}z$$

où D_j est le disque de centre $h\,a_j\,\mathrm{e}^{-2S_j/h}$ et de rayon $h^{3/2}\mathrm{e}^{-2S_j/h}$. On peut alors utiliser les estimations de résolvante fournies par le Théorème 4 de [4] pour montrer que ces derniers sont bien définis et $O(1)$.

1.5 Un modèle jouet mettant en défaut les quasimodes gaussiens usuels

Dans cette section, on va exhiber un opérateur semiclassique à première vue proche de ceux qui ont déjà été étudiés ou de ceux que l'on pourrait vouloir étudier mais pour lequel on va constater que la méthode des QG usuels de la section 1.3 ne fonctionne pas. En utilisant les notations (1.1.10), (1.1.12)

$$b_h = h\partial_v + \frac{v}{2} \qquad \text{et} \qquad H_0 = b_h^* b_h \tag{1.5.1}$$

il est défini par

$$P_h = X_0^h + H_0 + b_h^* \circ (\Pi_h \otimes \mathrm{Id}) \circ b_h \tag{1.5.2}$$

agissant sur $L^2(\mathbb{R}^d_x \times \mathbb{R}^d_v) = L^2(\mathbb{R}^{2d})$. Avec les notations de l'Appendice A, cet opérateur peut être interprété comme une perturbation de l'opérateur de Fokker-Planck $X_0^h + H_0$ par un terme non local et dans la classe $S^{1/2}(1)$. Cette perturbation préserve l'accrétivité de l'opérateur ainsi que son noyau. On peut également voir l'opérateur P_h comme un approximation (grossière) de l'opérateur de Boltzmann de relaxation linéaire. En effet, d'après l'écriture intégrale donnée à la Proposition 4.2.1 et le Lemme 7.6.1, le terme “de collisions” $H_0 + b_h^*(\Pi_h \otimes \mathrm{Id})b_h$ correspond aux constantes près à l'approximation par une somme de Riemann à deux termes de l'opérateur de relaxation linéaire défini en (1.1.11).

Pour étudier cet opérateur, en supposant pour simplifier que l'Hypothèse “double puits” 1.3.1 est vérifiée, l'approche des “QG usuels” de la section 1.3 consisterait à se donner une forme linéaire convenable (cf (1.3.9)) $\ell(x,v) = \ell_x \cdot x + \ell_v \cdot v$ où ℓ_x, $\ell_v \in \mathbb{R}^{2d}$ et à introduire le quasimode gaussien essentiellement doné par

$$f_\ell(x,v) = \theta(x,v)\mathrm{e}^{-W(x,v)/h} \tag{1.5.3}$$

où

$$\theta(x,v) = \frac{1}{\sqrt{h}}\int_0^{\ell(x,v)} \mathrm{e}^{-\frac{s^2}{2h}}\mathrm{d}s$$

et

$$W(x,v) = \frac{V(x)}{2} + \frac{v^2}{4}. \tag{1.5.4}$$

Il s'agit ensuite de calculer $P_h f_\ell$.

Pour les deux premiers termes, en remarquant que

$$X_0^h(\mathrm{e}^{-W/h}) = H_0(\mathrm{e}^{-W/h}) = 0$$

et en utilisant

$$\nabla\theta = \frac{1}{\sqrt{h}}\, \mathrm{e}^{-\frac{\ell^2/2}{h}}\nabla\ell$$

ainsi que

$$b_h f_\ell = h\,\partial_v\theta\,\mathrm{e}^{-W/h}, \tag{1.5.5}$$

on peut assez facilement voir (cf par exemple [4, Lemme 3.1 et la discussion qui s'en suit]) que

$$(X_0^h + H_0)f_\ell(x,v) = \sqrt{h}\left[\begin{pmatrix}\ell_v^2\,\mathrm{Id} & -\mathrm{Hess}_0 V\\ \mathrm{Id} & (1+\ell_v^2)\,\mathrm{Id}\end{pmatrix}\begin{pmatrix}\ell_x\\ \ell_v\end{pmatrix}\cdot\begin{pmatrix}x\\ v\end{pmatrix} + O\Big((x,v)^2\Big)\right]\mathrm{e}^{-\widetilde{W}_\ell(x,v)/h} \tag{1.5.6}$$

avec

$$\widetilde{W}_\ell = W + \frac{1}{2}\ell(x)^2.$$

Pour le dernier terme, en se rappelant que Π_h agit trivialement en x et a pour noyau distributionnel

$$(2\pi h)^{-d/2}\mathrm{e}^{-\frac{v^2+w^2}{4h}},$$

on obtient à l'aide de

$$b_h^*\Big(a(x)\mathrm{e}^{-v^2/4h}\Big) = a(x)\cdot v\,\mathrm{e}^{-v^2/4h},$$

et de (1.5.5) que

$$\begin{aligned} b_h^*(\Pi_h\otimes\mathrm{Id})b_h f_\ell(x,v) &= \sqrt{h}\,b_h^*\Big(\Pi_h\big(\mathrm{e}^{-\widetilde{W}_\ell/h}\big)\ell_v\Big)\\ &= \sqrt{h}(2\pi h)^{-d/2}\mathrm{e}^{-\frac{W+(\ell_x\cdot x)^2/2}{h}}\ell_v\cdot v\\ &\qquad\times\int_{w\in\mathbb{R}^d}\exp\Big[\frac{-1}{h}\Big(\frac{w^2}{2}+\frac{(\ell_v\cdot w)^2}{2}+\ell_x\cdot x\,\ell_v\cdot w\Big)\Big]\mathrm{d}w. \end{aligned} \tag{1.5.7}$$

En complétant le carré en w dans l'intégrale (1.5.7), on obtient (voir le Lemme 7.2.4 pour plus de détails)

$$b_h^*(\Pi_h\otimes\mathrm{Id})b_h f_\ell(x,v) = \sqrt{h}\,\frac{\ell_v}{(1+\ell_v^2)^{1/2}}\cdot v\,\mathrm{e}^{-\frac{\widetilde{W}_{L_1}(x,v)}{h}} \tag{1.5.8}$$

où

$$L_1(x) = (1+\ell_v^2)^{-1/2}\ell_x\cdot x. \tag{1.5.9}$$

On constate que l'action de l'opérateur Π_h a entrainé la suppression de la partie en v dans le terme $\ell^2(x,v)$ que l'on était parvenu à faire apparaitre dans l'exponentielle en appliquant b_h à f_ℓ. Ainsi, contrairement aux cas présentés dans la méthode des QG usuels de la section 1.3, la phase apparaissant dans le dernier terme de $P_h f_\ell$ n'est pas $\widetilde{W}_\ell$ mais $\widetilde{W}_{L_1}$, i.e celle associée à la forme linéaire L_1. Cela empêche d'obtenir les compensations attendues entre $(X_0^h + H_0)f_\ell$ et $b_h^*(\Pi_h\otimes\mathrm{Id})b_h f_\ell$.

Pour espérer parvenir à compenser ce dernier terme, l'idée (développée dans la section suivante) est alors d'ajouter à f_ℓ le quasimode gaussien f_{L_1} pour faire apparaitre, grâce à l'action de $X_0^h + H_0$, un nouveau terme en $\exp(-\widetilde{W}_{L_1}/h)$. Par ailleurs, puisque L_1 ne dépend pas de v, on a $f_{L_1}\in\mathrm{Ker}\,b_h$ et par conséquent l'action de l'opérateur $b_h^*(\Pi_h\otimes\mathrm{Id})b_h$ sur f_{L_1} ne va pas produire de nouveau terme.

1.6 Superposition de QG

On poursuit ici l'étude de l'opérateur "jouet" (1.5.2), toujours avec l'Hypothèse "double puits" 1.3.1. On vient de voir que la méthode des QG usuels ne peut fonctionner pour étudier cet opérateur. Comme indiqué dans la section précédente, on va alors chercher à déterminer si l'implémentation d'un quasimode de la forme

$$f = \alpha f_\ell + \beta f_{L_1} \tag{1.6.1}$$

permet de surmonter les difficultés précédemment rencontrées ; avec α, $\beta > 0$ à determiner, $\ell = (\ell_x, \ell_v) \in \mathbb{R}^{2d}$ une forme linéaire convenable au sens de (1.3.9), L_1 définie en (1.5.9) et les fonctions f_ℓ, f_{L_1} introduites en (1.5.3). Le calcul de $P_h f$ découle directement de (1.5.6) et (1.5.8) :

$$P_h f(x,v) = \sqrt{h}\left[\alpha \begin{pmatrix} \ell_v^2 \,\mathrm{Id} & -\mathrm{Hess}_0 V \\ \mathrm{Id} & (1+\ell_v^2)\,\mathrm{Id} \end{pmatrix} \begin{pmatrix} \ell_x \\ \ell_v \end{pmatrix} \cdot \begin{pmatrix} x \\ v \end{pmatrix} + O\Big((x,v)^2\Big)\right] \mathrm{e}^{-\frac{\widetilde{W}_\ell(x,v)}{h}} + \sqrt{h}\, \frac{\alpha\, \ell_v + \beta\, \ell_x}{(1+\ell_v^2)^{1/2}} \cdot v\, \mathrm{e}^{-\frac{\widetilde{W}_{L_1}(x,v)}{h}}. \tag{1.6.2}$$

Ne pouvant pas obtenir de compensations entre les différentes exponentielles, on cherche alors dans l'optique d'appliquer une méthode de Laplace (voir Appendice B) à annuler (au moins au permier ordre en (x,v)) les préfacteurs de chacune. Pour la première, en notant τ la valeur propre négative de $\mathrm{Hess}_0 V$, cela impose de prendre

$$\ell_x = -(1+\ell_v^2)\ell_v$$

et ℓ_v comme un vecteur propre de $\mathrm{Hess}_0 V$ associé à τ satisfaisant

$$(1+\ell_v^2)\ell_v^2 = |\tau|.$$

Cela détermine (là encore au signe près) la forme linéaire de départ ℓ qui s'avère effectivement être convenable (c'est en fait celle qui apparait lors de l'étude à l'aide des QG usuels de l'équation de Fokker-Planck, cf [4]). En injectant ces nouvelles informations dans le deuxième préfacteur de (1.6.2), la condition pour l'annuler devient

$$\beta = \frac{\alpha}{1+\ell_v^2}.$$

En implémentant tous ces choix dans (1.6.1), on a donc réussi à construire un quasimode f obtenu en superposant les QG usuels f_ℓ et f_{L_1} et pour lequel on a

$$P_h f = \sqrt{h}\, a(x,v)\mathrm{e}^{-\frac{\widetilde{W}_\ell(x,v)}{h}} + \sqrt{h}\, b(x,v)\mathrm{e}^{-\frac{\widetilde{W}_{L_1}(x,v)}{h}}$$

avec

$$a(x,v) = O\Big((x,v)^2\Big), \quad b(x,v) = O\Big((x,v)^2\Big),$$

ce qui permet alors bien d'obtenir l'analogue de (1.3.10) (ici on a même $b = 0$, mais de façon générale, on pourrait avoir un préfacteur b non identiquement nul).

Les obstacles rencontrés dans la section 1.5 pour la mise en place de la méthode des QG usuels donnent un très bon aperçu des phénomènes se produisant lorsque l'on essaie d'appliquer cette méthode à l'étude de l'équation de Boltzmann de relaxation linéaire. De même, la résolution présentée dans cette section via une superposition de QG fournit les idées de base qui permettront de traiter le cas de l'opérateur de relaxation linéaire de Boltzmann aux chapitres 4 et 7.

Première partie

Présentation des résultats obtenus

Chapitre 2

Metastabilité pour une classe d'équations de Boltzmann linéaires type "relaxation douce"

2.1 Contexte et résultats principaux

On s'intéresse dans ce chapitre aux résultats établis dans [36] sur l'équation de Boltzmann linéaire

$$\begin{cases} h\partial_t f + v \cdot h\partial_x f - \partial_x V \cdot h\partial_v f + Q_h(f) = 0 \\ f_{|t=0} = f_0 \end{cases} \tag{2.1.1}$$

et plus précisément sur l'opérateur associé

$$\begin{aligned} P_h &= v \cdot h\partial_x - \partial_x V \cdot h\partial_v + Q_h \\ &= X_0^h + Q_h \end{aligned} \tag{2.1.2}$$

agissant sur $L^2(\mathbb{R}^{2d})$.

L'objectif de ce travail est double. Dans un premier temps, on cherche à établir un résultat similaire à celui obtenu par Robbe dans [39] mais pour une large classe d'opérateurs de collision. Le deuxième but est de fournir une asymptotique complète des petites valeurs propres de P_h comme cela a pu être fait dans [18] pour le Laplacien de Witten ou dans [22, 23] avec de récentes améliorations par Bony et al. dans [4] pour le cas d'opérateurs différentiels de type Fokker-Planck. On parvient à montrer de tels résultats pour notre équation (2.1.1) dans le cadre d'une classe d'opérateurs de collisions pseudo-différentiels présentant de bonnes propriétés symboliques ainsi qu'une structure factorisée.

Plus précisément, avec les notions de l'Appendice A, on considère ici des opérateurs de collisions Q_h satisfaisant l'hypothèse suivante.

Hypothèse 2.1.1. *Il existe $\tau > 0$ et une matrice symétrique de symboles analytiques*

$$M^h(x, v, \eta) = \big(m_{p,q}(x, v, \eta)\big)_{1\le p,q\le d} \in \mathcal{M}_d\big(S^0_\tau(\langle(v, \eta)\rangle^{-2})\big)$$

qui envoie $\mathbb{R}^{3d}$ dans $\mathcal{M}_d(\mathbb{R})$ et telle que, avec la notation (1.5.1), l'opérateur de collisions Q_h satisfait

a) $Q_h = b_h^* \circ Op_h(M^h) \circ b_h$

b) $M^h \sim \sum_{n\ge 0} h^n M_n$ dans $\mathcal{M}_d\big(S^0_\tau(\langle(v, \eta)\rangle^{-2})\big)$

c) Pour tout $(x, v, \eta) \in \mathbb{R}^{3d}$, $M^h(x, v, \eta) = M^h(x, v, -\eta)$

d) Pour tout $(x, v, \eta) \in \mathbb{R}^{3d}$, $M_0(x, v, \eta) \ge \frac{1}{C}\langle(v, \eta)\rangle^{-2}\,\mathrm{Id}$.

On peut montrer (voir Lemme 5.1.3) que cette hypothèse couvre notamment le cas d'opérateurs de collisions donnés par certaines fonctions de l'oscillateur harmonique en vitesse H_0. Un exemple est fourni par l'opérateur de collisions dit *de relaxation douce*, donné avec la notation (1.5.1) par

$$Q_h = H_0(1 + H_0)^{-1} \tag{2.1.3}$$

et étudié par Robbe dans [38].

On se place également sous des hypothèses de confinement pour le potentiel V, assurant entre autres que le bas du spectre du Laplacien de Witten associé est discret.

Hypothèse 2.1.2. *Le potentiel V est une fonction lisse de Morse dépendant uniquement de la variable spatiale $x \in \mathbb{R}^d$, à valeurs dans $\mathbb{R}$, bornée inférieurement et telle que*

$$|\partial_x V(x)| \geq \frac{1}{C} \qquad \text{pour } |x| > C.$$

De plus, pour tout $\alpha \in \mathbb{N}^d$ avec $|\alpha| \geq 2$, il existe C_α tel que

$$|\partial_x^\alpha V| \leq C_\alpha.$$

En particulier, pour tout $0 \leq k \leq d$, l'ensemble des points critiques d'indice k de V qu'on note $\mathcal{U}^{(k)}$ est fini et on note également

$$n_0 = \#\mathcal{U}^{(0)}. \tag{2.1.4}$$

Enfin, on suppose que $n_0 \geq 2$.

Cette hypothèse implique notamment (cf [30], Lemma 3.14) que $\mathrm{e}^{-V/2h} \in L^2(\mathbb{R}_x^d)$. On peut alors facilement vérifier que

$$\mathrm{e}^{-W/h} \in \operatorname{Ker} P_h$$

avec W le potentiel global associé à (2.1.1) et défini en (1.5.4).

Pour un opérateur tel que P_h, qui n'est par exemple pas auto-adjoint à résolvante compacte, on ne dispose a priori pas d'informations sur son spectre (à part dans notre cas que ce dernier est contenu dans $\{z \in \mathbb{C}\,;\, \operatorname{Re} z \geq 0\}$). On commence donc par établir une première description du spectre près de 0 de P_h qui, de façon similaire à d'autres opérateurs non auto-adjoints étudiés dans [22, 39], se trouve en particulier être discret :

Théorème 2.1.3. *Supposons que les Hypothèses 2.1.1 et 2.1.2 sont satisfaites et rappelons la notation (2.1.4). Alors l'opérateur P_h (muni d'un domaine convenable) admet 0 comme valeur propre simple. De plus, il existe $c > 0$ et $h_0 > 0$ tels que pour tout $0 < h \leq h_0$, $\operatorname{Spec}(P_h) \cap \{\operatorname{Re} z \leq ch^2\}$ est constitué d'exactement n_0 valeurs propres (comptées avec multiplicité algébrique) qui sont exponentiellement petites par rapport à $1/h$ et pour tout $0 < \tilde{c} \leq c$, l'estimée de résolvante*

$$(P_h - z)^{-1} = O(h^{-2})$$

est vérifiée uniformément sur $\{\operatorname{Re} z \leq ch^2\} \backslash B(0, \tilde{c}h^2)$. Enfin, 0 mis à part, les parties réelles de ces petites valeurs propres sont strictement positives.

Ce résultat peut se voir comme une généralisation du Théorème 3.0.2 de [38] qui traitait le cas de la relaxation douce, c'est à dire le cas où Q_h est donné par (2.1.3).

Afin d'étudier le comportement en temps long des solutions de (2.1.1), on a besoin d'une description précise du spectre près de 0 de P_h, dans l'esprit de celles données dans [4, 18, 22] pour des opérateurs différentiels. On va réussir à en donner une en appliquant la méthode des QG usuels présentée à la section 1.3 qui fournira une asymptotique précise des petites valeurs propres de P_h. Cela constitue le résultat principal de [36] et fait l'objet du Théorème 2.1.5. Dans ce Théorème, afin de simplifier grandement l'énoncé et les notations, on choisit de se placer sous l'Hypothèse de non dégénérescence C.0.8 dont on verra aux chapitres 3 et 6 qu'on pourrait se passer. Cette dernière implique en particulier que V a un unique minimum global qu'on note $\underline{\mathbf{m}}$.
Dans le but d'énoncer le Théorème 2.1.5, on commence par rappeler que dans l'esprit de [6, 18, 23], les applications $\mathbf{j}$ et S de la Définition C.0.7 associent à chaque minimum respectivement les points d'échappement et la hauteur du puits qui lui correspond. On donne ensuite le Lemme suivant qui apparaitra comme un co-produit de la méthode des QG usuels qu'on va mettre en place.

Lemme 2.1.4. *On rappelle la matrice M_0 de l'Hypothèse 2.1.1. Soient $\mathbf{m} \in \mathcal{U}^{(0)} \backslash \{\underline{\mathbf{m}}\}$ ainsi que $\mathbf{s} \in \mathbf{j}(\mathbf{m})$ où $\mathbf{j}$ est l'application de la Définition C.0.7. La matrice*

$$\Phi^{\mathbf{s}} = \begin{pmatrix} 0 & -\mathrm{Hess}_{\mathbf{s}} V \\ \mathrm{Id} & M_0(\mathbf{s}, 0, 0) \end{pmatrix}$$

a uniquement une valeur propre dans $\{\mathrm{Re}\, z < 0\}$ qui est en fait réelle et qu'on note $-\alpha_0^{\mathbf{s}}$.

D'après le Théorème 2.1.3, on peut associer à chaque $\mathbf{m} \in \mathcal{U}^{(0)} \backslash \{\underline{\mathbf{m}}\}$ une valeur propre non nulle et exponentiellement petite de P_h qu'on appelle $\lambda(\mathbf{m}, h)$.

Théorème 2.1.5. *Supposons que les Hypothèses 2.1.1, 2.1.2 et C.0.8 sont satisfaites et rappelons la notation $\alpha_0^{\mathbf{s}}$ du Lemme 2.1.4. Les valeurs propres exponentiellement petites de P_h vérifient la formule suivante :*

$$\lambda(\mathbf{m}, h) = h e^{-2\frac{S(\mathbf{m})}{h}} \frac{\det(\mathrm{Hess}_{\mathbf{m}} V)^{1/2}}{2\pi} B_h(\mathbf{m})$$

où $B_h(\mathbf{m})$ admet un développement classique dont le premier terme est

$$\sum_{\mathbf{s} \in \mathbf{j}(\mathbf{m})} |\det(\mathrm{Hess}_{\mathbf{s}} V)|^{-1/2} \alpha_0^{\mathbf{s}}$$

et les applications S et $\mathbf{j}$ proviennent de la Définition C.0.7.

Grâce à la localisation précise obtenue au Théorème 2.1.5, on est en mesure de donner des informations quant au retour à l'équilibre et à la metastabilité des solutions de (2.1.1), dans l'esprit de [4]. Plus précisément, on donne une vitesse précise de convergence du semigroupe $e^{-tP_h/h}$ vers $\mathbb{P}_1$, le projecteur orthogonal sur Ker P_h : en appelant λ^* une des valeurs propres non nulles de P_h dont la partie réelle est minimale, on établit que la vitesse de convergence est essentiellement donnée par $\exp(-t\, \mathrm{Re}\, \lambda^*/h)$:

Corollaire 2.1.6. *Sous les hypothèses du Théorème 2.1.5, pour tout $N \geq 1$, il existe $C_N > 0$ et $h_0 > 0$ tels que pour tout $0 < h \leq h_0$ et $t \geq 0$,*

$$\|e^{-tP_h/h} - \mathbb{P}_1\| \leq C_N e^{-t\, \mathrm{Re}\, \lambda^*(1 - C_N h^N)/h}.$$

De plus, si λ^* ne partage son développement asymptotique fourni par le Théorème 2.1.5 avec aucune autre valeur propre de P_h (en particulier c'est une valeur propre simple), alors λ^* est réelle et on a même

$$\|e^{-tP_h/h} - \mathbb{P}_1\| \leq C e^{-t\lambda^*/h}.$$

Corollaire 2.1.7. *Supposons que les hypothèses du Théorème 2.1.5 sont satisfaites. Considérons des minima locaux $\mathbf{m}_1 = \underline{\mathbf{m}}$, $\mathbf{m}_2$, ..., $\mathbf{m}_K$ tels que*

$$S(\mathcal{U}^{(0)}) = \{+\infty = S(\mathbf{m}_1) > S(\mathbf{m}_2) > \cdots > S(\mathbf{m}_K)\}$$

pour l'application S de la Définition C.0.7. Pour $2 \leq k \leq K$, notons $\mathbb{P}_k$ le projecteur spectral (qui n'est pas nécessairement orthogonal) associé aux valeurs propres qui sont $O\big(e^{-2\frac{S(\mathbf{m}_k)}{h}}\big)$. Alors pour tout temps $(t_k^{\pm})_{1 \leq k \leq K}$ satisfaisant

$$t_K^- \geq h^{-1} |\ln(h^\infty)| \quad \text{et} \quad t_k^- \geq |\ln(h^\infty)| e^{2\frac{S(\mathbf{m}_{k+1})}{h}} \quad \text{pour} \quad k = 1, \ldots, K-1$$

ainsi que

$$t_1^+ = +\infty \qquad \text{et} \quad t_k^+ = O\Big(h^\infty e^{2\frac{S(\mathbf{m}_k)}{h}}\Big) \qquad \text{pour} \quad k = 2, \ldots, K$$

on a

$$e^{-tP_h/h} = \mathbb{P}_k + O(h^\infty) \qquad \text{sur } [t_k^-, t_k^+].$$

En d'autres termes, on a montré l'existence d'échelles de temps durant lesquelles, au cours de sa convergence vers l'équilibre global, la solution de (2.1.1) va essentiellement visiter les états metastables associés aux petites valeurs propres de P_h.

2.2 Estimations hypocoercives et première description du spectre près de 0

Le but de cette section est d'expliquer la démonstration du Théorème 2.1.3. Pour ce faire, on va s'inspirer de la preuve de Robbe dans [39] où un résultat analogue est démontré dans le cas de l'équation de Boltzmann de relaxation linéaire. La différence principale provient du fait que les opérateurs de collisions que l'on considère sont $O(1)$ alors que celui étudié dans [39] est de norme précisément égale à h. Cela explique notamment la présence dans le Théorème 2.1.3 des h^2 et h^{-2} qui ne sont sûrement pas optimaux et devraient pouvoir être remplacés par des h et h^{-1} (on sait en tout cas que c'est le cas pour la relaxation douce, c'est à dire lorsque Q_h est donné par (2.1.3), cf [38] Théorème 3.0.2).

L'opérateur P_h n'étant pas auto-adjoint, on va avoir besoin pour récupérer des informations sur son spectre près de 0 de considérer le projecteur spectral

$$\Pi_0 = \frac{1}{2i\pi}\int_{|z|=ch^2}(z-P_h)^{-1}\mathrm{d}z.$$

Afin de montrer que ce dernier est bien défini et borné uniformément en h, on va avoir besoin d'estimations de résolvante. On est alors confronté à une nouvelle difficulté : P_h n'est par elliptique. Pour contourner celle-ci, on va employer des méthodes hypocoercives dans l'esprit de [13, 21, 45] ou par la suite [7, 39]. On aimerait disposer d'une minoration de

$$u \mapsto \frac{\operatorname{Re}\langle P_h u, u\rangle}{\|u\|^2}.$$

L'opérateur X_0^h étant anti-adjoint, on est donc amené à s'intéresser à l'opérateur Q_h, la partie auto-adjointe de P_h. Ce dernier a le mauvais goût de s'annuler sur le sous-espace de dimension infinie $\mu_h L^2(\mathbb{R}^d_x)$, où μ_h est définie en (1.1.5). Cependant, grâce à l'Hypothèse 2.1.1 on montre tout de même en utilisant le calcul symbolique qu'avec la notation (1.1.10),

$$Q_h \geq \frac{h}{C}(1-\Pi_h).$$

Il nous manque donc seulement un gain en Π_h. On va voir que l'on peut obtenir un tel gain, non pas exactement pour l'opérateur P_h ni sur $L^2(\mathbb{R}^{2d})$ tout entier, mais pour une petite perturbation de P_h et sur un sous-espace de codimension finie. Cet espace est défini comme l'orthogonal de quasimodes pour P_h construits à partir de quasimodes connus pour le Laplacien de Witten associé au potentiel V, dans l'esprit de [18]. Plus précisément, on sait qu'il existe une famille orthonormale de fonctions $(\varphi_j)_{1\leq j\leq n_0} \subset \mathcal{C}_c^\infty(\mathbb{R}^d_x)$ de la forme

$$\varphi_j = \chi_j \mathrm{e}^{-\frac{V-V(x_j)}{2h}}$$

où x_j est un minimum local de V et χ_j est une fonction plateau localisant autour de x_j ; et telles que pour toute fonction $w \in (\varphi_j)_j^\perp$ et avec la notation (1.2.2),

$$\langle \Delta_V w, w\rangle \geq \frac{h\|w\|^2}{C}. \tag{2.2.1}$$

On définit alors les quasimodes annoncés pour P_h en accolant à φ_j la Maxwellienne en vitesse :

$$g_j = \varphi_j \mu_h$$

et on vérifie facilement qu'il s'agit bien de quasimodes pour P_h au sens où

$$P_h g_j = O(\mathrm{e}^{-c/h}).$$

La perturbation de P_h qui nous fournira le gain en Π_h attendu est donnée par $N_{h,\varepsilon}P_h$ où

$$N_{h,\varepsilon} = \mathrm{Id} + \varepsilon h(L+L^*)$$

et L est un opérateur auxiliaire, borné et choisi de sorte que le terme de la forme

$$\langle [L, X_0^h]u, u\rangle$$

fourni par le calcul de

$$\langle N_{h,\varepsilon} P_h u, u\rangle$$

fasse apparaitre le Laplacien de Witten Δ_V. C'est grâce à ce dernier que l'on récupère le gain en Π_h annoncé sur $(g_j)^{\perp}_{1\leq j\leq n_0}$, en utilisant (2.2.1) après avoir remarqué que $\Pi_h u \in \mu_h(\varphi_j)^{\perp}_j$ dès lors que $u \in (g_j)^{\perp}_j$. On obtient alors le résultat central de cette section.

Proposition 2.2.1. *Il existe $\varepsilon > 0$ et $h_0 > 0$ tels que pour tout $h \in]0, h_0]$ et $u \in \mathcal{S}(\mathbb{R}^{2d}) \cap (g_j)^{\perp}_{1\leq j\leq n_0}$, on a*

$$\mathrm{Re}\langle N_{h,\varepsilon} P_h u, u\rangle \geq \frac{h^2}{C}\|u\|^2.$$

Ici contrairement à (2.2.1), le gain obtenu est en h^2 et non en h car il provient de la perturbation qui est elle-même d'ordre h. Grâce à cette Proposition, on parvient sans trop de difficulté à montrer les estimations de résolvantes annoncées au Théorème 2.1.3. On peut alors considérer le projecteur spectral Π_0 et il suffit essentiellement pour achever la preuve du Théorème 2.1.3 de montrer que ce dernier est de rang n_0.

2.3 QG pour l'opérateur P_h

Pour simplifier la présentation, on va de nouveau se placer sous l'Hypothèse “double puits” 1.3.1. C'est évidemment un cadre beaucoup plus restrictif que celui traité dans [36] où on suppose simplement que l'Hypothèse de non dégénérescence C.0.8 est vérifiée, mais il permet déjà de rendre compte des principales nouveautés et difficultés.

On va maintenant tâcher de construire des QG pour l'opérateur P_h défini en (2.1.2) et pour le potentiel W défini en (1.5.4). Dans l'esprit de [4], on va suivre la version raffinée de la démarche présentée dans la section 1.3 qui consiste à prendre pour ℓ non pas une forme linéaire mais une fonction lisse admettant un développement classique $\ell^h \sim \sum_n h^n \ell_n$ que l'on notera parfois encore simplement ℓ.

On calcule sans difficulté l'action de X_0^h sur un QG ; en réutilisant les notations (1.3.5) et (1.3.8) où la variable x est remplacée par les variables (x, v), celle-ci est essentiellement donnée par

$$X_0^h f_{\mathbf{m},h} = \sqrt{h}\, H_W \cdot \nabla\ell\, \mathrm{e}^{-\widetilde{W}_\ell/h} \tag{2.3.1}$$

où

$$H_W = \begin{pmatrix} v \\ -\partial_x V \end{pmatrix}.$$

La véritable difficulté, qui représente une des nouveautés principales de [36] est de calculer l'action de l'opérateur non local Q_h sur notre QG. C'est notamment ici que l'on va utiliser la factorisation de Q_h fournie par l'Hypothèse 2.1.1. En notant

$$g^h \text{ le symbole de } b_h^* \circ \mathrm{Op}_h(M^h),$$

celle-ci nous permet d'écrire

$$\begin{aligned} Q_h f_{\mathbf{m},h} &= \mathrm{Op}_h(g^h)\Big((h\partial_v + v/2) f_{\mathbf{m},h}\Big) \\ &= h\,\mathrm{Op}_h(g^h)\Big(\partial_v\theta\, \mathrm{e}^{-W/h}\Big) \\ &= \sqrt{h}\,\mathrm{Op}_h(g^h)\Big(\mathrm{e}^{-\widetilde{W}_\ell/h}\partial_v\ell\Big) \end{aligned}$$

et on peut alors utiliser l'analyticité du symbole g^h garantie par cette même hypothèse pour obtenir par une déformation de contour d'intégration une formule de la forme

$$Q_h f_{\mathbf{m},h}(x, v) = \sqrt{h}\, I^h(x, v)\, \mathrm{e}^{-\widetilde{W}_\ell(x,v)/h} \tag{2.3.2}$$

où I^h est donnée par une intégrale oscillante faisant intervenir g^h et $\partial_v\ell$. En mettant (2.3.1) et (2.3.2) ensemble, on a au final, à des termes d'erreur près

$$P_h f_{\mathbf{m},h}(x, v) = \sqrt{h}\, \omega^h(x, v)\, \mathrm{e}^{-\widetilde{W}_\ell(x,v)/h}. \tag{2.3.3}$$

Ainsi, de façon analogue à (1.3.7), on obtient bien une formule dans laquelle apparait uniquement l'exponentielle de la phase $\widetilde{W}_\ell$. Ici, le préfacteur ω^h dépend de la fonction ℓ et grâce au développement asymptotique de cette dernière ainsi qu'à celui du symbole M^h garanti par l'Hypothèse 2.1.1, on montre que ω^h admet également un développement classique $\omega^h \sim \sum_j h^j \omega_j$ dont on parvient à calculer les coefficients :

$$\omega_0 = H_W \cdot \nabla \ell_0 + M_0\Big(x, v, i\big(\frac{v}{2} + \ell_0\, \partial_v \ell_0\big)\Big)\big(v + \ell_0 \partial_v \ell_0\big) \cdot \partial_v \ell_0 \tag{2.3.4}$$

et pour $j \geq 1$,

$$\omega_j = T(x,v) \cdot \nabla \ell_j + F(x,v)\ell_j + R_j(\ell_0, \dots, \ell_{j-1}) \tag{2.3.5}$$

où T et F sont des fonctions lisses à valeurs respectivement dans $\mathbb{R}^{2d}$ et $\mathbb{R}$ faisant intervenir V, ℓ_0 et la matrice M_0, tandis que R_j est une fonction lisse de $\mathbb{R}^j$ dans $\mathbb{C}$. Notre but est alors de choisir la fonction ℓ (ou plutôt les coefficients de son développement classique) de façon à annuler tous les ω_j. Là encore, on est confronté à une difficulté qui n'apparaissait pas dans [4] ; puisqu'on veut une fonction ℓ à valeurs réelles, il s'agit de montrer que les équations sur les ℓ_j fournies par (2.3.4) et (2.3.5) sont réelles, ce qui nécessite notamment d'expliciter la fonction R_j. On y parvient à grands coups de calculs et en utilisant l'analycité combinée à la parité du symbole M^h.

2.4 Résolution des équations sur le préfacteur ω^h

Au vu de (2.3.4), on cherche une fonction ℓ_0 satisfaisant l'équation dite *eikonale* suivante

$$H_W \cdot \nabla \ell_0 + M_0\Big(x, v, i\big(\frac{v}{2} + \ell_0\, \partial_v \ell_0\big)\Big)\big(v + \ell_0 \partial_v \ell_0\big) \cdot \partial_v \ell_0 = 0. \tag{2.4.1}$$

On part de l'observation de [4] : en notant $\tilde{p}$ la complexification du symbole principal de P_h donnée par

$$\tilde{p}(x, v, \xi, \eta) = \xi \cdot v - \eta \cdot \partial_x V + (\eta^t + v^t/2) M_0(x, v, i\eta)(\eta - v/2),$$

et telle que

$$\tilde{p}\big(x, v, \nabla W\big) = 0,$$

on a au voisinage du point selle 0

$$\ell_0 \text{ est solution de (2.4.1)} \iff \tilde{p}\Big(x, v, \nabla\big(W + \ell_0^2/2\big)\Big) = 0.$$

Cette réécriture permet d'utiliser les méthodes de géométrie symplectique présentées dans [12, 22] pour trouver une phase ϕ telle que

$$\tilde{p}\Big(x, v, \nabla\phi(x,v)\Big) = 0 \qquad \text{et} \qquad \mathrm{Hess}_{(0,0)}\phi > 0. \tag{2.4.2}$$

On vérifie ensuite que la solution de (2.4.1)

$$\ell_0 = \sqrt{2(\phi - W)}$$

est bien définie et lisse. Par ailleurs, on a par (2.4.2) et la définition de ℓ_0 que la fonction ℓ admettant ℓ_0 comme premier terme est convenable au sens de (1.3.9).

Pour les $(\ell_j)_{j\geq 1}$, au vu de (2.3.5), on procède par récurrence sur $j \geq 1$. En supposant $\ell_0, \dots, \ell_{j-1}$ connues, on cherche ℓ_j satisfaisant l'équation de transport

$$T(x,v) \cdot \nabla \ell_j + F(x,v)\ell_j + R_j(\ell_0, \dots, \ell_{j-1}) = 0. \tag{2.4.3}$$

On peut là aussi procéder comme dans [4], c'est-à-dire utiliser des séries formelles pour trouver une solution approchée de (2.4.3) avant de la raffiner en une véritable solution (voir aussi [12], chapitre 3). Pour ce faire, l'étape principale consiste à montrer l'inversibilité sur les polynômes homogènes de degré fixé de l'approximation de l'opérateur de transport de (2.4.3) que l'on obtient en remplaçant les coefficients de (2.4.3) par le premier terme non nul de leur développement de Taylor, c'est à dire de l'opérateur

$$\mathcal{L}_0 = D_{(0,0)}T\begin{pmatrix} x \\ v \end{pmatrix} \cdot \nabla + \alpha_0 \quad \in \bigcap_{n\geq 1} \mathscr{L}(\mathcal{P}^n_{hom})$$

où

$$\alpha_0 = F(0,0) = M_0(0,0,0)\nu_2 \cdot \nu_2$$

avec

$$\nu = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} \partial_x \ell_0(0,0) \\ \partial_v \ell_0(0,0) \end{pmatrix}.$$

Grâce à un Lemme de [4], on y parvient simplement en montrant que le spectre de $D_{(0,0)}T + \alpha_0$ est inclus dans $\{\mathrm{Re}\, z > 0\}$. C'est l'étude de ce spectre qui permet au passage de montrer le Lemme 2.1.4.

A l'issue de cette section, on dispose de fonctions $(\ell_j)_{j\geq 0}$ satisfaisant (2.4.1) et (2.4.3). On peut alors à l'aide d'une procédure de Borel construire une fonction $\ell^h \sim \sum_{j\geq 0} h^j \ell_j$ convenable au sens de (1.3.9) et telle que le préfacteur dans (2.3.3) satisfait

$$\omega^h = O(h^\infty).$$

Avec ce choix de fonction ℓ^h, on a par la méthode de Laplace (voir Appendice B)

$$\|P_h f_{\mathbf{m},h}\|^2 = O\big(h^\infty \mathrm{e}^{-2W(0)/h}\big)\|f_{\mathbf{m},h}\|^2,$$

soit l'analogue dans le cas du raffinement de la méthode des QG usuels de (1.3.10).

2.5 Calcul des petites valeurs propres et metastabilité

Toujours en suivant la méthode présentée à la section 1.3, on va maintenant calculer la valeur propre approchée $\tilde{\lambda}_h$ définie en (1.3.11). Ici, puisque X_0^h est un opérateur différentiel anti-adjoint et que $f_{\mathbf{m},h}$ est à valeurs réelles, on a

$$\langle X_0^h f_{\mathbf{m},h}, f_{\mathbf{m},h}\rangle = 0 \tag{2.5.1}$$

et par suite

$$\tilde{\lambda}_h = \langle Q_h f_{\mathbf{m},h}, f_{\mathbf{m},h}\rangle \|f_{\mathbf{m},h}\|^{-2}.$$

En utilisant la factorisation donnée par l'Hypothèse 2.1.1 et par un calcul similaire à celui effectué pour établir (2.3.2), on obtient

$$\begin{aligned}\tilde{\lambda}_h &= \Big\langle \mathrm{Op}_h(M^h)\Big((h\partial_v + v/2)f_{\mathbf{m},h}\Big), (h\partial_v + v/2)f_{\mathbf{m},h}\Big\rangle \|f_{\mathbf{m},h}\|^{-2} \\ &= h\int_{(x,v)} \tilde{I}^h \cdot \partial_v \ell\, \mathrm{e}^{-2\widetilde{W}_\ell/h} \mathrm{d}(x,v)\|f_{\mathbf{m},h}\|^{-2}\end{aligned}$$

où $\tilde{I}^h$ est donnée par une intégrale oscillante faisant intervenir M^h et $\partial_v \ell$ dont on réussit à calculer un développement asymptotique grâce à la méthode de la phase stationnaire. Puisque ℓ est convenable au sens de (1.3.9), on obtient finalement par la méthode de Laplace (voir Appendice B) l'analogue de (1.3.12) suivant :

$$\tilde{\lambda}_h = h\,\tilde{B}_h \frac{\det(\mathrm{Hess}_{\mathbf{m}} V)^{1/2}}{2\pi}\,\mathrm{e}^{-2W(0)/h}$$

avec $\tilde{B}_h$ qui admet un développement classique dont le premier terme est

$$|\det(\mathrm{Hess}_0 V)|^{-1/2}\,\alpha_0.$$

En suivant alors pas à pas la fin de la méthode des QG usuels dont on conserve les notations, on montre dans notre cas

$$\lambda_h = \hat{\lambda}_h\Big(1 + O(h^\infty)\Big) = \tilde{\lambda}_h\Big(1 + O(h^\infty)\Big) = h\,\frac{\det(\mathrm{Hess}_{\mathbf{m}} V)^{1/2}}{2\pi}\Big(\tilde{B}_h + O(h^\infty)\Big)\mathrm{e}^{-2W(0)/h}.$$

Comme indiqué à la fin de la section 1.3, lorsque l'Hypothèse "double puits" 1.3.1 est remplacée par l'Hypothèse de non dégénérescence C.0.8 (comme c'est le cas dans [36]), la matrice d'interaction n'est plus aussi simple et il devient nettement plus compliqué d'obtenir une formule du type

$$\lambda_h = \hat{\lambda}_h \Big(1 + O\big(h^\infty\big) \Big).$$

Une solution est alors d'employer un résultat d'algèbre linéaire issu de [4] dont la preuve repose en partie sur l'utilisation des compléments de Schur.
Enfin, les Corollaires 2.1.6 et 2.1.7 découlent directement du contenu de la section 1.4.

Chapitre 3

Bas du spectre d'un opérateur non local et factorisé : le cas d'un potentiel général

3.1 Contexte et résultat principal

On s'intéresse à l'étude spectrale d'un opérateur semiclassique P_h associé à un potentiel W, agissant sur $L^2(\mathbb{R}^d)$ et admettant une factorisation de la forme

$$P_h = d_W^* \circ \widehat{Q} \circ d_W \tag{3.1.1}$$

où $\widehat{Q}$ est un opérateur pseudo-différentiel et d_W est l'opérateur de dérivation

$$d_W = h\nabla + \nabla W.$$

De tels opérateurs ont déjà été intensivement étudiés, comme par exemple le Laplacien de Witten

$$\Delta_W = d_W^* d_W,$$

ou encore les opérateurs cinétiques de Fokker-Planck ou de Boltzmann linéaire donnés en séparant les variables de position et de vitesse par

$$d_W^* \circ \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & \mathrm{Op}_h(M^h) \end{pmatrix} \circ d_W$$

pour des choix convenables du symbole M^h.

Les résultats fournis dans la littérature sur le spectre de ces opérateurs sont couramment démontrés pour des potentiels particuliers ou au moins vérifiant l'Hypothèse de non dégénérescence C.0.8, à l'exception de [31] où le cas de potentiels généraux a été traité pour le Laplacien de Witten et de [4] pour des opérateur différentiels de type Fokker-Planck, où la deuxième partie de cette hypothèse a été relaxée. Dans cet esprit, le but de ce travail est d'adapter les idées introduites dans [31] à un cadre non local et à l'utilisation de la méthode des QG afin de décrire le spectre près de 0 de P_h sans avoir recours à l'Hypothèse de non dégénérescence C.0.8.

Dans le but de séparer les difficultés et de concentrer ces dernières sur la topologie du potentiel W, on va condidérer un opérateur P_h présentant des propriétés agréables qu'on énumère dans l'hypothèse suivante.

Hypothèse 3.1.1. *L'opérateur P_h est borné et satisfait (3.1.1). De plus, l'opérateur pseudo-différentiel $\widehat{Q} = \mathrm{Op}_h(q_h)$ est auto-adjoint, positif et son symbole q_h vérifie les points suivants (on utilise ici certaines notions et notations issues de l'Appendice A avec (x, v) remplacé par x et η remplacé par ξ).*

a) Pour tout x, $\xi \in \mathbb{R}^d$, on a $q_h(x, -\xi) = q_h(x, \xi)$.

b) Le symbole q_h est analytique dans la variable ξ. Plus précisément,

$$q_h \sim \sum_{n\geq 0} h^n q_n \text{ dans l'espace de symboles analytiques } \mathcal{M}_d\big(S_1^0(\langle \xi_i\rangle^{-1}\langle \xi_j\rangle^{-1})\big).$$

c) Pour tout point selle $\mathbf{s}$ de W, on a $q_0(\mathbf{s},0) = \mathrm{Id}$.

On pourrait assez aisément traiter le cas où la matrice identité est remplacée par une matrice définie positive dans le point c) (comme c'est le cas dans [36]), mais ce sont bien les considérations liées au potentiel qui nous intéressent dans cet article. Un exemple d'opérateur satisfaisant cette hypothèse est l'opérateur de marche aléatoire étudié dans [3] (voir notamment Lemme 3.2 et Lemme 3.4). Dans cette référence, les auteurs étudient des potentiels satisfaisant les hypothèses suivantes.

Hypothèse 3.1.2. *Le potentiel W est tel que*

a) W est une fonction lisse, de Morse et à valeurs réelles.

b) Pout tout $|x| \geq C$ et $\beta \in \mathbb{N}^d\backslash\{0\}$, on a

$$W(x) \geq -C, \qquad |\nabla W(x)| \geq \frac{1}{C} \quad \text{et} \quad |\partial^\beta W(x)| \leq C_\beta.$$

c) W vérifie l'Hypothèse de non dégénérescence C.0.8

En particulier, pour tout $0 \leq k \leq d$, l'ensemble des points critiques d'indice k de W qu'on note $\mathcal{U}^{(k)}$ est fini et on définit $n_0 = \#\mathcal{U}^{(0)}$. Enfin, on suppose que

$$n_0 \in \mathbb{N}_{\geq 2}. \tag{3.1.2}$$

On peut montrer (voir [30], Lemme 3.14) que pour une fonction W satisfaisant l'Hypothèse 3.1.2, on a $W(x) \geq |x|/C$ en dehors d'un compact. En particulier, sous l'Hypothèse 3.1.2, on a

$$\mathrm{e}^{-W/h} \in L^2(\mathbb{R}^d) \quad \text{et} \quad \lim_{|x|\to+\infty} W(x) = +\infty. \tag{3.1.3}$$

On va se placer ici dans un cadre plus général que celui de l'Hypothèse 3.1.2.

Hypothèse 3.1.3. *On suppose que P_h est un opérateur satisfaisant l'Hypothèse 3.1.1 et associé à un potentiel W vérifiant le point a) de l'Hypothèse 3.1.2 ainsi que (3.1.2) et (3.1.3). On suppose de plus que*

- *P_h admet 0 comme valeur propre simple.*
- *Il existe $c > 0$ et $h_0 > 0$ tels que pour tout $0 < h \leq h_0$, on a que $\mathrm{Spec}(P_h) \cap [0, ch]$ est constitué d'exactement n_0 valeurs propres (avec multiplicité algébrique) qui sont exponentiellement petites par rapport à $1/h$.*

En particulier, l'estimation de résolvante

$$(z - P_h)^{-1} = O(h^{-1}) \tag{3.1.4}$$

est vérifiée sur $|z| = ch$.

Il est démontré dans [3] (Lemmes 3.2 et 3.4) que l'opérateur de marche aléatoire étudié dans cette référence fournit un exemple d'opérateur non local vérifiant l'Hypothèse 3.1.3.

Le résultat principal du présent travail est le Théorème 6.7.4 dans lequel on donne une description précise des petites valeurs propres de P_h dans le cadre où ce dernier satisfait l'Hypothèse 3.1.3, ce qui couvre le cas de potentiels ne satisfaisant non pas l'Hypothèse de non dégénérescence C.0.8 mais seulement le point a) de l'Hypothèse 3.1.2 ainsi que (3.1.2) et (3.1.3). On y parvient grâce à la méthode des QG usuels après avoir adapté les constructions de quasimodes au cas d'un potentiel général en suivant [31].

Pour simplifier cette présentation, on va expliquer la démarche dans un cas simple de potentiel ne vérifiant pas l'Hypothèse de non dégénérescence C.0.8 et contenant l'essentiel des idées qui permettent de traiter le cas général. On se place donc maintenant sous l'hypothèse qui suit.

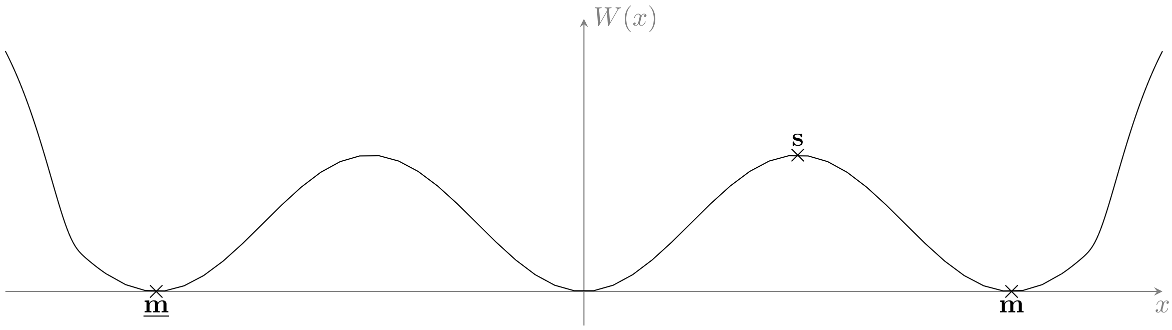
Hypothèse "triple puits" 3.1.4. *Le potentiel W est de Morse, s'annule en 0 et vérifie les points suivants :*

- *Pour tout $x \in \mathbb{R}^d$, on a $W(-x) = W(x)$.*
- *On a $\mathcal{U}^{(0)} = \{0, \mathbf{m}, -\mathbf{m}\}$ et il existe un voisinage $\mathcal{O}_0$ de 0 tel que pour tout $x \in \mathcal{O}_0$,*

$$W(\mathbf{m} + x) = W(x).$$

  On posera $\underline{\mathbf{m}} = -\mathbf{m}$.
- *W a exactement 2 points selles qu'on note $\mathbf{s}$ et $-\mathbf{s}$.*

Voilà un exemple de potentiel en dimension $d = 1$ satisfaisant cette hypothèse :



Avec les notations de l'Appendice C, on a donc $\mathbf{j}(0) = \{-\mathbf{s}, \mathbf{s}\}$ et on peut supposer sans perte de généralité que $\mathbf{j}(\mathbf{m}) = \{\mathbf{s}\}$. Ainsi, le potentiel W ne vérifie aucun point de l'Hypothèse de non dégénérescence C.0.8 : il possède 3 minima globaux et $\mathbf{j}(0) \cap \mathbf{j}(\mathbf{m}) = \{\mathbf{s}\} \neq \emptyset$.

3.2 Orthogonalité

On continue de prendre

$$f_{\underline{\mathbf{m}},h} = \mathrm{e}^{-W/h}$$

puisqu'il s'agit d'une véritable fonction propre de P_h. La première difficulté, dûe au fait que le point a) de l'Hypothèse de non dégénérescence C.0.8 n'est pas vérifié est que les constructions de QG usuels présentées à la section 1.3 ne fournissent plus une famille (presque) orthogonale. En effet, l'exponentiellement petit dans (1.3.6) provenait de la différence de hauteur entre les minima. Ainsi, si on conserve la définition donné en (1.3.5) du quasimode $f_{\mathbf{m},h}$ (avec $\theta_{\mathbf{m}}$ localisant près du puits contenant $\mathbf{m}$), alors la propriété (1.3.6) n'est ici plus vérifiée entre $f_{\underline{\mathbf{m}},h}$ et $f_{\mathbf{m},h}$. Il en va de même entre $f_{\underline{\mathbf{m}},h}$ et $f_{0,h}$.

Il apparait que $-\mathbf{m}$ est le seul des 3 minima à appartenir uniquement au support du quasimode $f_{\underline{\mathbf{m}},h}$. Or, en suivant la procédure décrite au Lemme C.0.5, on constate que l'on aurait pu poser $\underline{\mathbf{m}} = \mathbf{m}$ ou même $\underline{\mathbf{m}} = 0$ et que dans ces cas, $-\mathbf{m}$ aurait appartenu au support de deux des trois quasimodes. La symétrie des rôles entre les 3 minima suggère, dans le but de rétablir (1.3.6), de faire en réalité porter les supports de $f_{\mathbf{m},h}$ et $f_{0,h}$ sur les trois puits contenant les différents minima. Pour ce faire, notons $\theta_{\underline{\mathbf{m}}}(x) = \theta_{\mathbf{m}}(-x)$ qui est une fonction plateau localisant près du puits contenant $\underline{\mathbf{m}}$ et mettons à jour la définition de nos quasimodes en posant

$$\begin{aligned} f_{\mathbf{m},h} &= \big(\alpha\theta_{\mathbf{m}} + \beta\theta_0 + \gamma\theta_{\underline{\mathbf{m}}}\big)\mathrm{e}^{-W/h} \\ f_{0,h} &= \big(\tilde{\alpha}\theta_{\mathbf{m}} + \tilde{\beta}\theta_0 + \tilde{\gamma}\theta_{\underline{\mathbf{m}}}\big)\mathrm{e}^{-W/h} \end{aligned} \tag{3.2.1}$$

où α, β, γ, $\tilde{\alpha}$, $\tilde{\beta}$ et $\tilde{\gamma}$ sont des réels à déterminer. En rappelant que la notation $E(\mathbf{m})$ de l'Appendice C désigne essentiellement le puits contenant $\mathbf{m}$ et en utilisant l'Hypothèse 3.1.4 ainsi que des changements de variables affines, on a alors

$$\langle f_{\underline{\mathbf{m}},h}, f_{\mathbf{m},h}\rangle = (\alpha + \beta + \gamma) \int_{E(\mathbf{m})} \theta_{\mathbf{m}}(x)\mathrm{e}^{-2W(x)/h}\mathrm{d}x + O(\beta \mathrm{e}^{-c/h}),$$

$$\langle f_{\underline{\mathbf{m}},h}, f_{0,h}\rangle = (\tilde\alpha + \tilde\beta + \tilde\gamma)\int_{E(\mathbf{m})} \theta_{\mathbf{m}}(x)\mathrm{e}^{-2W(x)/h}\mathrm{d}x + O(\tilde\beta \mathrm{e}^{-c/h}) \quad \text{et}$$

$$\langle f_{\mathbf{m},h}, f_{0,h}\rangle = (\alpha\tilde\alpha + \beta\tilde\beta + \gamma\tilde\gamma)\int_{E(\mathbf{m})} \theta_{\mathbf{m}}(x)\mathrm{e}^{-2W(x)/h}\mathrm{d}x + O(\beta\tilde\beta \mathrm{e}^{-c/h}).$$

On peut annuler les premiers termes de ces trois dernières expressions en prenant par exemple

$$\alpha = 1, \quad \beta = -1, \quad \gamma = 0,$$

$$\tilde\alpha = \frac{1}{2}, \quad \tilde\beta = \frac{1}{2}, \quad \tilde\gamma = -1$$

de sorte que

$$\Big((1,1,1);\, (\alpha,\beta,\gamma);\, (\tilde\alpha,\tilde\beta,\tilde\gamma)\Big)$$

soit une famille orthogonale de $\mathbb{R}^3$. Ainsi, en prenant pour $f_{\mathbf{m},h}$ et $f_{0,h}$ des combinaisons linéaires bien choisies de troncatures sur les puits contenant les trois minima qui se trouvent à la même hauteur, on est parvenu à rétablir la propriété d'orthogonalité (1.3.6) entre nos quasimodes et ce faisant à essentiellement se ramener au cas où le point a) de l'Hypothèse de non dégénérescence C.0.8 est vérifié.

3.3 Interaction entre les quasimodes

On a vu en (1.3.7) que dans le cas des QG usuels, on a grâce à la factorisation de P_h que $P_h f_{\mathbf{m},h}$ est essentiellement localisé près des points selles qui se trouvent sur le bord du (ou maintenant des) puits contenant le support de $f_{\mathbf{m},h}$. Lorsque l'Hypothèse de non dégénérescence C.0.8 est vérifiée, cela permet de montrer que les valeurs propres de la matrice d'interaction entre les quasimodes donnée par

$$\left(\frac{\langle P_h f_{\mathbf{m},h}, f_{\mathbf{m}',h}\rangle}{\|f_{\mathbf{m},h}\|\|f_{\mathbf{m}',h}\|}\right)_{\mathbf{m},\mathbf{m}'\in\mathcal{U}^{(0)}}$$

sont essentiellement ses éléments diagonaux.
Ce n'est plus le cas sous l'Hypothèse 3.1.4 puisque dans le calcul de

$$m_{0,\mathbf{m}} := \frac{\langle P_h f_{\mathbf{m},h}, f_{0,h}\rangle}{\|f_{0,h}\|\|f_{\mathbf{m},h}\|},$$

des contributions non négligeables apparaissent au voisinage de $\mathbf{j}(0)\cap\mathbf{j}(\mathbf{m})$ (et même au voisinage de $-\mathbf{s}$ par (3.2.1)). Dans cette situation, toutes les entrées de la matrice

$$\begin{pmatrix} \langle P_h f_{\mathbf{m},h}, f_{\mathbf{m},h}\rangle \|f_{\mathbf{m},h}\|^{-2} & m_{0,\mathbf{m}} \\ m_{0,\mathbf{m}} & \langle P_h f_{0,h}, f_{0,h}\rangle \|f_{0,h}\|^{-2}\end{pmatrix}, \tag{3.3.1}$$

sont d'ordre $h\mathrm{e}^{-2W(\mathbf{s})/h}$ et cette dernière admet un développement asymptotique lorsque h tend vers 0. Le résultat que l'on obtient alors est que les 2 petites valeurs propres non nulles de P_h ont le même développement asymptotique que celles de la matrice (3.3.1), ce qui explique la forme de l'énoncé du Théorème 6.7.4.

Chapitre 4

Petites valeurs propres et metastabilité de l'équation de relaxation linéaire de Boltzmann

4.1 Contexte et résultats principaux

On s'intéresse ici aux résultats de [34] qui traite de nouveau de l'opérateur de Boltzmann associé à (1.1.8)

$$\begin{aligned} P_h &= v \cdot h\partial_x - \partial_x V \cdot h\partial_v + Q_h \\ &= X_0^h + Q_h \end{aligned}$$

agissant sur $L^2(\mathbb{R}^{2d})$, mais cette fois ci dans le cadre du modèle de BGK pour lequel, avec les notations (1.1.5) et (1.1.10), l'opérateur de collision

$$Q_h = h(\mathrm{Id} - \Pi_h) \tag{4.1.1}$$

correspond à une simple relaxation vers la Maxwellienne μ_h. On va ici aussi travailler sous l'hypothèse de confinement 2.1.2 pour le potentiel V ainsi que l'Hypothèse de non dégénérescence C.0.8 afin de concentrer les difficultés sur les nouvelles propriétés de l'opérateur et éviter les considérations du chapitre 3.

On dispose déjà grâce à Robbe [39] d'estimations de résolvante pour cet opérateur, ainsi que d'une première description de son spectre près de 0 qui s'avère être constitué d'un nombre fini de valeurs propres réelles et exponentiellement petites par rapport à $1/h$.

Théorème 4.1.1. *Supposons que l'Hypothèse 2.1.2 est satisfaite et rappelons la notation (2.1.4). Alors l'opérateur P_h (muni d'un domaine convenable) admet 0 comme valeur propre simple. De plus, il existe $c > 0$ et $h_0 > 0$ tels que pour tout $0 < h \leq h_0$, $\mathrm{Spec}(P_h) \cap \{\mathrm{Re}\, z \leq ch\}$ est constitué d'exactement n_0 valeurs propres (comptées avec multiplicité algébrique) qui sont réelles, positives et $O(\mathrm{e}^{-c/h})$. Par ailleurs, pour tout $0 < \tilde{c} \leq c$, l'estimée de résolvante*

$$(P_h - z)^{-1} = O(h^{-1})$$

est vérifiée uniformément sur $\{\mathrm{Re}\, z \leq ch\} \backslash B(0, \tilde{c}h)$.

Le but de ce travail est donc d'améliorer cette description en fournissant un résultat similaire à celui obtenu dans [36] dans le cas de la relaxation douce. Les principales difficultés proviennent de l'opérateur non local Q_h. On montrera qu'il s'agit d'un opérateur pseudo-différentiel mais qui présente de "mauvaises" propriétés microlocales (avec les notations de l'Appendice A, ce dernier est dans la classe critique $S^{1/2}$). On verra notamment que, de façon analogue au phénomène décrit dans la section 1.5 et contrairement aux opérateurs de collisions considérés dans [36], l'opérateur Q_h fait échouer la méthode des QG usuels pour l'étude du spectre de P_h. C'est en introduisant une superposition de QG dans l'esprit de la section 1.6

qu'on va pouvoir établir le résultat principal de ce travail. Notre approche repose sur une factorisation de Q_h qu'on va exhiber ainsi que la connaissance explicite de son symbole. Pour des raisons techniques liées à l'échec de la méthode des QG usuels, on ne donnera ici pour chaque petite valeur propre non pas un développement asymptotique complet mais seulement un équivalent. On note là encore $\lambda(\mathbf{m},h)$ la valeur propre non nulle et exponentiellement petite de P_h associée au minimum $\mathbf{m}\in\mathcal{U}^{(0)}\backslash\{\underline{\mathbf{m}}\}$ de V.

Théorème 4.1.2. *Supposons que les Hypothèses 2.1.2 et C.0.8 sont satisfaites. Les petites valeurs propres de P_h satisfont l'équivalent suivant dans la limite $h\to 0$:*

$$\lambda(\mathbf{m},h)\sim h\varrho(\mathbf{m})\,\mathrm{e}^{\frac{-2S(\mathbf{m})}{h}}$$

avec

$$\varrho(\mathbf{m})=\frac{1}{\pi}\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right)^{\frac{1}{\sqrt{|\tau_{\mathbf{s}}|}}}\left(\frac{\det \mathrm{Hess}_{\mathbf{m}}V}{|\det \mathrm{Hess}_{\mathbf{s}}V|}\right)^{1/2}\int_{\gamma_1\le z\le\gamma<1}k_0^{\mathbf{s}}(\gamma)k_0^{\mathbf{s}}(z)\ln\left(2\,\frac{(1+z)(1+\gamma)}{1+3z+3\gamma+z\gamma}\right)\mathrm{d}z\mathrm{d}\gamma$$

où

$$k_0^{\mathbf{s}}(z)=\frac{2\sqrt{2}}{\sqrt{|\tau_{\mathbf{s}}|}(z-\gamma_2)^2}\left(\frac{z-\gamma_1}{z-\gamma_2}\right)^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}-1}\quad;\quad\gamma_1=-3+2\sqrt{2}\quad;\quad\gamma_2=-3-2\sqrt{2},$$

les applications S et $\mathbf{j}$ proviennent de la Définition C.0.7 et τ_s est la valeur propre négative de $\mathrm{Hess}_{\mathbf{s}}V$.

Ce théorème nous permet ici aussi de façon analogue au chapitre sur la relaxation douce de déduire des informations sur le comportement en temps long des solutions de (1.1.8) et plus précisément sur les phénomènes de retour à l'équilibre et de metastabilité. On les regroupe dans les corollaires qui suivent.

Corollaire 4.1.3. *Sous les hypothèses du Théorème 4.1.2 et en notant λ^* la plus petite valeur propre non nulle de P_h, il existe $h_0>0$ tel que pour tout $0<h\le h_0$ et $t\ge 0$,*

$$\|\mathrm{e}^{-tP_h/h}-\mathbb{P}_1\|\le C\mathrm{e}^{-t\lambda^*/h}.$$

Corollaire 4.1.4. *Supposons que les hypothèses du Théorème 4.1.2 sont satisfaites. Considérons des minima locaux $\mathbf{m}_1=\underline{\mathbf{m}}$, $\mathbf{m}_2$, ..., $\mathbf{m}_K$ tels que*

$$S(\mathcal{U}^{(0)})=\{+\infty=S(\mathbf{m}_1)>S(\mathbf{m}_2)>\cdots>S(\mathbf{m}_K)\}$$

pour l'application S de la Définition C.0.7. Pour $2\le k\le K$, notons $\mathbb{P}_k$ le projecteur spectral (qui n'est pas nécessairement orthogonal) associé aux valeurs propres qui sont $O\big(\mathrm{e}^{-2\frac{S(\mathbf{m}_k)}{h}}\big)$. Alors pour tout temps $(t_k^{\pm})_{1\le k\le K}$ satisfaisant

$$t_K^-\ge h^{-1}|\ln(h^\infty)|\quad et\quad t_k^-\ge|\ln(h^\infty)|\mathrm{e}^{2\frac{S(\mathbf{m}_{k+1})}{h}}\quad pour\quad k=1,\dots,K-1$$

ainsi que

$$t_1^+=+\infty\qquad et\quad t_k^+=O\Big(h^\infty\mathrm{e}^{2\frac{S(\mathbf{m}_k)}{h}}\Big)\qquad pour\quad k=2,\dots,K$$

on a

$$\mathrm{e}^{-tP_h/h}=\mathbb{P}_k+O(h^\infty)\qquad sur\ [t_k^-,t_k^+].$$

4.2 Échec des QG usuels

Plaçons nous pour cette présentation une fois de plus dans le cadre simplifié donné par l'Hypothèse "double puits" 1.3.1. Une première approche "naïve" de notre problème consiste à essayer de mettre en place la méthode des QG usuels pour P_h. On considère donc de nouveau une forme linéaire $\ell=(\ell_x,\ell_v)$ dans les variables (x,v) convenable au sens de (1.3.9) ainsi que le QG f_ℓ pour le potentiel W définis en (1.5.3) et (1.5.4). Il s'agit maintenant de calculer l'action de l'opérateur P_h sur f_ℓ. On connait déjà le terme principal de $X_0^hf_\ell$ grâce à (2.3.1). Reste à calculer Q_h appliqué à f_ℓ.

Bien que l'opérateur Q_h défini en (4.1.1) puisse apparaitre comme plutôt simple (c'est un multiple d'un projecteur orthogonal), un premier pas pour notre étude va être d'adopter un point de vue plus microlocal. Pour ce faire, on va commencer par montrer que Q_h admet une factorisation semblable à celle du cas de la relaxation douce (cf Hypothèse 2.1.1).

Proposition 4.2.1. *On rappelle les notations de l'Appendice A ainsi que (1.5.1). Il existe un symbole $m_h \in S^{1/2}\big(\langle v, \eta\rangle^{-2}\big)$ donné par*

$$m_h(v,\eta) = 2\int_0^1 (y+1)^{d-2} \mathrm{e}^{-\frac{y}{h}\left(\frac{v^2}{2}+2\eta^2\right)} \mathrm{d}y$$

tel que

$$Q_h = b_h^* \circ \mathrm{Op}_h(m_h \,\mathrm{Id}) \circ b_h.$$

En particulier, par le calcul symbolique,

$$Q_h = \mathrm{Op}_h(g_h) \circ b_h$$

avec

$$g_h(v,\eta) = \int_0^1 (y+1)^{d-1} \mathrm{e}^{-\frac{y}{h}\left(\frac{v^2}{2}+2\eta^2\right)} \mathrm{d}y\,(-2i\eta^t + v^t) \in S^{1/2}\big(\langle v,\eta\rangle^{-1}\big).$$

On démontre ce résultat en utilisant le noyau distributionnel de Π_h pour établir que ce dernier est un opérateur pseudo-différentiel dont on calcule explicitement le symbole. On utilise ensuite le calcul symbolique pour montrer la factorisation annoncée.

On peut alors fournir le calcul fondamental suivant qui montre, de façon analogue à (1.5.8), que contrairement au cas de la relaxation douce, l'action de Q_h sur f_ℓ produit des exponentielles d'autres phases que $\widetilde{W}_\ell$, rendant impossible au vu de (2.3.1) les compensations avec le terme $X_0^h f_\ell$.

Lemme 4.2.2. *Rappelons la notation (1.3.8). On a*

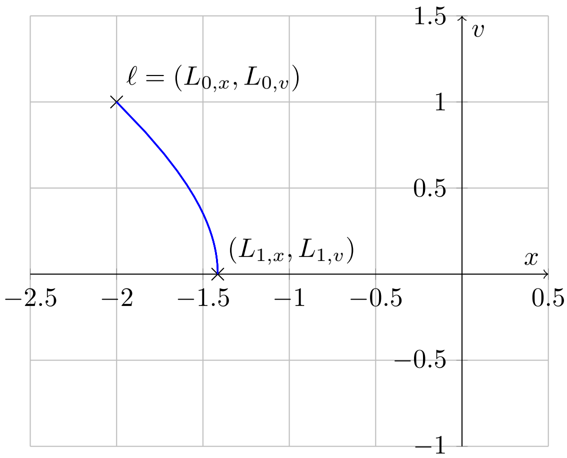
$$Q_h f_\ell(x,v) = -\sqrt{h}\int_0^1 \partial_y(L_y)\,\mathrm{e}^{-\frac{\widetilde{W}_{L_y}(x,v)}{h}}\,\mathrm{d}y \cdot \begin{pmatrix} x \\ v \end{pmatrix} \tag{4.2.1}$$

où par un léger abus de notation, on note L_y la forme linéaire

$$L_y(x,v) = \frac{(1+y)\ell_x \cdot x + (1-y)\ell_v \cdot v}{\Big(4y\ell_v^2 + (y+1)^2\Big)^{1/2}}$$

ainsi que le vecteur qui la représente.

Voici une représentation pour $d = 1$ de la courbe tracée par le vecteur L_y lorsque $y \in [0,1]$. On peut notamment remarquer que $L_0 = \ell$ et que la composante en v de L_1 est nulle.



4.3 Superposition intégrale de QG

Pour espérer compenser toutes les exponentielles apparaissant dans $Q_h f_\ell$ d'après le Lemme 4.2.2, on va suivre la résolution proposée dans la section 1.6. Dans le cas présent, il s'agit donc de considérer un nouveau quasimode donné par une superposition des QG associés aux formes linéaires $(L_y)_{y\in[0,1]}$ afin que l'action de X_0^h sur ce dernier produise un terme semblable à (4.2.1), tout en espérant que l'action de Q_h ne produise pas encore de nouvelles phases en dehors de $(\widetilde{W}_{L_y})_{y\in[0,1]}$.

Avec les notations du Lemme 4.2.2, on définit donc le quasimode essentiellement donné près de 0 par

$$F_\ell(x,v) = \int_0^1 k(\gamma) f_{L_\gamma}(x,v)\,\mathrm{d}\gamma \tag{4.3.1}$$

où par analogie avec (1.6.1), k est une densité de probabilité sur $[0,1]$ à déterminer. Pour connaitre l'action de Q_h sur ce nouveau quasimode, il suffit d'appliquer le Lemme 4.2.2 avec f_{L_γ} au lieu de f_ℓ et d'intégrer contre k. En notant

$$L_\gamma = (L_{\gamma,x}\,;\,L_{\gamma,v}),$$

on obtient

$$Q_h F_\ell = -\sqrt{h}\int_0^1 k(\gamma)\int_0^1 \partial_y\mathscr{L}(\gamma,y)\,\exp\Big[-\frac{1}{h}\Big(W(x,v)+\frac{1}{2}\big[\mathscr{L}(\gamma,y)\cdot(x,v)\big]^2\Big)\Big]\,\mathrm{d}y\,\mathrm{d}\gamma\cdot\begin{pmatrix}x\\ v\end{pmatrix} \tag{4.3.2}$$

où $\mathscr{L}(\gamma,y)$ désigne le vecteur

$$\left(\frac{1+y}{\big(4|L_{\gamma,v}|^2y+(y+1)^2\big)^{1/2}}\,L_{\gamma,x}\quad;\quad\frac{1-y}{\big(4|L_{\gamma,v}|^2y+(y+1)^2\big)^{1/2}}\,L_{\gamma,v}\right).$$

Tout l'enjeu est maintenant de savoir si les phases

$$\Big(W(x,v)+\frac{1}{2}\big[\mathscr{L}(\gamma,y)\cdot(x,v)\big]^2\Big)_{y,\gamma\in[0,1]}$$

appartiennent ou non à la famille $(\widetilde{W}_{L_y})_{y\in[0,1]}$. Il s'avère que c'est le cas puisqu'en notant

$$\Gamma_\gamma(y) = \frac{y+\gamma}{1+y\gamma}, \tag{4.3.3}$$

on peut assez facilement vérifier que

$$\mathscr{L}(\gamma,y) = L_{\Gamma_\gamma(y)}. \tag{4.3.4}$$

Formellement, en écrivant dans l'esprit de la Proposition 4.2.1

$$Q_h = \int_0^1 Q_{h,y}\,\mathrm{d}y\,, \tag{4.3.5}$$

la quantité $\Gamma_\gamma(y)$ donne l'indice de la phase sur laquelle est envoyé le quasimode associé à la forme linéaire d'indice γ sous l'action de $Q_{h,y}$. Ainsi, (4.3.2) devient

$$Q_h F_\ell = -\sqrt{h}\int_0^1 k(\gamma)\int_0^1 \partial_y\big(L_{\Gamma_\gamma(y)}\big)\,\exp\Big[-\frac{1}{h}\widetilde{W}_{L_{\Gamma_\gamma(y)}}(x,v)\Big]\,\mathrm{d}y\,\mathrm{d}\gamma\cdot\begin{pmatrix}x\\ v\end{pmatrix}$$

puis par le changement de variable $z=\Gamma_\gamma(y)$

$$Q_h F_\ell = -\sqrt{h}\int_0^1\int_0^z k(\gamma)\,\mathrm{d}\gamma\,\partial_z(L_z)\cdot\begin{pmatrix}x\\ v\end{pmatrix}\mathrm{e}^{-\frac{\widetilde{W}_{L_z}(x,v)}{h}}\mathrm{d}z.$$

À l'aide de (2.3.1), on obtient alors à des termes négligeables près

$$P_h F_\ell = \sqrt{h}\int_0^1\Big[k(z)\begin{pmatrix}0 & -\mathrm{Hess}_0 V\\ \mathrm{Id} & 0\end{pmatrix}L_z - \int_0^z k(\gamma)\,\mathrm{d}\gamma\,\partial_z L_z\Big]\cdot\begin{pmatrix}x\\ v\end{pmatrix}\mathrm{e}^{-\frac{\widetilde{W}_{L_z}(x,v)}{h}}\mathrm{d}z$$

que l'on peut voir comme l'analogue de (1.6.2). On va donc là aussi tâcher de choisir k et ℓ de sorte à annuler le préfacteur de chaque exponentielle, c'est à dire tels que

$$k(z)\begin{pmatrix} 0 & -\mathrm{Hess}_0 V \\ \mathrm{Id} & 0 \end{pmatrix} L_z - \int_0^z k(\gamma)\,\mathrm{d}\gamma\,\partial_z L_z = 0 \qquad \forall z \in [0,1]. \tag{4.3.6}$$

En notant

$$K(z) = \int_0^z k(\gamma)\,\mathrm{d}\gamma$$

la fonction de répartition associée à k, on obtient facilement des conditions nécessaires sur ℓ et K.

Lemme 4.3.1. *On rappelle que τ désigne la valeur propre négative de* $\mathrm{Hess}_0 V$*. Si ℓ et K sont tels que (4.3.6) est vérifiée, alors*

$$\mathrm{Hess}_0 V \ell_v = \tau \ell_v \qquad ; \qquad \ell_x = -\left(\frac{(1+\ell_v^2)|\tau|}{\ell_v^2}\right)^{1/2} \ell_v$$

et K est une fonction de répartition sur $[0,1]$ qui satisfait l'EDO

$$K'(z) - \frac{2\sqrt{\ell_v^2(1+\ell_v^2)}}{\sqrt{|\tau|}\Big(4z\ell_v^2 + (z+1)^2\Big)} K(z) = 0. \tag{4.3.7}$$

Ce lemme détermine le choix de ℓ, toujours au signe près, mais ici également à la norme de ℓ_v près. Malheureusement, il n'existe pas de fonction de répartition sur $[0,1]$ vérifiant (4.3.7). En revanche, en notant $\gamma_2(\ell_v^2) < \gamma_1(\ell_v^2) < 0$ les deux singularités de

$$\Big(4z\ell_v^2 + (z+1)^2\Big)^{-1},$$

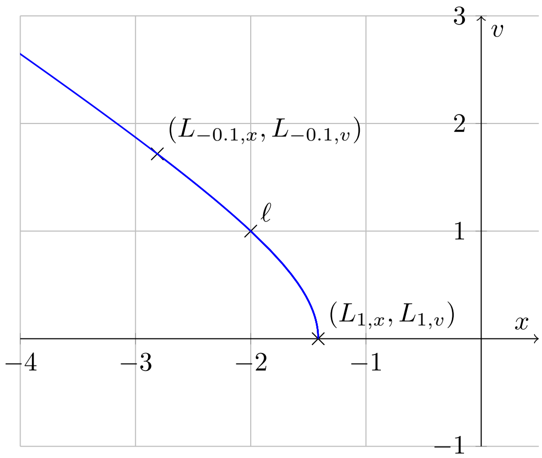
il existe une fonction de répartition sur $]\gamma_1(\ell_v^2), 1]$ qui satisfait (4.3.7). Une idée serait alors de reprendre toutes les constructions de cette section mais pour $\gamma \in]\gamma_1(\ell_v^2), 1]$ au lieu de $[0,1]$. Il se trouve notamment que (4.3.4) reste vraie dans ce cadre.

Mais un nouveau problème se présente alors : l'explosion en $\gamma_1(\ell_v^2)$ de la norme de L_γ entraîne que les termes de restes (que l'on a omis d'écrire dans cette présentation) issus des divers calculs de cette section ne sont pas uniformes en $\gamma \in]\gamma_1(\ell_v^2), 1]$. On s'en sort finalement en travaillant sur $[\gamma_1(\ell_v^2) + \nu, 1]$ où $\nu > 0$ est fixé suffisament petit avant de faire tendre h vers 0. On se contente alors de produire une solution approchée de (4.3.7) pour laquelle on a

$$\|P_h F_\ell\| = h\,\mathrm{e}^{-W(0)/h} \|F_\ell\| \Big(O_\nu\Big(h^{\frac{1}{2}}\Big) + O\Big(\nu^{\frac{1}{2\sqrt{|\tau|}}}\,|\ln(\nu)|\Big)\Big). \tag{4.3.8}$$

C'est cet analogue de (1.3.10) qui explique qu'on obtient seulement un équivalent dans le Théorème 4.1.2.

La courbe tracée par le vecteur L_γ apparaissant dans notre nouvelle superposition de quasimode a alors la forme suivante.



On comprend intuitivement sur cette figure que la superposition de quasimodes obtenue ne dépend pas de la norme de ℓ_v (le vecteur ℓ ne représente plus le “point de départ” de la courbe), ce qui est cohérent avec l'observation qui a suivi le Lemme 4.3.1. La convention choisie dans [34] est alors de prendre

$$\ell_v^2 = 1$$

afin d'obtenir de nombreuses simplifications. C'est ce qui explique par exemple les valeurs de k_0, γ_1 et γ_2 dans le Théorème 4.1.2.

4.4 Conclusion

La dernière étape de ce travail consiste à calculer la valeur propre approchée donnée grâce à (2.5.1) par

$$\tilde{\lambda}_{\nu,h} = \langle Q_h F_\ell, F_\ell \rangle \|F_\ell\|^{-2}$$

où on a utilisé la notation (4.3.1) en y remplaçant 0 par $\gamma_1 + \nu$. Ce calcul est rendu plus pénible par le fait de travailler avec une superposition de quasimodes ainsi qu'un opérateur de la forme (4.3.5) ; cela fait par exemple apparaitre 3 intégrales de plus que dans le cas de la relaxation douce. La démarche adoptée dans [34] consiste à commencer par ignorer ces difficultés en calculant grâce à la Proposition 4.2.1 et une méthode de Laplace (voir Appendice B) le produit scalaire

$$\langle Q_{h,y} f_{L_\gamma}, f_{L_\gamma} \rangle. \tag{4.4.1}$$

Ce n'est pas exactement la quantité qui va apparaitre dans les intégrales, celle-ci étant par bilinéarité plutôt de la forme

$$\langle Q_{h,y} f_{L_\gamma}, f_{L_z} \rangle$$

mais on montre qu'on peut se ramener à (4.4.1) par un changement de variable en remarquant qu'avec la notation (4.3.3), on a

$$Q_{h,y} f_{L_\gamma} = (\Gamma_z^{-1} \circ \Gamma_\gamma)'(y)\, Q_{h,\Gamma_z^{-1}\circ\Gamma_\gamma(y)} f_{L_z}.$$

On parvient alors à établir

$$\tilde{\lambda}_{\nu,h} = h\, \tilde{\varrho}_{\nu,h}\, \mathrm{e}^{-2W(0)/h} \tag{4.4.2}$$

où en utilisant les notations du Théorème 4.1.2

$$\lim_{\nu\to 0}\lim_{h\to 0} \tilde{\varrho}_{\nu,h} = \frac{1}{\pi}\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right)^{\frac{1}{\sqrt{|\tau|}}} \left(\frac{\det \mathrm{Hess}_{\mathbf{m}} V}{|\det \mathrm{Hess}_0 V|}\right)^{1/2} \int_{\gamma_1 \leq z \leq \gamma < 1} k_0(\gamma) k_0(z) \ln\left(2\,\frac{(1+z)(1+\gamma)}{1+3z+3\gamma+z\gamma}\right) \mathrm{d}z\mathrm{d}\gamma.$$

À partir de ce point, en combinant (4.3.8) et (4.4.2), la fin de la preuve du Théorème 4.1.2 est identique à celle du Théorème 2.1.5 du chapitre 2, en adaptant simplement la forme des restes. Enfin, là encore les Corollaires 4.1.3 et 4.1.4 découlent directement du contenu de la section 1.4.

Deuxième partie

Articles de thèse

Chapitre 5

Metastability results for a class of linear Boltzmann equations

On fournit dans ce chapitre des démonstrations (en anglais) issues de [36] des résultats présentés au chapitre 2.

5.1 Introduction

5.1.1 Motivations

We are interested in the linear Boltzmann equation :

$$\begin{cases} h\partial_t u + v\cdot h\partial_x u - \partial_x V\cdot h\partial_v u + Q_{\mathcal{H}}(h,u) = 0 \\ u_{|t=0} = u_0 \end{cases} \tag{5.1.1}$$

in a semiclassical framework (i.e in the limit $h \to 0$), where h is a *semiclassical parameter* and corresponds to the temperature of the system. Here we denoted for shortness ∂_x and ∂_v the partial gradients with respect to x and v. This equation is used to model the evolution of a system of charged particles in a gas on which acts an electrical force associated to the real valued potential V that only depends on the space variable x. The interactions between the particles are modelled by the linear operator $Q_{\mathcal{H}}$ which is called *collision operator*. Here the unknown is the function $u : \mathbb{R}_+ \to L^1(\mathbb{R}^{2d})$ giving the probability density of the system of particles at time $t \in \mathbb{R}_+$, position $x \in \mathbb{R}^d$ and velocity $v \in \mathbb{R}^d$. For our purpose, we introduce the square roots of the usual Maxwellian distributions

$$\mu_h(v) = \frac{e^{-\frac{v^2}{4h}}}{(2\pi h)^{d/4}} \qquad \text{and} \qquad \mathcal{M}_h = e^{-\frac{V}{2h}}\mu_h. \tag{5.1.2}$$

In many models, we have

$$Q_{\mathcal{H}}(h,\mathcal{M}_h^2) = 0 \qquad \text{and} \qquad Q_{\mathcal{H}}^*(h,1) = 0 \tag{5.1.3}$$

so in particular $\mathcal{M}_h^2$ is a stable state of (5.1.1). In order to do a perturbative study of the time independent operator associated to (5.1.1) near $\mathcal{M}_h^2$, we introduce the natural Hilbert space

$$\mathcal{H} = \left\{u \in \mathcal{D}' \,;\, \mathcal{M}_h^{-1}u \in L^2(\mathbb{R}^{2d})\right\}.$$

It is clear from the Cauchy Schwarz inequality that $\mathcal{H}$ is indeed a subset of $L^1(\mathbb{R}^{2d})$ provided that $e^{-\frac{V}{2h}} \in L^2(\mathbb{R}^d_x)$. In view of (5.1.3) and the definition of $\mathcal{H}$, it is more convenient to work with the new unknown

$$f = \mathcal{M}_h^{-1}u \,:\, \mathbb{R}_+ \to L^2(\mathbb{R}^{2d})$$

for which the new equation becomes

$$\begin{cases} h\partial_t f + v\cdot h\partial_x f - \partial_x V\cdot h\partial_v f + Q_h(f) = 0 \\ f_{|t=0} = f_0 \end{cases} \tag{5.1.4}$$

where

$$Q_h = \mathcal{M}_h^{-1} \circ Q_{\mathcal{H}}(h,\cdot) \circ \mathcal{M}_h.$$

Our study will be focused on the new time independent operator

$$\begin{aligned} P_h &= v \cdot h\partial_x - \partial_x V \cdot h\partial_v + Q_h \\ &= X_0^h + Q_h \end{aligned}$$

for some specific choices of the collision operator Q_h, where the notation X_0^h will stand for the operator $v \cdot h\partial_x - \partial_x V \cdot h\partial_v$, but also for the vector field $(x,v) \mapsto h(v, -\partial_x V(x))$. There are plenty of different collision operators studied in the literature, their main properties being that these are symmetric integral operators acting as multiplicators in the position variable x and canceling the Maxwellian distribution. Our work is in particular motivated by the study of the *mild relaxation* operator introduced in [38] and given by $H_0(1+H_0)^{-1}$ with H_0 the harmonic oscillator in velocity defined by

$$H_0 = -h^2\Delta_v + \frac{v^2}{4} - \frac{hd}{2}. \tag{5.1.5}$$

In this spirit, the collision operators we will be working with will always be bounded and self-adjoint so, $(X_0^h, \mathcal{C}_c^\infty(\mathbb{R}^{2d}))$ being essentially skew-adjoint, the operator P_h (endowed with the appropriate domain) is maximal accretive and (5.1.4) is well-posed. More generally, some interesting cases of collision operators are given by functions of H_0 (see for instance [21, 27–29, 38]) which is the setting that we will adopt.

This paper is concerned with the spectral study of the operator P_h. This type of questions has recently known some major progress on the impulse of microlocal methods. In the case of the linear Boltzmann equation (5.1.4), the use of hypocoercive techniques in 2015 in [39] enabled to get some resolvent estimates and establish a rough localization of the small spectrum of P_h which consists of exponentially small eigenvalues in correspondence with the minima of the potential V. This type of result is similar to the one obtained for example for the Witten Laplacian by Helffer and Sjöstrand in [19] in the 1980's. Such a localization already leads to return to equilibrium and metastability results which can be improved as the description of the small spectrum becomes more precise. For example, sharp asymptotics of the small eigenvalues of the Witten Laplacian were obtained later in the 2000's in [6] and [18] and later again for Kramers-Fokker-Planck type operators by Hérau et al. in [22]. In these papers, the idea was to exhibit a supersymmetric structure for the operator and then study both the derivative acting from 0-forms into 1-forms and its adjoint with the help of basic quasimodes. In [38], Robbe managed to show that the Boltzmann equation (5.1.4) with mild relaxation enjoys such a supersymmetric structure. However, in that case, the matrix appearing in the modification of the inner product does not obey good estimates with respect to the semiclassical parameter h. This is why our goal here will be to give precise spectral asymptotics for the operator P_h through a more recent approach which consists in directly constructing a family of accurate quasimodes for our operator in the spirit of [26] and [4].

The aim of this paper is twofold. Firstly, we want to prove a result similar to the one obtained by Robbe in [39] but for a large class of collision operators. The second goal is to provide complete asymptotics of the small eigenvalues of P_h as it was done in [18] for the Witten Laplacian or in [22, 23] with recent improvements by Bony et al. in [4] in the case of Fokker-Planck type differential operators. We manage to establish such results for the equation (5.1.4) for a class of pseudo-differential collision operators presenting nice symbol properties as well as a factorized structure.

5.1.2 Setting and main results

For $d' \in \mathbb{N}^*$ and $Z \in \mathbb{C}^{d'}$, we use the standard notation $\langle Z \rangle = (1+|Z|^2)^{1/2}$. In this paper, we will treat the case of collision operators of the form

$$Q_h = \varrho(H_0)$$

with ϱ satisfying the following :

Hypothesis 5.1.1. *The function $\varrho : \mathbb{R}_+ \to \mathbb{R}_+$ vanishes at the origin and for all $t \geq 0$,*

$$\varrho(t) \geq \frac{1}{C}\frac{t}{\langle t \rangle}.$$

Moreover, it admits an analytic extension to $\{\operatorname{Re} z > -\frac{1}{C}\}$ for which there exist $\varrho_\infty \in \mathbb{R}_+$ and $\alpha > 0$ such that $\varrho(z) = \varrho_\infty + O(\langle z\rangle^{-\alpha})$.

In particular, Q_h will be bounded uniformly in h and self-adjoint. An example of such collision operator is the *mild relaxation* operator introduced in [38] and given by $H_0(1+H_0)^{-1}$.

Using the material from Appendix A, we can actually introduce a new class of collision operators which appears to be more general that the one given by Hypothesis 5.1.1. Let us denote b_h the twisted derivative

$$b_h = h\partial_v + v/2 \tag{5.1.6}$$

so that in particular with the notation (5.1.5) we have $H_0 = b_h^* b_h$. We also use the standard notation $\mathcal{M}_d(\mathbb{R})$ for the set of all d-by-d real matrices.

Hypothesis 5.1.2. *Recall the material from Appendix A. There exists $\tau > 0$ and a symmetric matrix of analytic symbols*

$$M^h(x,v,\eta) = \big(m_{p,q}(x,v,\eta)\big)_{1\le p,q\le d} \in \mathcal{M}_d\big(S^0_\tau(\langle(v,\eta)\rangle^{-2})\big)$$

sending $\mathbb{R}^{3d}$ into $\mathcal{M}_d(\mathbb{R})$ and such that, with the notation (5.1.6), the collision operator Q_h satisfies

a) $Q_h = b_h^* \circ Op_h(M^h) \circ b_h$

b) $M^h \sim \sum_{n\ge 0} h^n M_n$ *in* $\mathcal{M}_d\big(S^0_\tau(\langle(v,\eta)\rangle^{-2})\big)$

c) For all $(x,v,\eta) \in \mathbb{R}^{3d}$, $M^h(x,v,\eta) = M^h(x,v,-\eta)$

d) For all $(x,v,\eta) \in \mathbb{R}^{3d}$, $M_0(x,v,\eta) \ge \frac{1}{C}\langle(v,\eta)\rangle^{-2}\,\mathrm{Id}$.

Since the $(M_n)_n$ do not depend on h, we easily get that these matrices of symbols are also even in η, symmetric, independent of ξ and with values in $\mathcal{M}_d(\mathbb{R})$; so in particular item d) makes sense. This will enable us to establish Lemma 5.2.1 which is sometimes reffered to as *microscopic coercivity* (see for instance [13]). As announced, we have the following Lemma which is proven in Appendix 5.7.1 :

Lemma 5.1.3. *Hypothesis 5.1.1 implies Hypothesis 5.1.2.*

We will also make a few confining assumptions on the function V, assuring for instance that the bottom spectrum of the associated Witten Laplacian is discrete. In particular, our potential will satisfy Assumption 2 from [26] and Hypothesis 1.1 from [39].

Hypothesis 5.1.4. *The potential V is a smooth Morse function depending only on the space variable $x \in \mathbb{R}^d$ with values in $\mathbb{R}$ which is bounded from below and such that*

$$|\partial_x V(x)| \ge \frac{1}{C} \qquad \text{for } |x| > C.$$

Moreover, for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \ge 2$, there exists C_α such that

$$|\partial_x^\alpha V| \le C_\alpha.$$

In particular, for every $0 \le k \le d$, the set of critical points of index k of V that we denote $\mathcal{U}^{(k)}$ is finite and we set

$$n_0 = \#\mathcal{U}^{(0)}. \tag{5.1.7}$$

Finally, we will suppose that $n_0 \ge 2$.

The last assumption comes from the fact that when $n_0 = 1$, the so-called *small spectrum* of the operator P_h (i.e its eigenvalues with exponentially small modulus) is trivial, so there is nothing to study. It is shown in [30], Lemma 3.14 that for a function V satisfying Hypothesis 5.1.4, we have $V(x) \ge |x|/C$ outside of a compact. In particular, under Hypothesis 5.1.4, it holds $\mathrm{e}^{-V/2h} \in L^2(\mathbb{R}^d_x)$. Moreover, in our setting, X_0^h is a smooth vector field whose differential is bounded on $\mathbb{R}^{2d}$, so the operator X_0^h endowed with the domain

$$D = \{u \in L^2(\mathbb{R}^{2d})\,;\, X_0^h u \in L^2(\mathbb{R}^{2d})\} \tag{5.1.8}$$

is skew-adjoint on $L^2(\mathbb{R}^{2d})$ and the set $\mathcal{S}(\mathbb{R}^{2d})$ is a core for this operator. Therefore, $(P_h, D)^* = (-X_0^h + Q_h, D)$ and (P_h, D) is m-accretive on $L^2(\mathbb{R}^{2d})$.

For an operator such as P_h, which is not for instance self-adjoint with compact resolvent, we do not have any information a priori on its spectrum (except here that it is contained in $\{z \in \mathbb{C}\,;\, \mathrm{Re}\, z \geq 0\}$). Section 5.2 is thus devoted to establishing a first description of the spectrum of P_h near 0 which, in the spirit of the case of other non self-adjoint operators studied in [22,39], appears in particular to be discrete :

Theorem 5.1.5. *Assume that Hypotheses 5.1.2 and 5.1.4 are satisfied and recall the notation (5.1.7). Then the operator (P_h, D) admits 0 as a simple eigenvalue. Moreover, there exists $c > 0$ and $h_0 > 0$ such that for all $0 < h \leq h_0$, $\mathrm{Spec}(P_h) \cap \{\mathrm{Re}\, z \leq ch^2\}$ consists of exactly n_0 eigenvalues (counted with algebraic multiplicity) that are exponentially small with respect to $1/h$ and for all $0 < \tilde{c} \leq c$, the resolvent estimate*

$$(P_h - z)^{-1} = O(h^{-2})$$

holds uniformly in $\{\mathrm{Re}\, z \leq ch^2\} \backslash B(0, \tilde{c}h^2)$. Finally, except for 0, the real parts of these small eigenvalues are positive.

This result can be seen as a generalization of Theorem 3.0.2 from [38] (up to the h^2 instead of h) as we saw that the *mild relaxation* operator (which is the collision operator studied in this reference) satisfies our hypotheses. In our case we get a localization of order h^2 because we adopt a simpler proof based on hypocoercivity (inspired by [39]) than the one presented in [38].

In order to study the long time behavior of the solutions of (5.1.4), we need a precise description of the small spectrum of P_h. To this aim, we construct in Sections 5.3 and 5.4 in the spirit of the WKB method a family of accurate quasimodes localized around the minima of V that enables us to establish sharp asymptotics of the small eigenvalues of P_h. This leads in Section 5.5 to the establishment of Theorem 5.1.7 which is the main result of this paper. For the sake of simplicity, we make in the statement an additionnal assumption (Hypothesis 5.3.5) on the topology of the potential V that could actually be omitted (see [31] or [4]). It implies in particular that V has a unique global minimum that we denote $\underline{\mathbf{m}}$. In order to be able to state our main result, we give the following Lemma which is actually a consequence of Proposition 5.4.7 and Lemma 5.4.8.

Lemma 5.1.6. *Recall the matrix M_0 from Hypothesis 5.1.2 and let $\mathbf{m} \in \mathcal{U}^{(0)} \backslash \{\underline{\mathbf{m}}\}$ and $\mathbf{s} \in \mathbf{j}(\mathbf{m})$ where $\mathbf{j}$ is the topological map defined in 5.3.4. The matrix*

$$\Phi^{\mathbf{s}} = \begin{pmatrix} 0 & -\mathrm{Hess}_{\mathbf{s}} V \\ \mathrm{Id} & M_0(\mathbf{s}, 0, 0) \end{pmatrix}$$

has only one eigenvalue in $\{\mathrm{Re}\, z < 0\}$ which is actually real and that we denote $-\alpha_0^{\mathbf{s}}$.

According to Theorem 5.1.5, we can associate to each $\mathbf{m} \in \mathcal{U}^{(0)} \backslash \{\underline{\mathbf{m}}\}$ a non zero exponentially small eigenvalue of P_h that we denote $\lambda(\mathbf{m}, h)$.

Theorem 5.1.7. *Suppose that Hypotheses 5.1.2, 5.1.4 and 5.3.5 are satisfied and recall the notation $\alpha_0^{\mathbf{s}}$ from Lemma 5.1.6. The exponentially small eigenvalues of P_h satisfy the following formula :*

$$\lambda(\mathbf{m}, h) = h e^{-2\frac{S(\mathbf{m})}{h}} \frac{\det(\mathrm{Hess}_{\mathbf{m}} V)^{1/2}}{2\pi} B_h(\mathbf{m})$$

where $B_h(\mathbf{m})$ admits a classical expansion whose first term is

$$\sum_{\mathbf{s} \in \mathbf{j}(\mathbf{m})} |\det(\mathrm{Hess}_{\mathbf{s}} V)|^{-1/2}\, \alpha_0^{\mathbf{s}}$$

and the maps S and $\mathbf{j}$ are defined in Definition 5.3.4.

When Hypothesis 5.1.2 is replaced by Hypothesis 5.1.1, we can give a slightly more precise statement. In that case, denoting $\boldsymbol{\mu}_{\mathbf{s}}$ the only negative eigenvalue of $\mathrm{Hess}_{\mathbf{s}} V$, the first term of $B_h(\mathbf{m})$ is

$$\frac{1}{2} \sum_{\mathbf{s} \in \mathbf{j}(\mathbf{m})} |\det(\mathrm{Hess}_{\mathbf{s}} V)|^{-1/2} \Big(-\varrho'(0) + \sqrt{\varrho'(0)^2 - 4\boldsymbol{\mu}_{\mathbf{s}}} \Big). \tag{5.1.9}$$

Indeed, under Hypothesis 5.1.1, it is shown in Appendix 5.7.1, more precisely in (5.7.13) that $M_0(\mathbf{s},0,0) = \tilde{\varrho}(0)\,\mathrm{Id} = \varrho'(0)\,\mathrm{Id}$. Thanks to Proposition 5.4.7 from which we keep the notations, we then have

$$\mathrm{Hess}_{\mathbf{s}} V \nu_2 = -\varrho'(0)^2(1+\nu_2^2)\nu_2^2\,\nu_2$$

and consequently

$$\alpha_0^{\mathbf{s}} = \varrho'(0)\nu_2^2 = -\frac{\varrho'(0)}{2} + \frac{\sqrt{\varrho'(0)^2 - 4\boldsymbol{\mu}_{\mathbf{s}}}}{2}$$

so the statement follows.

Remark 5.1.8. *The case of the Fokker-Planck operator, i.e when $\varrho(t) = t$ and $M^h = \mathrm{Id}$ is not covered by Theorem 5.1.7 as it does not fit its hypotheses. However, formally applying our formula (5.1.9) to this case, we still recover the one from [4,22] for this operator (be careful that our notation $\boldsymbol{\mu}_{\mathbf{s}}$ and the notation $\mu(\mathbf{s})$ from [4] do not stand for the same object).*

Finally, Section 5.6 consists in using the sharp localization obtained in Theorem 5.1.7 in order to discuss the phenomena of return to equilibrium and metastability for the solutions of (5.1.4). More precisely, we are able to give a sharp rate of convergence of the semigroup $\mathrm{e}^{-tP_h/h}$ towards $\mathbb{P}_1$, the orthogonal projector on $\mathrm{Ker}\,P_h$: denoting λ^* a non zero eigenvalue of P_h whose real part is minimal, we establish that the rate of return to equilibrium is essentially given by $\mathrm{Re}\,\lambda^*/h$:

Corollary 5.1.9. *Under the assumptions of Theorem 5.1.7, for any $N \geq 1$, there exists $C_N > 0$ and $h_0 > 0$ such that for all $0 < h \leq h_0$ and $t \geq 0$,*

$$\|\mathrm{e}^{-tP_h/h} - \mathbb{P}_1\| \leq C_N \mathrm{e}^{-t\,\mathrm{Re}\,\lambda^*(1-C_N h^N)/h}.$$

Moreover, if λ^* does not share its expansion given by Theoerm 5.1.7 with another eigenvalue of P_h (in particular it is a simple eigenvalue), then λ^* is real and we even have

$$\|\mathrm{e}^{-tP_h/h} - \mathbb{P}_1\| \leq C\mathrm{e}^{-t\lambda^*/h}.$$

Besides, in the spirit of [4], we also show the metastable behavior of the solutions of (5.1.4) :

Corollary 5.1.10. *Suppose that the assumptions of Theorem 5.1.7 hold true. Let us consider some local minima $\mathbf{m}_1 = \underline{\mathbf{m}}$, $\mathbf{m}_2, \dots, \mathbf{m}_K$ such that*

$$S(\mathcal{U}^{(0)}) = \{+\infty = S(\mathbf{m}_1) > S(\mathbf{m}_2) > \dots > S(\mathbf{m}_K)\}$$

for the map S from Definition 5.3.4. For $2 \leq k \leq K$, denote $\mathbb{P}_k$ the spectral projection associated to the eigenvalues that are $O\big(\mathrm{e}^{-2\frac{S(\mathbf{m}_k)}{h}}\big)$. Then for any times $(t_k^{\pm})_{1\leq k\leq K}$ satisfying

$$t_K^- \geq h^{-1}|\ln(h^\infty)| \quad \text{and} \quad t_k^- \geq |\ln(h^\infty)|\mathrm{e}^{2\frac{S(\mathbf{m}_{k+1})}{h}} \quad \text{for} \quad k = 1,\dots,K-1$$

as well as

$$t_1^+ = +\infty \qquad \text{and} \quad t_k^+ = O\Big(h^\infty \mathrm{e}^{2\frac{S(\mathbf{m}_k)}{h}}\Big) \qquad \text{for} \quad k = 2,\dots,K$$

one has

$$\mathrm{e}^{-tP_h/h} = \mathbb{P}_k + O(h^\infty) \qquad \text{on } [t_k^-, t_k^+].$$

In other words, we have shown the existence of timescales on which, during its convergence towards the global equilibrium, the solution of (5.1.4) will essentially visit the metastable spaces associated to the small eigenvalues of P_h.

The results presented in this paper should be reasonably easy to adapt to the case of collision operators satisfying Hypothesis 5.1.2 with the space S^0 replaced by S^κ for $\kappa \in [0, 1/2[$ (we should get some expansions in powers of $h^{1-2\kappa}$ instead of just h). Another perspective would then be to study the critical case $\kappa = 1/2$ which should in particular cover the *linear relaxation* collision operator corresponding to the linear BGK model

$$Q_h = h(1 - \Pi_h) \tag{5.1.10}$$

where, using the notation (5.1.2),

$$\Pi_h : L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^{2d}) \tag{5.1.11}$$

denotes the orthogonal projection on

$$E_h = \mu_h \, L^2(\mathbb{R}^d_x) \tag{5.1.12}$$

and for which Robbe gave a first localization of the small spectrum of the associated operator $X_0^h + Q_h$ in [39].

5.2 Rough description of the small spectrum

Throughout the paper, we assume that Hypotheses 5.1.2 and 5.1.4 hold true. This implies in particular that Q_h is bounded uniformly in h and self-adjoint in $L^2(\mathbb{R}^{2d})$. Let us begin with a Lemma which consists in comparing our collision operator with the one introduced in (5.1.10) and studied in [39]. This will in particular enable us to use some computations from [39] later on.

Lemma 5.2.1. *There exists $h_0 > 0$ such that for all $0 < h < h_0$,*

$$Q_h \geq \frac{h}{C}(1 - \Pi_h)$$

where Π_h is the projection introduced in (5.1.11). In particular, Q_h is non negative.

Proof. Since the space E_h defined in (5.1.12) is contained in $\operatorname{Ker} Q_h$, it is enough to prove that $\langle Q_h u, u\rangle \geq \frac{h}{C}\|u\|^2$ for $u \in E_h^\perp$. Let $u \in E_h^\perp$ and recall the notations H_0 and H_1 from (5.1.5) and (5.7.4). Let us consider an approximate square root A of $(1 + H_1)$ given by

$$A = \operatorname{Op}_h\Big(\big(1 + v^2/4 + \eta^2 + h(1 - d/2)\big)^{1/2}\mathrm{Id}\Big) \in \Psi^0\big(\langle(v,\eta)\rangle\big).$$

By symbolic calculus, we easily have $A^2 = 1 + H_1 + h^2 R_1$ with $R_1 \in \Psi^0\big(\langle(v,\eta)\rangle^2\big)$. Besides, the symbol of A is clearly elliptic so A is invertible and its inverse is also a pseudo-differential operator satisfying $A^{-2} = (1 + H_1)^{-1} + h^2 R_2$ with $R_2 \in \Psi^0\big(\langle(v,\eta)\rangle^{-2}\big)$ (see for instance [12], chapter 8). Thus, using the factorization from Hypothesis 5.1.2 and the self-adjointness of A, we get

$$\langle Q_h u, u\rangle = \big\langle A\operatorname{Op}_h(M^h)A\, A^{-1}b_h u\,,\, A^{-1}b_h u\big\rangle.$$

Now according to Hypothesis 5.1.2 and symbolic calculus again, the principal symbol of $A\operatorname{Op}_h(M^h)A$ is elliptic so we can use the Gårding inequality to write

$$\begin{aligned}\langle Q_h u, u\rangle &\geq \frac{1}{C}\big\langle A^{-2}b_h u\,, b_h u\big\rangle \\ &\geq \frac{1}{C}\big\langle b_h^*(1 + H_1)^{-1}b_h u\,, u\big\rangle - \frac{h^2}{C}\big|\big\langle b_h^*\, R_2\, b_h u\,, u\big\rangle\big|.\end{aligned}$$

Still using symbolic calculus, we get $b_h^*\, R_2\, b_h = O(1)$ so applying (5.7.5) we finally have

$$\langle Q_h u, u\rangle \geq \frac{1}{C}\big\langle H_0(1 + H_0)^{-1}u\,, u\big\rangle - O(h^2)\|u\|^2$$

and the conclusion comes from the fact that the spectrum of $H_0(1 + H_0)^{-1}|_{E_h^\perp}$ is contained in $[h/C, +\infty[$. □

We can already prove that 0 is a simple eigenvalue of (P_h, D) and that the other eigenvalues have positive real part. It is easy to check that $\mathcal{M}_h$ defined in (5.1.2) is in $\operatorname{Ker} P_h$. Now let $\lambda \in \mathbb{R}$ and let us prove that for $u \in \operatorname{Ker}\,(P_h - i\lambda)$, one has $u \in \mathbb{C}\,\mathcal{M}_h$. Since X_0^h is skew-adjoint and Q_h is self-adjoint and non-negative, we have

$$0 = \operatorname{Re}\langle(P_h - i\lambda)u, u\rangle = \|Q_h^{1/2}u\|^2$$

so in particular $u \in \operatorname{Ker}\, Q_h = E_h$ according to Lemma 5.2.1. Therefore, $u = w\mu_h$ with $w \in L^2(\mathbb{R}^d_x)$ and using that $\mu_h^{-1}X_0^h u = i\lambda w$ does not depend on v, we get in the sense of distributions $\partial_x(\mathrm{e}^{V/2h}w) = 0$ which yields the desired result.

5.2.1 Hypocoercivity

Let us now use the dilatation operators

$$S_h : \begin{cases} L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^{2d}) \\ \qquad u \mapsto h^{-d/2} u\Big(\frac{\cdot}{\sqrt{h}}\,,\,\frac{\cdot}{\sqrt{h}}\Big) \end{cases} \qquad\qquad T_h : \begin{cases} L^2(\mathbb{R}^{d}_x) \to L^2(\mathbb{R}^{d}_x) \\ \qquad u \mapsto h^{-d/4} u\Big(\frac{\cdot}{\sqrt{h}}\Big) \end{cases}$$

that were introduced in [39] in which these were combined with a scaling of Π_h to conjugate P_h to a non-semiclassical operator with h-dependent potential. In our case, it will enable us to use some computations and results already established in [39].

Lemma 5.2.2. *Recall the notation (5.1.8). Denoting*

$$X_0 = v \cdot \partial_x - \partial_x V_h(x) \cdot \partial_v$$

where $V_h = h^{-1} V(\sqrt{h}\,\cdot)$,

$$\tilde{Q}_1 = h^{-1} S_h^{-1} Q_h S_h$$

and

$$\mathrm{Dom}\,(P) = \{u \in L^2(\mathbb{R}^{2d})\,;\, X_0 u \in L^2(\mathbb{R}^{2d})\}, \qquad P = X_0 + \tilde{Q}_1,$$

one has

$$(hP\,,\,\mathrm{Dom}(P)) = (S_h^{-1} P_h S_h\,,\,S_h^{-1} D).$$

Moreover,

$$(hP\,,\,\mathrm{Dom}(P))^* = (S_h^{-1} P_h^* S_h\,,\,S_h^{-1} D).$$

Proof. We have for $u \in L^2(\mathbb{R}^{2d})$

$$hX_0 u = S_h^{-1} X_0^h S_h u$$

so using that S_h is bounded we get $\mathrm{Dom}\,(P) = S_h^{-1} D$. Consequently,

$$(hP\,,\,\mathrm{Dom}(P)) = (S_h^{-1} P_h S_h\,,\,S_h^{-1} D)$$

and the result for the adjoint follows immediately. □

We also recall the notations of the following differential operators from [21] and [39] :

$$a = \partial_x + \frac{\partial_x V_h}{2} \quad ; \quad b = \partial_v + \frac{v}{2} \quad \text{and} \quad \Lambda^2 = a^* a + b^* b + 1.$$

The operator $(\Lambda^2, \mathcal{C}_c^\infty(\mathbb{R}^{2d}))$ is essentially self-adjoint. The Schwartz space $\mathcal{S}(\mathbb{R}^{2d})$ is included in the domain of its self-adjoint extension $(\Lambda^2, D(\Lambda^2))$ which is invertible. We can then define the operator

$$L = \Lambda^{-2} a^* b \tag{5.2.1}$$

which is bounded uniformly in h (see [39], Lemma 2.7), as well as the perturbation $h\varepsilon(L + L^*) = O(h)$ where $\varepsilon > 0$ will be chosen small enough later.
Besides, notice that $a^* a = -\Delta_x + |\partial_x V_h|^2/4 - \Delta V_h/2 =: \Delta_{V_h/2}$ is the Witten Laplacian in x associated to the potential $V_h/2$ and that

$$\begin{aligned} \Delta^h_{V/2} &:= h T_h a^* a T_h^{-1} \\ &= -h^2 \Delta_x + |\partial_x V|^2/4 - h\Delta V/2 \end{aligned}$$

is the semi-classical Witten Laplacian associated to the potential $V/2$. The small spectrum of this operator was first studied by Helffer and Sjöstrand in [19] and we now know (see for instance [18], Definition 4.3) that we can construct an orthonormal family $(\varphi_j)_{1 \leq j \leq n_0} \subset \mathcal{C}_c^\infty(\mathbb{R}^d_x)$ of quasimodes associated to this operator given by

$$\varphi_j = \chi_j \mathrm{e}^{-\frac{V - V(x_j)}{2h}}$$

where x_j is one of the local minima of V and χ_j is a cut-off function localizing around x_j. Recall the notation μ_h from (5.1.2) and let us now define the families of functions

$$g_j^h = \varphi_j \mu_h \qquad \text{and} \qquad g_j = S_h^{-1} g_j^h$$

for $1 \leq j \leq n_0$. These are actually quasimodes for our operators P_h and P_h^* :

Lemma 5.2.3. *The family $(g_j^h)_{1\le j\le n_0}$ is orthonormal and there exists $\alpha>0$ such that for all $1\le j\le n_0$,*

$$P_h g_j^h = O_{L^2}(\mathrm{e}^{-\frac{\alpha}{h}}), \qquad P_h^* g_j^h = O_{L^2}(\mathrm{e}^{-\frac{\alpha}{h}}).$$

Moreover, $P_h g_j^h$ and $P_h^* g_j^h$ are in $\mathcal{S}(\mathbb{R}^{2d})\subseteq D$ and we have

$$P_h^* P_h g_j^h = O_{L^2}(\mathrm{e}^{-\frac{\alpha}{h}}), \qquad P_h P_h^* g_j^h = O_{L^2}(\mathrm{e}^{-\frac{\alpha}{h}}).$$

Proof. The proof is the same as the one of Lemma 2.4 from [39] since with the notation (5.1.12) and Lemma 5.2.1 we also have $E_h = \mathrm{Ker}\ Q_h$. □

One of the key results of this section is that the real part of the perturbation of our operator is bounded from below on a subspace of finite codimension given by the orthogonal of the quasimodes. In order to state it, recall the notation (5.2.1) and denote $N_{h,\varepsilon}^{\pm}$ the bounded self-adjoint operator

$$\mathrm{Id} \pm \varepsilon h(L+L^*).$$

Proposition 5.2.4. *There exists $\varepsilon>0$ and $h_0>0$ such that for all $h\in]0,h_0]$ and $u\in\mathcal{S}(\mathbb{R}^{2d})\cap(g_j)^{\perp}_{1\le j\le n_0}$, one has*

$$\mathrm{Re}\langle N_{h,\varepsilon}^{+} P u, u\rangle \ge \frac{h}{C}\|u\|^2$$

as well as

$$\mathrm{Re}\langle N_{h,\varepsilon}^{-} P^* u, u\rangle \ge \frac{h}{C}\|u\|^2.$$

Proof. One has for $u\in\mathcal{S}(\mathbb{R}^{2d})$, using the fact that X_0 is skew-adjoint :

$$\begin{aligned}
\mathrm{Re}\langle N_{h,\varepsilon}^{+} P u, u\rangle &= \mathrm{Re}\langle Pu, N_{h,\varepsilon}^{+} u\rangle \\
&= \mathrm{Re}\langle \tilde{Q}_1 u, N_{h,\varepsilon}^{+} u\rangle + \mathrm{Re}\langle X_0 u, N_{h,\varepsilon}^{+} u\rangle \\
&= \|\tilde{Q}_1^{1/2} u\|^2 + h\varepsilon \mathrm{Re}\langle \tilde{Q}_1 u, (L+L^*)u\rangle + h\varepsilon\mathrm{Re}\langle X_0 u, (L+L^*)u\rangle \\
&= \|\tilde{Q}_1^{1/2} u\|^2 + h\varepsilon \mathrm{Re}\langle \tilde{Q}_1 u, (L+L^*)u\rangle + h\varepsilon\mathrm{Re}\langle [L,X_0]u, u\rangle \\
&= I + hII + hIII
\end{aligned}$$

Note that if we replace P by P^* and $N_{h,\varepsilon}^{+}$ by $N_{h,\varepsilon}^{-}$, we get $I-hII+hIII$. Besides, it is also proven in [39] that

$$[L,X_0] = \mathcal{A} + \Lambda^{-2}a^*a$$

where $\mathcal{A}$ is also bounded uniformly in h. Since $\|Q_h\|\le C$ and $Q_h \ge \frac{h}{C}(1-\Pi_h)$ according to Lemma 5.2.1, we get $\|\tilde{Q}_1\|\le \frac{C}{h}$ and $\tilde{Q}_1 \ge \frac{1}{C}(1-\Pi_1)$. Hence

$$\begin{aligned}
I \pm hII &\ge I - h|II| \\
&\ge \|\tilde{Q}_1^{1/2}u\|^2 - h\varepsilon\|\tilde{Q}_1 u\|\|(L+L^*)u\| \\
&\ge \|\tilde{Q}_1^{1/2}u\|^2 - \sqrt{C}h^{\frac{1}{2}}\varepsilon\|\tilde{Q}_1^{1/2}u\|\|(L+L^*)u\| \\
&\ge \frac{1}{2}\|\tilde{Q}_1^{1/2}u\|^2 - 2Ch\varepsilon^2\|L\|^2\|u\|^2 \\
&\ge \frac{1}{2C}\|(1-\Pi_1)u\|^2 - 2Ch\varepsilon^2\|L\|^2\|u\|^2 \qquad (5.2.2)
\end{aligned}$$

We can combine this with the following estimate from [39] (proof of Proposition 2.5) : there exists $\delta>0$ such that for $u\in(g_j)^{\perp}_{1\le j\le n_0}$,

$$III \ge -\frac{1}{4}\|(\mathrm{Id}-\Pi_1)u\|^2 - \varepsilon^2\|\mathcal{A}\|^2\|u\|^2 + \frac{\varepsilon\delta}{4}\|\Pi_1 u\|^2 - \varepsilon\|(\mathrm{Id}-\Pi_1)u\|^2.$$

This yields for $\varepsilon < \frac{\delta}{4(\|\mathcal{A}\|^2+C\|L\|^2)}$ that

$$
\begin{aligned}
I \pm hII + hIII &\geq \frac{1}{C}\|(\mathrm{Id}-\Pi_1)u\|^2 + h\frac{\varepsilon\delta}{4}\|\Pi_1 u\|^2 - h\varepsilon^2\Big(\|\mathcal{A}\|^2 + C\|L\|^2\Big)\|u\|^2 \\
&\geq \frac{h}{C}\|u\|^2. \qquad (5.2.3)
\end{aligned}
$$

so the proof is complete. □

This result extends to $u \in (g_j)^{\perp}_{1\leq j\leq n_0} \cap \mathrm{Dom}\,(P)$ since $\mathcal{S}(\mathbb{R}^{2d})$ is a core for both $(P, \mathrm{Dom}\,(P))$ and $(P^*, \mathrm{Dom}\,(P^*))$. It only differs from Proposition 2.5 in [39] by a factor h in the estimate. This comes from the fact that in our case, $\tilde{Q}_1 = O(h^{-1})$ and not $O(1)$ (because $Q_h = O(1)$ and not $O(h)$) so we have to use a perturbation of order h (the operator $N^{\pm}_{h,\varepsilon}$) to obtain the gain in $\|(1-\Pi_1)u\|^2$ in (5.2.2). As a consequence, the gain in $\|\Pi_1 u\|^2$ from (5.2.3) is of order h and not of order 1.

Corollary 5.2.5. *There exists $c > 0$ and $h_0 > 0$ such that for all $h \in]0, h_0]$, $u \in D \cap (g^h_j)^{\perp}_{1\leq j\leq n_0}$ and $z \in \mathbb{C}$ with $\mathrm{Re}\, z \leq ch^2$*

$$
\|(P_h - z)u\| \geq ch^2\|u\| \qquad \text{and} \qquad \|(P_h^* - z)u\| \geq ch^2\|u\|.
$$

Proof. Recall that $N^+_{h,\varepsilon} = 1 + O(h)$. Hence, for $u \in D \cap (g^h_j)^{\perp}_{1\leq j\leq n_0}$, we have by putting $u = S_h w$ and using that S_h is unitary

$$
\begin{aligned}
\|(P_h - z)u\|\|u\| &\geq \frac{1}{2}\|(P_h - z)u\|\|N^+_{h,\varepsilon}w\| \\
&\geq \frac{1}{2}\mathrm{Re}\langle (P_h - z)u, S_h N^+_{h,\varepsilon} w\rangle \\
&= \frac{1}{2}\mathrm{Re}\langle N^+_{h,\varepsilon}(hP - z)w, w\rangle \\
&\geq \frac{h^2}{C}\|u\|^2 - \mathrm{Re}\, z\|N^+_{h,\varepsilon}\|\|u\|^2 \\
&\geq \frac{h^2}{2C}\|u\|^2
\end{aligned}
$$

if $\mathrm{Re}\, z \leq h^2/2C$. The same proof holds when replacing P by P^* and $N^+_{h,\varepsilon}$ by $N^-_{h,\varepsilon}$. □

5.2.2 Resolvent estimates and first localization of the small eigenvalues

Using Lemma 5.2.3, it is clear that for $u \in \mathrm{Span}\big((g^h_j)_{1\leq j\leq n_0}\big)$ and $A \in \{P_h, P_h^*, P_h^* P_h, P_h P_h^*\}$ we have

$$
\|Au\|^2 = O(\mathrm{e}^{-\frac{2\alpha}{h}})\|u\|^2.
$$

Now if we denote $\mathbb{P}$ the orthogonal projection on $\mathrm{Span}\big((g^h_j)_{1\leq j\leq n_0}\big)$, we get by using Corollary 5.2.5 that for $z \in \mathbb{C}$ such that $\mathrm{Re}\ z \leq ch^2$ and $u \in D$

$$
\begin{aligned}
\|(P_h - z)u\|^2 &= \|(P_h - z)(\mathrm{Id} - \mathbb{P})u + (P_h - z)\mathbb{P}u\|^2 \\
&= \|(P_h - z)(\mathrm{Id} - \mathbb{P})u\|^2 + \|(P_h - z)\mathbb{P}u\|^2 + 2\mathrm{Re}\langle (P_h - z)(\mathrm{Id} - \mathbb{P})u, (P_h - z)\mathbb{P}u\rangle \\
&\geq c^2h^4\|(\mathrm{Id} - \mathbb{P})u\|^2 + |z|^2\|\mathbb{P}u\|^2 - O(\mathrm{e}^{-\frac{\alpha}{h}})\|u\|^2 + 2\mathrm{Re}\langle (P_h - z)(\mathrm{Id} - \mathbb{P})u, (P_h - z)\mathbb{P}u\rangle.
\end{aligned}
$$

The last term equals

$$
2\mathrm{Re}\Big[\langle (\mathrm{Id} - \mathbb{P})u, P_h^* P_h \mathbb{P}u\rangle - z\langle (\mathrm{Id} - \mathbb{P})u, P_h\mathbb{P}u\rangle - \bar{z}\langle (\mathrm{Id} - \mathbb{P})u, P_h^*\mathbb{P}u\rangle\Big] = (1 + |z|)O(\mathrm{e}^{-\frac{\alpha}{h}})\|u\|^2.
$$

Therefore choosing $\tilde{c} \leq c$, there exists $h_0 > 0$ such that for $h \leq h_0$ and z such that $\tilde{c}h^2 \leq |z| \leq ch^2$

$$
\|(P_h - z)u\|^2 \geq \Big(|z|^2 + O(\mathrm{e}^{-\frac{\alpha}{h}})\Big)\|u\|^2 \geq \frac{\tilde{c}^2h^4}{2}\|u\|^2.
$$

Once again, the same estimate holds with P_h^* instead of P_h and since the annulus we are working on is invariant by complex conjugation we also have

$$\|(P_h - z)^* u\| \geq \frac{\tilde{c}h^2}{2}\|u\|\,.$$

Therefore, we get the following resolvent estimate on the annulus centered in 0 and of radiuses $\tilde{c}h^2$ and ch^2 :

$$\|(P_h - z)^{-1}\| = O(h^{-2}) \quad \text{for } \tilde{c}h^2 \leq |z| \leq ch^2. \tag{5.2.4}$$

We can now consider the spectral projection

$$\Pi_0 = \frac{1}{2i\pi}\int_{|z|=ch^2}(z - P_h)^{-1}\mathrm{d}z \tag{5.2.5}$$

and its range that we denote H. This operator will yield some information on $\mathrm{Spec}(P_h) \cap B(0, ch^2)$ and therefore enable us to prove the main statement from Theorem 5.1.5.

The main point is that H is of dimension n_0. It can be obtained by a direct adaptation of the proof of Proposition 3.1 from [39]. Hence $\mathrm{Spec}(P_h) \cap B(0, ch^2)$ which is the same as $\mathrm{Spec}(P_h|_H)$ consists of n_0 eigenvalues (counted with algebraic multiplicity). Here again, our result slightly differs from the one in [39] as we do not rule out the possibilities that $P_h|_H$ contains some Jordan blocks and that some of its eigenvalues are not real. It only remains to prove that these are exponentially small with respect to $1/h$. We begin by noticing that thanks to Lemma 5.2.3, we have $(z - P_h)g_j^h = zg_j^h + O(\mathrm{e}^{-\frac{\alpha}{h}})$ and $(z - P_h^*)g_j^h = zg_j^h + O(\mathrm{e}^{-\frac{\alpha}{h}})$ from which we easily deduce

$$\Pi_0 g_j^h = g_j^h + O(\mathrm{e}^{-\frac{\alpha}{h}}) \qquad \text{and} \qquad \Pi_0^* g_j^h = g_j^h + O(\mathrm{e}^{-\frac{\alpha}{h}}). \tag{5.2.6}$$

In particular, $(\Pi_0 g_j^h)_{1\leq j\leq n_0}$ is almost orthonormal so for $u = \sum u_j \Pi_0 g_j^h \in H$, we have

$$\|u\|^2 = \big(1 + O(\mathrm{e}^{-\alpha/h})\big)\sum_{j=1}^{n_0}|u_j|^2.$$

Therefore it is enough to prove that P_h is exponnentially small on $(\Pi_0 g_j^h)_{1\leq j\leq n_0}$. But thanks to the resolvent estimate (5.2.4), it is easy to see that $\Pi_0 = O(1)$ and since P_h and Π_0 commute, we get the desired result.

To complete the proof of Theorem 5.1.5, it only remains to show the existence of the resolvent on $\{\mathrm{Re}\, z \leq ch^2\}\backslash B(0, \tilde{c}h^2)$ as well as the estimate in $O(h^{-2})$.

Lemma 5.2.6. *Denote $\hat{\Pi}_0 = 1 - \Pi_0$. For all $u \in L^2(\mathbb{R}^{2d})$, we have*

$$\hat{\Pi}_0 u = w + r$$

whith $w \in (g_j^h)^{\perp}_{1\leq j\leq n_0}$ and $r \in \mathrm{Span}\big((g_j^h)_{1\leq j\leq n_0}\big)$ satisfying $r = O(\mathrm{e}^{-\frac{\alpha}{h}})\|\hat{\Pi}_0 u\|$.

Proof. First we take for r the orthogonal projection of $\hat{\Pi}_0 u$ on $\mathrm{Span}\big((g_j^h)_{1\leq j\leq n_0}\big)$. Then we notice that using (5.2.6), we get

$$\langle g_j^h, \hat{\Pi}_0 u\rangle = \langle \hat{\Pi}_0^* g_j^h, \hat{\Pi}_0 u\rangle = O(\mathrm{e}^{-\frac{\alpha}{h}})\|\hat{\Pi}_0 u\|$$

which implies the announced estimate. □

Lemma 5.2.7. *For all $r' \in \mathrm{Span}\big((g_j)_{1\leq j\leq n_0}\big)$, we have $N_{h,\varepsilon}^{\pm} r' \in \mathrm{Dom}\,(P^*) = \mathrm{Dom}\,(P)$. Moreover, the restrictions to the finite dimensional subspace $\mathrm{Span}\big((g_j)_{1\leq j\leq n_0}\big)$ of the operators $PN_{h,\varepsilon}^{\pm}$ and $P^*N_{h,\varepsilon}^{\pm}$ are all $O(1)$.*

Proof. For the first statement, it is sufficient to show that for $1 \leq j \leq n_0$, the functions Lg_j and L^*g_j are both in $\mathrm{Dom}\,(P)$. But we have in the sense of distributions

$$X_0 L g_j = [X_0, L]g_j + LX_0 g_j \tag{5.2.7}$$

and we saw in the proof of Proposition 5.2.4 that $[X_0, L]$ is a bounded operator on $L^2(\mathbb{R}^{2d})$ so it is then clear that $X_0 L g_j \in L^2(\mathbb{R}^{2d})$ i.e $Lg_j \in \mathrm{Dom}\,(P)$. The same goes easily for L^*g_j. For the second statement, using Lemma 5.2.3 and the fact that $\tilde{Q}_1 = O(h^{-1})$, it suffices to notice that for $1 \leq j \leq n_0$, (5.2.7) implies that $X_0 L g_j$ and $X_0 L^* g_j$ are both $O(1)$ as we saw that L and $[X_0, L]$ are $O(1)$. □

Proposition 5.2.8. *Consider $\hat{P}_h$ the restriction of P_h to $\hat{\Pi}_0 D$ acting on $\hat{\Pi}_0 L^2(\mathbb{R}^{2d})$. Then for all $z \in \mathbb{C}$ such that* Re $z \leq ch^2$*, the resolvent $(\hat{P}_h - z)^{-1}$ exists and we have the uniform estimate*

$$(\hat{P}_h - z)^{-1} = O(h^{-2}).$$

Proof. We actually prove that the result of Proposition 5.2.4 remains true when replacing the set $(g_j)^{\perp}_{1\leq j\leq n_0} \cap \mathrm{Dom}\,(P)$ by $S_h^{-1}\hat{\Pi}_0 D$. We will deduce that the result of Corollary 5.2.5 also remains true when taking $u \in \hat{\Pi}_0 D$ instead of $(g_j^h)^{\perp}_{1\leq j\leq n_0} \cap D$, which is precisely the statement that we want to prove. Let $u \in D$, using the notations from Lemma 5.2.6 we have

$$\begin{aligned}\mathrm{Re}\,\langle PS_h^{-1}\hat{\Pi}_0 u, N^+_{h,\varepsilon}S_h^{-1}\hat{\Pi}_0 u\rangle &= \mathrm{Re}\,\langle PS_h^{-1}w, N^+_{h,\varepsilon}S_h^{-1}w\rangle + \mathrm{Re}\,\langle PS_h^{-1}w, N^+_{h,\varepsilon}S_h^{-1}r\rangle \\ &\quad + \mathrm{Re}\,\langle PS_h^{-1}r, N^+_{h,\varepsilon}S_h^{-1}w\rangle + \mathrm{Re}\,\langle PS_h^{-1}r, N^+_{h,\varepsilon}S_h^{-1}r\rangle\,.\end{aligned}$$

Now let us denote $w' = S_h^{-1}w \in (g_j)^{\perp}_{1\leq j\leq n_0} \cap \mathrm{Dom}\,(P)$ and $r' = S_h^{-1}r \in \mathrm{Span}\big((g_j)_{1\leq j\leq n_0}\big)$. We can use Proposition 5.2.4 as well as Lemmas 5.2.6 and 5.2.7 to get

$$\begin{aligned}\mathrm{Re}\,\langle N^+_{h,\varepsilon}PS_h^{-1}\hat{\Pi}_0 u, S_h^{-1}\hat{\Pi}_0 u\rangle &= \mathrm{Re}\,\langle Pw', N^+_{h,\varepsilon}w'\rangle + \mathrm{Re}\,\langle w', P^* N^+_{h,\varepsilon}r'\rangle + \mathrm{Re}\,\langle N^+_{h,\varepsilon}Pr', w'\rangle + \mathrm{Re}\,\langle Pr', N^+_{h,\varepsilon}r'\rangle \\ &\geq \frac{h}{C}\|w\|^2 - O\big(\|w\|\,\|r\|\big) - O(\mathrm{e}^{-\frac{\alpha}{h}}\|r\|) \\ &\geq \frac{h}{2C}\|S_h^{-1}\hat{\Pi}_0 u\|^2.\end{aligned}$$

As usual, all of the above remains true with P^* and $N^-_{h,\varepsilon}$ instead of P and $N^+_{h,\varepsilon}$ so the proof is now complete. ☐

End of Proof of Theorem 5.1.5 : Let $z \in \mathbb{C}$ satisfying $\mathrm{Re}\, z \leq ch^2$ and $|z| \geq \tilde{c}h^2$ and recall the notation $H = \mathrm{Ran}\,\Pi_0$. We already know from Proposition 5.2.8 that $\hat{P}_h - z$ is invertible, but it is clearly also the case of $P_h|_H - z$ since $P_h|_H = O(\mathrm{e}^{-\alpha/h})$. Therefore $P_h - z$ is invertible and we have

$$(P_h - z)^{-1} = (\hat{P}_h - z)^{-1}\hat{\Pi}_0 + (P_h|_H - z)^{-1}\Pi_0. \tag{5.2.8}$$

Besides, we easily have for such z that $\|(P_h|_H - z)u\| \geq \frac{1}{C}h^2\|u\|$ which combined with (5.2.8), Proposition 5.2.8 and the fact that $\|\Pi_0\| = O(1)$ yields the estimate $(P_h - z)^{-1} = O(h^{-2})$. ☐

5.3 Accurate quasimodes

5.3.1 General form

Let us denote

$$W(x,v) = \frac{V(x)}{2} + \frac{v^2}{4}$$

the global potential on $\mathbb{R}^{2d}$. The introduction of our quasimodes will rely on the topological constructions described in Appendix C. In our case, it has to be done for the global potential, i.e the function W. However, by the definition of W, a strong connection between these constructions for W and the ones for V will appear, leading to simplifications. In that spirit, let us state a Lemma that will enable us to show that, roughly speaking, the previous constructions for $Y = V/2$ are the projections on $\mathbb{R}^d_x$ of the ones for $Y = W$. First, we give the following easy observation.

Remark 5.3.1. *By definition of W, we have $V/2 = W(\cdot, 0)$. Moreover, if $(x_0, v_0) \in \{W < \sigma\}$, then $\{x_0\} \times B(0, |v_0|) \subseteq \{W < \sigma\}$.*

Using the notations from Appendix C, we denote for shortness $\mathcal{C}_\sigma = \mathcal{C}^{V/2}_\sigma$ and $\widetilde{\mathcal{C}}_\sigma = \mathcal{C}^W_\sigma$ as well as $\mathcal{U}^{(k)} = \mathcal{U}^{(k),V/2}$ and $\widetilde{\mathcal{U}}^{(k)} = \mathcal{U}^{(k),W}$ (we do similarly with $\mathtt{V}$ or $\mathcal{U}_k$ instead of $\mathcal{U}$). Notice that $\widetilde{\mathcal{U}}^{(k)} = \mathcal{U}^{(k)} \times \{0\}$. We introduce the natural projection $\pi_x : \mathbb{R}^{2d} \to \mathbb{R}^d_x$ sending (x, v) on x that we also consider as a map from $\mathcal{P}(\mathbb{R}^{2d})$ to $\mathcal{P}(\mathbb{R}^d_x)$.

Lemma 5.3.2. *For all $\sigma \in \mathbb{R}$, the projection π_x sends $\widetilde{\mathcal{C}}_\sigma$ in $\mathcal{C}_\sigma$. Moreover, the map $\pi_x : \widetilde{\mathcal{C}}_\sigma \to \mathcal{C}_\sigma$ is bijective.*

Proof. The proof of the first statement is an easy consequence of Remark 5.3.1. For the second statement, let $x \in E \in \mathcal{C}_\sigma$ and denote $\tilde{E}$ the element of $\widetilde{\mathcal{C}}_\sigma$ containing $(x, 0)$. By the first statement, we necessarily have $\pi_x(\tilde{E}) = E$ so we have shown the surjectivity. Now let $\tilde{E}_1$, $\tilde{E}_2 \in \widetilde{\mathcal{C}}_\sigma$ such that $\pi_x(\tilde{E}_1) = \pi_x(\tilde{E}_2) = E_1$. Let also $(x_1, v_1) \in \tilde{E}_1$ and $(x_2, v_2) \in \tilde{E}_2$. Since x_1, $x_2 \in E_1$, there exists a path $(\gamma(t), 0)$ from $(x_1, 0)$ to $(x_2, 0)$ contained in $\{W < \sigma\}$. Thus, the concatenation of the paths $(x_1, (1-t)v_1)$, $(\gamma(t), 0)$ and (x_2, tv_2) yields a path linking (x_1, v_1) and (x_2, v_2) in $\{W < \sigma\}$. Hence $\tilde{E}_1 = \tilde{E}_2$ and we get the injectivity. □

Proposition 5.3.3. *a) We have $\widetilde{\mathtt{V}}^{(1)} = \mathtt{V}^{(1)} \times \{0\}$. In particular, $V/2$ and W have the same separating saddle values.*

b) A set $\tilde{E} \in \widetilde{\mathcal{C}}_\sigma$ is critical if and only if $\pi_x(\tilde{E})$ is critical.

c) A labeling $((\mathbf{m}, 0)_{k,j})_{k,j}$ is adapted to W if and only if $(\mathbf{m}_{k,j})_{k,j}$ is adapted to $V/2$.

Moreover, given an adapted labeling, the mappings from Definition C.0.7 satisfy

$$E^{V/2}(\mathbf{m}_{k,j}) = \pi_x\big(E^W(\mathbf{m}_{k,j}, 0)\big) \quad \text{and} \quad \mathbf{j}^W(\mathbf{m}_{k,j}, 0) = \mathbf{j}^{V/2}(\mathbf{m}_{k,j}) \times \{0\}.$$

Proof. Let $\tilde{E} \in \widetilde{\mathcal{C}}_\sigma$. Thanks to Remark 5.3.1, we easily have

$$(x, 0) \in \partial\tilde{E} \iff x \in \partial\big(\pi_x(\tilde{E})\big). \tag{5.3.1}$$

a) : We already know that $\widetilde{\mathcal{U}}^{(1)} = \mathcal{U}^{(1)} \times \{0\}$. Besides, we easily deduce from (5.3.1) and Lemma 5.3.2 that $(\mathbf{s}, 0) \in \widetilde{\mathcal{U}}^{(1)}$ is in the closure of two distinct CCs of $\{W < W(\mathbf{s}, 0)\}$ if and only if $\mathbf{s} \in \mathcal{U}^{(1)}$ is in the closure of two distinct CCs of $\{V < V(\mathbf{s})\}$ so the first item is proven.
b) : This is also a straightforward consequence of (5.3.1) and Lemma 5.3.2 combined with item *a*).
c) : Let $\tilde{E} \in \widetilde{\mathcal{C}}_{\sigma_k}$. By Remark 5.3.1, we easily have

$$(\mathbf{m}, 0) \text{ is a global minimum of } W|_{\tilde{E}} \iff \mathbf{m} \text{ is a global minimum of } V|_{\pi_x(\tilde{E})}. \tag{5.3.2}$$

Besides, since $\widetilde{\mathtt{U}}_k^{(0)} = \mathtt{U}_k^{(0)} \times \{0\}$, we have that π^k defined as $\pi_x : \widetilde{\mathtt{U}}_k^{(0)} \to \mathtt{U}_k^{(0)}$ is bijective. We can then conclude as

$$T_k^W = \pi_x^{-1} \circ T_k^{V/2} \circ \pi^k \tag{5.3.3}$$

where π_x denotes the bijective map from Lemma 5.3.2.
The last statement is a direct consequence of (5.3.3), (5.3.1) and item *a*). □

From now on, we fix a labeling $(\mathbf{m}_{k,j})_{k,j}$ adapted to V.

Definition 5.3.4. *Recall the maps from Definition C.0.7. In the rest of the paper, we set*

$$\mathbf{j} = \mathbf{j}^{V/2}.$$

Moreover, in view of Proposition 5.3.3, we can also set

$$\boldsymbol{\sigma}(\mathbf{m}) = \boldsymbol{\sigma}^{V/2}(\mathbf{m}) = \boldsymbol{\sigma}^W(\mathbf{m}, 0) \qquad \text{and} \qquad S(\mathbf{m}) = S^{V/2}(\mathbf{m}) = S^W(\mathbf{m}, 0).$$

However, be careful that we choose to denote $E = \pi_x^{-1} \circ E^{V/2}$ so that the range of E is in $\mathcal{P}(\mathbb{R}^{2d})$. Following [6, 18, 23, 26], we can now state our last assumption that allows us to treat the generic case. As mentionned in the introduction, this assumption could actually be omitted (see [31] or [4]) but this would introduce additionnal difficulties that are not the main concern of this paper.

Hypothesis 5.3.5. *For all $\mathbf{m} \in \mathcal{U}^{(0)}$, we have*

a) $\mathbf{m}$ is the only global minimum of $V|_{E^{V/2}(\mathbf{m})}$

b) for any $\mathbf{m}' \in \mathcal{U}^{(0)} \backslash \{\mathbf{m}\}$, the sets $\mathbf{j}(\mathbf{m})$ and $\mathbf{j}(\mathbf{m}')$ do not intersect.

According to Proposition 5.3.3 and (5.3.2), this hypothesis is equivalent to the facts that $(\mathbf{m},0)$ is the only global minimum of $W|_{E(\mathbf{m})}$ and $\mathbf{j}^W(\mathbf{m},0)\cap\mathbf{j}^W(\mathbf{m'},0)=\emptyset$ which is what we use in practice.

Recall the notation (A.0.1) and let us extend our notions of asymptotic expansions to smooth functions that are not necessarily symbols. Throughout the paper, for $d'\in\mathbb{N}^*$, $\Omega\subseteq\mathbb{R}^{d'}$ and $a\in\mathcal{C}^\infty(\Omega)$ a function depending on h and such that for all $\beta\in\mathbb{N}^{d'}$ we have $\partial^\beta a=O_{L^\infty}(1)$, we will denote

$$a\sim_h\sum_{j\geq 0}h^ja_j, \tag{5.3.4}$$

where $(a_j)_{j\geq 0}\subset\mathcal{C}^\infty(\Omega)$ are allowed to depend on h, provided that for all $\beta\in\mathbb{N}^{d'}$ and $N\in\mathbb{N}$, there exists $C_{\beta,N}$ such that

$$\Big\|\partial^\beta\Big(a-\sum_{j=0}^{N-1}h^ja_j\Big)\Big\|_{\infty,\Omega}\leq C_{\beta,N}h^N.$$

It implies in particular that $\partial^\beta a_j=O_{L^\infty}(1)$. We will also say that $a\in\mathcal{C}^\infty(\Omega)$ admits a classical expansion on Ω and we will denote $a\sim\sum_{j\geq 0}h^ja_j$ if $a\sim_h\sum_{j\geq 0}h^ja_j$ and the (a_j) are independent of h. From now on, the letter r will denote a small universal positive constant whose value may decrease as we progress in this paper (one can think of r as $1/C$). For $x\in\mathbb{R}^d$, we denote $B_0(x,r)=B(x,r)\times B(0,r)\subseteq\mathbb{R}^{2d}$. We essentially follow the quasimodal construction from [4]. We will also denote

$$H_W=h^{-1}X_0^h=\begin{pmatrix}v\\-\partial_xV\end{pmatrix},$$

where we allowed ourselves to identify the differential operator X_0^h and the vector field representing it.

Let $\mathbf{m}\in\mathcal{U}^{(0)}\backslash\{\underline{\mathbf{m}}\}$; for each $\mathbf{s}\in\mathbf{j}(\mathbf{m})$ we introduce a function $\ell^{\mathbf{s},h}$ that will appear in our quasimodes. Note that thanks to item b) from Hypothesis 5.3.5, each $\ell^{\mathbf{s},h}$ corresponds to a unique $\mathbf{m}\in\mathcal{U}^{(0)}\backslash\{\underline{\mathbf{m}}\}$. Our goal will be to find some functions $\ell^{\mathbf{s},h}$ such that our quasimodes are the most accurate possible. In order to begin the computations that will yield the equations that the function $\ell^{\mathbf{s},h}$ should satisfy, we will for the moment assume that it satisfies the following :

(5.3.5)

a) $\ell^{\mathbf{s},h}$ is a smooth real valued function on $\mathbb{R}^{2d}$ whose support is contained in $B_0(\mathbf{s},3r)$
b) $\ell^{\mathbf{s},h}$ admits a classical expansion $\ell^{\mathbf{s},h}(x,v)\sim\sum h^j\ell^{\mathbf{s}}_j(x,v)$ on $B_0(\mathbf{s},2r)$
c) $\ell^{\mathbf{s}}_0$ vanishes at $(\mathbf{s},0)$
d) $(\mathbf{s},0)$ is a local minimum of the function $W+(\ell^{\mathbf{s}}_0)^2/2$ which is non degenerate
e) the functions $\theta_{\mathbf{m},h}$ (which depends on $\ell^{\mathbf{s},h}$) and $\chi_{\mathbf{m}}$ that we will introduce in (5.3.7)-(5.3.10) are such that $\theta_{\mathbf{m},h}$ is smooth on a neighborhood of supp $\chi_{\mathbf{m}}$.

Once we will have found the desired function $\ell^{\mathbf{s},h}$, we will see in Proposition 5.5.2 that these assumptions are actually satisfied. Denote $\zeta\in\mathcal{C}_c^\infty(\mathbb{R},[0,1])$ an even cut-off function supported in $[-\gamma,\gamma]$ that is equal to 1 on $[-\gamma/2,\gamma/2]$ where $\gamma>0$ is a parameter to be fixed later and

$$A_h=\frac12\int_{\mathbb{R}}\zeta(s)\mathrm{e}^{-\frac{s^2}{2h}}\mathrm{d}s=\int_0^\gamma\zeta(s)\mathrm{e}^{-\frac{s^2}{2h}}\mathrm{d}s=\frac{\sqrt{\pi h}}{\sqrt2}(1+O(\mathrm{e}^{-\alpha/h}))\qquad\text{for some }\alpha>0. \tag{5.3.6}$$

We now define for each $\mathbf{m}\in\mathcal{U}^{(0)}\backslash\{\underline{\mathbf{m}}\}$ a function $\theta_{\mathbf{m},h}$ as follows : if $(x,v)\in B_0(\mathbf{s},r)\cap\{|\ell^{\mathbf{s},h}|\leq 2\gamma\}$ for some $\mathbf{s}\in\mathbf{j}(\mathbf{m})$,

$$\theta_{\mathbf{m},h}(x,v)=\frac12\Big(1+A_h^{-1}\int_0^{\ell^{\mathbf{s},h}(x,v)}\zeta(s)\mathrm{e}^{-s^2/2h}\mathrm{d}s\Big) \tag{5.3.7}$$

whereas we set

$$\theta_{\mathbf{m},h}=1\quad\text{on }\Big(E(\mathbf{m})+B(0,\varepsilon)\Big)\backslash\Big(\bigsqcup_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\big(B_0(\mathbf{s},r)\cap\{|\ell^{\mathbf{s},h}|\leq 2\gamma\}\big)\Big) \tag{5.3.8}$$

with $\varepsilon(r) > 0$ to be fixed later and

$$\theta_{\mathbf{m},h} = 0 \quad \text{everywhere else.} \tag{5.3.9}$$

Note that $\theta_{\mathbf{m},h}$ takes values in $[0,1]$. Denote Ω the CC of $\{W \leq \boldsymbol{\sigma}(\mathbf{m})\}$ containing $\mathbf{m}$. The CCs of $\{W \leq \boldsymbol{\sigma}(\mathbf{m})\}$ are separated so for $\varepsilon > 0$ small enough, there exists $\tilde{\varepsilon} > 0$ such that

$$\min\big\{W(x,v)\,;\, \mathrm{d}\big((x,v),\Omega\big) = \varepsilon\big\} = \boldsymbol{\sigma}(\mathbf{m}) + 2\tilde{\varepsilon}.$$

Thus the distance between $\{W \leq \boldsymbol{\sigma}(\mathbf{m}) + \tilde{\varepsilon}\} \cap \big(\Omega + B(0,\varepsilon)\big)$ and $\partial\big(\Omega + B(0,\varepsilon)\big)$ is positive and we can consider a cut-off function

$$\chi_{\mathbf{m}} \in \mathcal{C}_c^\infty(\mathbb{R}^{2d}, [0,1]) \tag{5.3.10}$$

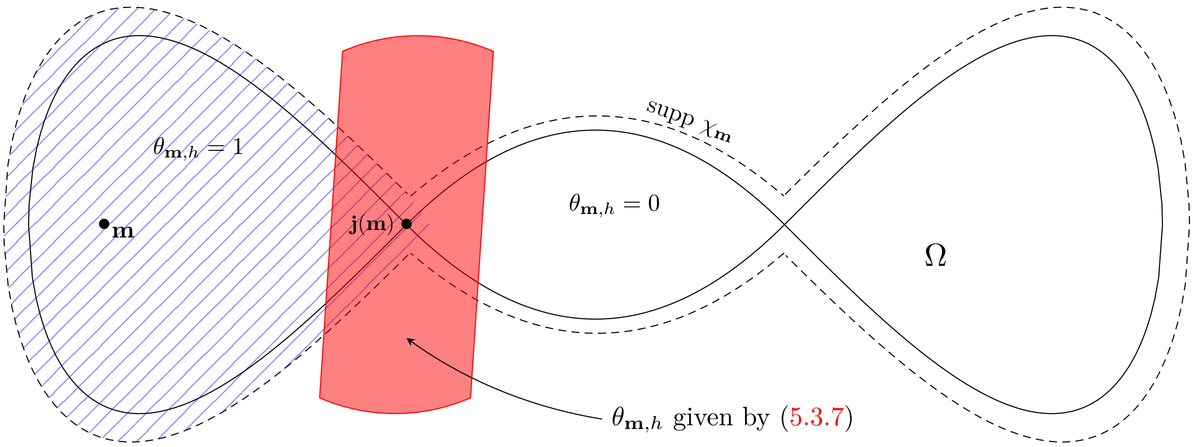
such that

$$\chi_{\mathbf{m}} = 1 \text{ on } \{W \leq \boldsymbol{\sigma}(\mathbf{m}) + \tilde{\varepsilon}\} \cap \big(\Omega + B(0,\varepsilon)\big)$$

and

$$\operatorname{supp} \chi_{\mathbf{m}} \subset \big(\Omega + B(0,\varepsilon)\big).$$

To sum up, we have the following picture :



We also denote

$$W_{\mathbf{m}}(x,v) = W(x,v) - V(\mathbf{m})/2$$

and it is clear that on the support of $\nabla\chi_{\mathbf{m}}$, we have

$$W_{\mathbf{m}} \geq S(\mathbf{m}) + \tilde{\varepsilon}.$$

Our quasimodes will be the L^2-renormalizations of the functions

$$f_{\mathbf{m},h}(x,v) = \chi_{\mathbf{m}}(x,v)\theta_{\mathbf{m},h}(x,v)\mathrm{e}^{-W_{\mathbf{m}}(x,v)/h} \quad ; \quad \mathbf{m} \in \mathcal{U}^{(0)} \backslash \{\underline{\mathbf{m}}\} \tag{5.3.11}$$

and for $\mathbf{m} = \underline{\mathbf{m}}$,

$$f_{\underline{\mathbf{m}},h}(x,v) = \mathrm{e}^{-W_{\underline{\mathbf{m}}}(x,v)/h} \in \operatorname{Ker} P_h.$$

Note that these functions belong to $\mathcal{C}_c^\infty(\mathbb{R}^{2d})$ thanks to our assumption on the $(\ell^{\mathbf{s},h})_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}$ and that for $\mathbf{m} \neq \underline{\mathbf{m}}$, we have

$$\operatorname{supp} f_{\mathbf{m},h} \subseteq E(\mathbf{m}) + B(0,\varepsilon') \tag{5.3.12}$$

where $\varepsilon' = \max(\varepsilon, r)$.

5.3.2 Action of the operator P_h

Let us fix $\mathbf{m} \in \mathcal{U}^{(0)} \backslash \{\underline{\mathbf{m}}\}$. We will denote

$$\widetilde{W}_{\mathbf{m},h} = W_{\mathbf{m}} + \sum_{\mathbf{s} \in \mathbf{j}(\mathbf{m})} (\ell^{\mathbf{s},h})^2/2 \tag{5.3.13}$$

and

$$\psi^{\mathbf{m},h}(x,v,v') = \int_0^1 \partial_v \widetilde{W}_{\mathbf{m},h}(x, v' + t(v-v'))\mathrm{d}t. \tag{5.3.14}$$

Remark 5.3.6. *Using Hypothesis 5.1.2, it is easy to see that $b_h^* Op_h(M^h) = Op_h(g^h)$, with*

$$g^h = (-i\,{}^t\eta + {}^tv/2)M^h - \frac{h}{2}({}^t\nabla_v - \frac{i}{2}{}^t\nabla_\eta)M^h \in \mathcal{M}_{1,d}\big(S^0_\tau(\langle(v,\eta)\rangle^{-1})\big)$$

where

$${}^t\nabla_v M^h = \Big(\sum_{k=1}^d \partial_{v_k} m_{k,j}\Big)_{1\leq j\leq d}$$

and ${}^t\nabla_\eta$ is defined similarly.

Proposition 5.3.7. *Let $f_{\mathbf{m},h}$ be the quasimode defined in (5.3.11). With the notations introduced in (5.3.6) and (5.3.13), one has*

$$P_h f_{\mathbf{m},h} = \frac{h}{2} A_h^{-1} \omega^{\mathbf{m},h}\, \mathrm{e}^{\frac{-\widetilde{W}_{\mathbf{m},h}}{h}} \mathbf{1}_{\mathbf{j}^W(\mathbf{m})+B_0(0,2r)} + O_{L^2}\Big(h^\infty \mathrm{e}^{-\frac{S(\mathbf{m})}{h}}\Big)$$

where $\omega^{\mathbf{m},h}$ is a function bounded uniformly in h and defined on $\mathbf{j}^W(\mathbf{m}) + B_0(0,2r)$ by

$$\omega^{\mathbf{m},h} = \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} \Big(H_W \cdot \nabla \ell^{\mathbf{s},h} + I^{\mathbf{s},h}\Big)$$

with $I^{\mathbf{s},h}(x,v)$ given for $(x,v) \in \mathbf{j}^W(\mathbf{m}) + B_0(0,2r)$ by the oscillatory integral

$$(2\pi h)^{-d} \int_{\mathbb{R}^d} \int_{|v'|\leq 2r} \mathrm{e}^{\frac{i}{h}\eta\cdot(v-v')} g^h\Big(x, \frac{v+v'}{2}, \eta + i\psi^{\mathbf{m},h}(x,v,v')\Big)\partial_v \ell^{\mathbf{s},h}(x,v')\, \mathrm{d}v'\mathrm{d}\eta.$$

Proof. In order to lighten the notations, we will drop some of the exponents and indexes $\mathbf{m}$, $\mathbf{s}$ and h in the proof. By (5.3.5), we have on the support of χ that θ is smooth and

$$\nabla\theta = \frac{A_h^{-1}}{2} \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} \mathrm{e}^{-(\ell^{\mathbf{s}})^2/2h} \zeta(\ell^{\mathbf{s}})\nabla\ell^{\mathbf{s}}\, \mathbf{1}_{B_0(\mathbf{s},r)}.$$

Here we have to put the indicator function because $\zeta(\ell)\nabla\ell$ might have some support in $B_0(\mathbf{s},3r)\backslash B_0(\mathbf{s},r)$. We can then begin by computing

$$\begin{aligned} X_0^h f &= hH_W \cdot \nabla f \\ &= hH_W \cdot \nabla\theta\, \chi \mathrm{e}^{-W_{\mathbf{m}}/h} + hH_W \cdot \nabla\chi\, \theta \mathrm{e}^{-W_{\mathbf{m}}/h} \\ &= \frac{h}{2}A_h^{-1}\chi\mathrm{e}^{-\widetilde{W}/h} \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} \zeta(\ell^{\mathbf{s}}) H_W \cdot \nabla\ell^{\mathbf{s}}\, \mathbf{1}_{B_0(\mathbf{s},r)} + O\Big(h\mathrm{e}^{-\frac{S(\mathbf{m})+\tilde{\varepsilon}}{h}}\Big). \end{aligned} \tag{5.3.15}$$

since $W_{\mathbf{m}} \geq S(\mathbf{m}) + \tilde{\varepsilon}$ on the support of $\nabla\chi$. Now we can use Remark 5.3.6 to write

$$\begin{aligned} Q_h(f) &= h\mathrm{Op}_h(g)\big((\partial_v\theta)\chi\mathrm{e}^{-W_{\mathbf{m}}/h} + (\partial_v\chi)\theta\mathrm{e}^{-W_{\mathbf{m}}/h}\big) \\ &= \frac{h}{2}A_h^{-1} \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} \mathrm{Op}_h(g)\Big(\zeta(\ell^{\mathbf{s}})\chi\mathrm{e}^{-\widetilde{W}/h}\partial_v\ell^{\mathbf{s}}\, \mathbf{1}_{B_0(\mathbf{s},r)}\Big) + O\Big(h\mathrm{e}^{-\frac{S(\mathbf{m})+\tilde{\varepsilon}}{h}}\Big) \end{aligned} \tag{5.3.16}$$

since $g \in S(\langle (v,\eta)\rangle^{-1})$ and thus $\mathrm{Op}_h(g)$ is bounded uniformly in h. But since g does not depend on ξ, we have for $\mathbf{s} \in \mathbf{j}(\mathbf{m})$

$$(2\pi h)^d \mathrm{Op}_h(g)\Big(\zeta(\ell)\chi \mathrm{e}^{-\widetilde{W}/h}\partial_v \ell\, \mathbf{1}_{B_0(\mathbf{s},r)}\Big)(x,v) = \int_{\mathbb{R}^d}\int_{|v'|\le r} \mathrm{e}^{\frac{i}{h}\eta\cdot(v-v')} g\Big(x,\frac{v+v'}{2},\eta\Big)$$
$$\times\, \chi(x,v')\zeta\big(\ell(x,v')\big)\mathrm{e}^{-\widetilde{W}(x,v')/h}\partial_v\ell(x,v')\,\mathrm{d}v'\mathrm{d}\eta\, \mathbf{1}_{B(\mathbf{s},r)}(x). \tag{5.3.17}$$

Let us now treat separately the cases $|v| \ge 2r$ and $|v| < 2r$.
When $|v| \ge 2r$, we have $|v - v'| \ge r$ so we can apply the non stationnary phase to the integral in η to get that for all $x \in B(\mathbf{s}, r)$ and $N \ge 1$, there exists $C_N > 0$ such that

$$\Big|\int_{\mathbb{R}^d}\int_{|v'|\le r} \mathrm{e}^{\frac{i}{h}\eta\cdot(v-v')} g\Big(x,\frac{v+v'}{2},\eta\Big)\chi(x,v')\zeta\big(\ell(x,v')\big)\mathrm{e}^{-\widetilde{W}(x,v')/h}\partial_v\ell(x,v')\,\mathrm{d}v'\mathrm{d}\eta\Big| \le C_N h^N |v|^{-N}\mathrm{e}^{-\frac{S(\mathbf{m})}{h}}$$

where we used item d) from (5.3.5), the fact that $W_{\mathbf{m}}(\mathbf{s},0)+\ell_0^2(\mathbf{s},0)/2 = S(\mathbf{m})$ and the estimate $|v-v'| \ge |v|/2$. Hence we have shown that

$$Q_h f\, \mathbf{1}_{\{|v|\ge 2r\}} = O\Big(h^\infty \mathrm{e}^{-\frac{S(\mathbf{m})}{h}}\Big) \quad \text{and} \quad P_h f\, \mathbf{1}_{\{|v|\ge 2r\}} = O\Big(h^\infty \mathrm{e}^{-\frac{S(\mathbf{m})}{h}}\Big). \tag{5.3.18}$$

Now for the case $|v| < 2r$, let us denote $J_1^{\mathbf{s}}(x,v)$ the RHS of (5.3.17). Proceeding as in [33] in order to take the $\mathrm{e}^{-\widetilde{W}(x,v')/h}$ in front of the oscillatory integral,we get that for any $x \in B(\mathbf{s},r)$,

$$J_1^{\mathbf{s}}(x,v) = \mathrm{e}^{-\widetilde{W}(x,v)/h} J_2^{\mathbf{s}}(x,v) \tag{5.3.19}$$

where

$$J_2^{\mathbf{s}}(x,v) = \int_{\mathbb{R}^d}\int_{|v'|\le r} \mathrm{e}^{\frac{i}{h}\big(\eta - i\psi(x,v,v')\big)\cdot\big(v-v'\big)} g\Big(x,\frac{v+v'}{2},\eta\Big)\chi(x,v')\zeta\big(\ell(x,v')\big)\partial_v\ell(x,v')\,\mathrm{d}v'\mathrm{d}\eta\, \mathbf{1}_{B(\mathbf{s},r)}(x)$$

and ψ is the function defined in (5.3.14). For $K \subset \{1,\dots,d\}$ and $z \in \mathbb{C}^d$, denote $z_K = (z_j)_{j\in K}$. We also denote for $d' \in \mathbb{N}$ and $1 \le j \le d'$

$$e_j = (\delta_{k,j})_{1\le k\le d'} \in \mathbb{N}^{d'} \tag{5.3.20}$$

the elements of the canonical basis of $\mathbb{C}^{d'}$. Now notice that ψ is a smooth function and that using the expansion of ℓ and (5.3.13), we get on $B_0(\mathbf{s},2r)\times\{|v'|\le 2r\}$,

$$\psi(x,v,v') = \frac{v+v'}{4} + \int_0^1 (\ell_0\partial_v\ell_0)(x, v'+t(v-v'))\mathrm{d}t + O(h).$$

In particular, we can choose r small enough so that $|\psi| < \tau$ on $B_0(\mathbf{s},2r)\times\{|v'|\le 2r\}$. Besides, since $g \in S^0_\tau(\langle (v,\eta)\rangle^{-1})$, we have for all $K \subset \{1,\dots,d\}$ and $k \in \{1,\dots,d\}\backslash K$ that the symbol

$$\eta_k \mapsto g\Big(x,\frac{v+v'}{2},\eta + i\sum_{j\in K}[\psi(x,v,v')]_j e_j\Big)$$

has an analytic continuation to $\{|\eta_k| < \tau\}$ for any $x \in B(\mathbf{s},r)$, v, $v' \in B(0,2r)$ and $\eta \in \mathbb{R}^d$. Hence, one can use the Cauchy formula which combined with the decay of g yields

$$\int_{\mathbb{R}} \mathrm{e}^{\frac{i}{h}\big(\eta_k - i[\psi(x,v,v')]_k\big)(v_k-v'_k)} g\Big(x,\frac{v+v'}{2},\eta + i\sum_{j\in K}[\psi(x,v,v')]_j e_j\Big)\,\mathrm{d}\eta_k =$$
$$\int_{\mathbb{R}} \mathrm{e}^{\frac{i}{h}\eta_k(v_k-v'_k)} g\Big(x,\frac{v+v'}{2},\eta + i\sum_{j\in K\cup\{k\}}[\psi(x,v,v')]_j e_j\Big)\,\mathrm{d}\eta_k.$$

Applying this successively for each component of η on the integrals in $J_2^{\mathbf{s}}$ finally gives $J_2^{\mathbf{s}} = J_3^{\mathbf{s}}$ where

$$J_3^{\mathbf{s}}(x,v) = \int_{\mathbb{R}^d}\int_{|v'|\le r} \mathrm{e}^{\frac{i}{h}\eta\cdot(v-v')} g\Big(x,\frac{v+v'}{2},\eta + i\psi(x,v,v')\Big)\chi(x,v')\zeta\big(\ell(x,v')\big)\partial_v\ell(x,v')\,\mathrm{d}v'\mathrm{d}\eta\, \mathbf{1}_{B(\mathbf{s},r)}(x).$$

Combined with (5.3.17) and (5.3.19), this yields for $|v|<2r$

$$(2\pi h)^d \mathrm{Op}_h(g)\Big(\zeta(\ell)\chi \mathrm{e}^{-\widetilde{W}/h}\partial_v\ell\,\mathbf{1}_{B_0(\mathbf{s},r)}\Big)(x,v)=\mathrm{e}^{-\widetilde{W}(x,v)/h}J_3^{\mathbf{s}}(x,v). \tag{5.3.21}$$

Therefore, setting on $\mathbf{j}^W(\mathbf{m})+B_0(0,2r)$

$$\tilde{\omega}=\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\Big(\chi\zeta(\ell^{\mathbf{s}})H_W\cdot\nabla\ell^{\mathbf{s}}\,\mathbf{1}_{B_0(\mathbf{s},r)}+(2\pi h)^{-d}J_3^{\mathbf{s}}(x,v)\Big),$$

we have according to (5.3.15), (5.3.16), (5.3.18) and (5.3.21)

$$P_hf=\frac{h}{2}A_h^{-1}\tilde{\omega}\,\mathrm{e}^{-\widetilde{W}/h}\mathbf{1}_{\mathbf{j}^W(\mathbf{m})+B_0(0,2r)}+O\Big(h^\infty\mathrm{e}^{-\frac{S(\mathbf{m})}{h}}\Big).$$

Hence it is sufficient to check that on $\mathbf{j}^W(\mathbf{m})+B_0(0,2r)$

$$(\tilde{\omega}-\omega)\mathrm{e}^{-\widetilde{W}/h}=O\Big(h^\infty\mathrm{e}^{-\frac{S(\mathbf{m})}{h}}\Big).$$

This can be done easily using again the non stationary phase on an h-independent neighborhood of $(\mathbf{s},0)$ on which $\chi\zeta(\ell)-1$ vanishes since item d) from (5.3.5) implies that $\mathrm{e}^{-\widetilde{W}/h}=O(\mathrm{e}^{-(S(\mathbf{m})+\delta)/h})$ outside of this neighborhood for some $\delta>0$. □

Remark 5.3.8. *Since $P_h^*=-X_0^h+Q_h$, it is clear from the previous proof that*

$$P_h^*f_{\mathbf{m},h}=\frac{h}{2}A_h^{-1}\overset{*}{\omega}{}^{\mathbf{m},h}\,\mathrm{e}^{\frac{-\widetilde{W}_{\mathbf{m},h}}{h}}\mathbf{1}_{\mathbf{j}^W(\mathbf{m})+B_0(0,2r)}+O_{L^2}\Big(h^\infty\mathrm{e}^{-\frac{S(\mathbf{m})}{h}}\Big)$$

with

$$\overset{*}{\omega}{}^{\mathbf{m},h}=\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\Big(-H_W\cdot\nabla\ell^{\mathbf{s},h}+I^{\mathbf{s},h}\Big).$$

5.4 Equations on $\ell^{\mathbf{s},h}$

From now on, we also fix $\mathbf{s}\in\mathbf{j}(\mathbf{m})$.

Lemma 5.4.1. *The function $\omega^{\mathbf{m},h}$ admits the classical expansion $\omega^{\mathbf{m},h}\sim\sum_{j\geq0}h^j\omega_j^{\mathbf{m}}$ on $B_0(\mathbf{s},2r)$ where*

$$\omega_0^{\mathbf{m}}=H_W\cdot\nabla\ell_0^{\mathbf{s}}+M_0\Big(x,v,i\big(\frac{v}{2}+\ell_0^{\mathbf{s}}\,\partial_v\ell_0^{\mathbf{s}}\big)\Big)\big(v+\ell_0^{\mathbf{s}}\partial_v\ell_0^{\mathbf{s}}\big)\cdot\partial_v\ell_0^{\mathbf{s}}$$

and for $j\geq1$,

$$\begin{aligned}\omega_j^{\mathbf{m}}=\;&H_W\cdot\nabla\ell_j^{\mathbf{s}}+M_0\Big(x,v,i\big(\frac{v}{2}+\ell_0^{\mathbf{s}}\,\partial_v\ell_0^{\mathbf{s}}\big)\Big)\big(v+2\ell_0^{\mathbf{s}}\partial_v\ell_0^{\mathbf{s}}\big)\cdot\partial_v\ell_j^{\mathbf{s}}\\&+i\,\ell_0^{\mathbf{s}}\big({}^tv+\ell_0^{\mathbf{s}}\,{}^t(\partial_v\ell_0^{\mathbf{s}})\big)D_\eta M_0\big(x,v,i(v/2+\ell_0^{\mathbf{s}}\partial_v\ell_0^{\mathbf{s}})\big)\big(\partial_v\ell_j^{\mathbf{s}}\big)\,\partial_v\ell_0^{\mathbf{s}}\\&+M_0\Big(x,v,i\big(\frac{v}{2}+\ell_0^{\mathbf{s}}\,\partial_v\ell_0^{\mathbf{s}}\big)\Big)\partial_v\ell_0^{\mathbf{s}}\cdot\partial_v\ell_0^{\mathbf{s}}\,\ell_j^{\mathbf{s}}\\&+i\big({}^tv+\ell_0^{\mathbf{s}}\,{}^t(\partial_v\ell_0^{\mathbf{s}})\big)D_\eta M_0\big(x,v,i(v/2+\ell_0^{\mathbf{s}}\partial_v\ell_0^{\mathbf{s}})\big)\big(\partial_v\ell_0^{\mathbf{s}}\big)\,\partial_v\ell_0^{\mathbf{s}}\,\ell_j^{\mathbf{s}}\\&+R_j(\ell_0^{\mathbf{s}},\dots,\ell_{j-1}^{\mathbf{s}})\end{aligned} \tag{5.4.1}$$

where $R_j:\big(\mathcal{C}^\infty(B_0(\mathbf{s},2r))\big)^j\to\mathcal{C}^\infty(B_0(\mathbf{s},2r))$ and D_η denotes the partial differential with respect to the variable η.

Proof. Once again, we drop some of the exponents and indexes $\mathbf{m}$, $\mathbf{s}$ and h in the proof. Denote $B_\infty(0,2r)=\{v',\eta\in\mathbb{R}^{2d}\,;\,\max(|v'|,|\eta|)<2r\}$. The first terms of ω_0 and ω_j are both easily obtained thanks to the expansion of ℓ on $B_0(\mathbf{s},2r)$. Hence it remains to get an expansion of $g(x,v/2+v'/2,\eta+i\psi(x,v,v'))$ that we will then be able to combine with the stationnary phase to get an expansion of the whole term $I^{\mathbf{s},h}$ of ω. Let us start with an expansion of ψ : the expansion of ℓ yields

$$\partial_v \widetilde{W} - v/2 \sim \sum_{j\geq 0} h^j \sum_{k=0}^{j} \ell_k \partial_v \ell_{j-k} \quad \text{on } B_0(\mathbf{s}, 2r)$$

so using (5.3.14), we get

$$\psi \sim \sum_{j\geq 0} h^j \psi_j \quad \text{on } B_0(\mathbf{s}, 2r) \times \{|v'| \leq 2r\}$$

where

$$\psi_0(x, v, v') = \frac{v+v'}{4} + \int_0^1 (\ell_0 \partial_v \ell_0)(x, v' + t(v-v'))\mathrm{d}t \tag{5.4.2}$$

and for $j \geq 1$,

$$\psi_j(x, v, v') = \int_0^1 \sum_{k=0}^{j} \big(\ell_k \partial_v \ell_{j-k}\big)(x, v' + t(v-v'))\mathrm{d}t. \tag{5.4.3}$$

Besides, since $M^h \sim \sum_{n\geq 0} h^n M_n$ in $\mathcal{M}_d\big(S^0_\tau(\langle(v,\eta)\rangle^{-2})\big)$, we deduce thanks to Proposition 5.7.3 and Remark 5.3.6 that g also has a classical expansion $g \sim \sum_{n\geq 0} h^n g_n$ in $\mathcal{M}_{1,d}\big(S^0_\tau(\langle(v,\eta)\rangle^{-1})\big)$, where the (g_n) are given by

$$g_0(x, v, \eta) = \Big(-i\,{}^t\eta + \frac{{}^t v}{2}\Big) M_0(x, v, \eta) \tag{5.4.4}$$

and

$$g_n(x, v, \eta) = \Big(-i\,{}^t\eta + \frac{{}^t v}{2}\Big) M_n(x, v, \eta) - \frac{1}{2}({}^t\nabla_v - \frac{i}{2}{}^t\nabla_\eta) M_{n-1}(x, v, \eta) \tag{5.4.5}$$

for $n \geq 1$. According to Corollary 5.7.7, we have

$$g_n\Big(x, \frac{v+v'}{2}, \eta + i\psi(x, v, v')\Big) \sim \sum_{j\geq 0} h^j g_{n,j}(x, v, v', \eta) \quad \text{on } B_0(\mathbf{s}, 2r) \times B_\infty(0, 2r)$$

with

$$g_{n,0}(x, v, v', \eta) = g_n\Big(x, \frac{v+v'}{2}, \eta + i\psi_0(x, v, v')\Big) \tag{5.4.6}$$

and for $j \geq 1$

$$g_{n,j}(x, v, v', \eta) = iD_\eta g_n\Big(x, \frac{v+v'}{2}, \eta + i\psi_0(x, v, v')\Big)\big(\psi_j(x, v, v')\big) + R^1_j(\ell_0, \ldots, \ell_{j-1}) \tag{5.4.7}$$

where $R^1_j : \big(\mathcal{C}^\infty(B_0(\mathbf{s}, 2r))\big)^j \to \mathcal{C}^\infty(B_0(\mathbf{s}, 2r))$. Using the expansion of g itself and Proposition 5.7.2, we get

$$g\Big(x, \frac{v+v'}{2}, \eta + i\psi(x, v, v')\Big) \sim_h \sum_{n\geq 0} h^n g_n\Big(x, \frac{v+v'}{2}, \eta + i\psi(x, v, v')\Big)$$

on $B_0(\mathbf{s}, 2r) \times B_\infty(0, 2r)$ so we can use Proposition 5.7.4 which yields

$$g\Big(x, \frac{v+v'}{2}, \eta + i\psi(x, v, v')\Big) \sim \sum_{j\geq 0} h^j \sum_{n=0}^{j} g_{n,j-n}(x, v, v', \eta) \tag{5.4.8}$$

on $B_0(\mathbf{s},2r)\times B_\infty(0,2r)$. Thus, using the expansion (5.4.8) that we just got, the one of $\partial_v\ell$, and the one for an oscillatory integral given by the stationnary phase (see for instance [47], Theorem 3.17) as well Proposition 5.7.4, we finally get

$$I^{\mathbf{s},h}\sim\sum_{j\geq 0}h^jI_j\quad\text{on } B_0(\mathbf{s},2r),\tag{5.4.9}$$

where

$$I_j(x,v)=\sum_{n_1+n_2+n_3+n_4=j}\frac{1}{i^{n_1}n_1!}\big(\partial_{v'}\cdot\partial_\eta\big)^{n_1}\Big(g_{n_2,n_3}(x,v,v',\eta)\partial_v\ell_{n_4}(x,v')\Big)\Big|_{\substack{v'=v\\ \eta=0}}.$$

We can already use (5.4.6) to deduce the expression of ω_0 by noticing that according to (5.4.2), $\psi_0(x,v,v)=v/2+\ell_0\partial_v\ell_0$. For $j\geq 1$, the terms of I_j in which the function ℓ_j appears are obviously the one given by $n_4=j$, but also the one given by $n_3=j$ according to (5.4.7). Indeed, in that case, we have using (5.4.3) that

$$\begin{aligned}g_{0,j}(x,v,v,0)=i\ell_0D_\eta g_0\big(x,v,i(v/2{+}\ell_0\partial_v\ell_0)\big)\big(\partial_v\ell_j\big)&\\ +iD_\eta g_0\big(x,v,i(v/2+\ell_0\partial_v\ell_0)\big)\big(\partial_v\ell_0\big)\,\ell_j+R^2_j(\ell_0,\dots,\ell_{j-1})&\end{aligned}$$

where $R_j^2:\big(\mathcal{C}^\infty(B_0(\mathbf{s},2r))\big)^j\to\mathcal{C}^\infty(B_0(\mathbf{s},2r))$. We can now conclude as for any $X\in\mathbb{R}^d$,

$$\begin{aligned}D_\eta g_0\big(x,v,i(v/2+\ell_0\partial_v\ell_0)\big)(X)=-i\,{}^tXM_0\big(x,v,i(v/2+\ell_0\partial_v\ell_0)\big)&\\ +\big({}^tv+\ell_0\,{}^t(\partial_v\ell_0)\big)D_\eta M_0\big(x,v,i(v/2+\ell_0\partial_v\ell_0)\big)(X)&\end{aligned}$$

according to (5.4.4). □

Denote $(m^n_{p,q})_{p,q}$ the entries of the matrix M_n from Hypothesis 5.1.2. Since we have for $X\in\mathbb{R}^d$

$$D_\eta M_0\big(x,v,i(v/2+\ell_0\partial_v\ell_0)\big)(X)=\Big(\partial_\eta m^0_{p,q}\big(x,v,i(v/2+\ell_0\partial_v\ell_0)\big)\cdot X\Big)_{1\leq p,q\leq d}$$

we get by putting

$$\begin{aligned}U(x,v)=M_0\Big(x,v,i\big(\frac{v}{2}+\ell_0\,\partial_v\ell_0\big)\Big)\partial_v\ell_0&\\ +\sum_{1\leq p,q\leq d}(v_p+\ell_0\partial_{v_p}\ell_0)i\partial_\eta m^0_{p,q}\Big(x,v,i\big(\frac{v}{2}+\ell_0\,\partial_v\ell_0\big)\Big)\partial_{v_q}\ell_0&\end{aligned}\tag{5.4.10}$$

that equation (5.4.1) reads

$$\omega_j=\left[H_W+\begin{pmatrix}0\\ M_0\Big(x,v,i\big(\frac{v}{2}+\ell_0\,\partial_v\ell_0\big)\Big)(v+\ell_0\partial_v\ell_0)+\ell_0\,U\end{pmatrix}\right]\cdot\nabla\ell_j+U\cdot\partial_v\ell_0\,\ell_j+R_j(\ell_0,\dots,\ell_{j-1}).$$

Lemma 5.4.2. *Let $(x,v)\in B_0(\mathbf{s},2r)$ and $|v'|<2r$. For any $n\in\mathbb{N}$, $\beta\in\mathbb{N}^d$ and $1\leq p,q\leq d$, we have*

$$\partial^\beta_\eta m^n_{p,q}\Big(x,\frac{v+v'}{2},i\psi_0^{\mathbf{m}}(x,v,v')\Big)\in i^{|\beta|}\mathbb{R}$$

and

$$\partial^\beta_\eta g_n\Big(x,\frac{v+v'}{2},i\psi_0^{\mathbf{m}}(x,v,v')\Big)\in i^{|\beta|}\mathbb{R}^d.$$

In particular, U defined in (5.4.10) sends $B_0(\mathbf{s},2r)$ in $\mathbb{R}^d$.

Proof. Since ℓ_0 vanishes at $(\mathbf{s},0)$, we can suppose that r is such that $i\psi_0(x,v,v')$ is in

$$D(0,\tau)^d=\{z\in\mathbb{C};\,|z|<\tau\}^d\tag{5.4.11}$$

so by analyticity and using the parity of $m^n_{p,q}$, we have

$$\partial^\beta_\eta m^n_{p,q}\Big(x,\frac{v+v'}{2},i\psi_0(x,v,v')\Big)=\sum_{\substack{\gamma\in\mathbb{N}^d;\\ |\gamma|+|\beta|\in 2\mathbb{N}}}i^{|\gamma|}\frac{\partial_\eta^{\gamma+\beta}m^n_{p,q}(x,\frac{v+v'}{2},0)}{\gamma!}\psi_0(x,v,v')^\gamma\in i^{|\beta|}\mathbb{R}.$$

The result for g_n follows easily using (5.4.4) and (5.4.5). □

We also have the following result whose proof is postponed to Appendix 5.7.4 as it involves tedious calculations.

Lemma 5.4.3. *The term $R_j(\ell_0^{\mathbf{s}},\dots,\ell_{j-1}^{\mathbf{s}})$ from Lemma 5.4.1 is real valued. Moreover, it satisfies $R_j(\ell_0^{\mathbf{s}},\dots,\ell_{j-1}^{\mathbf{s}}) = -R_j(-\ell_0^{\mathbf{s}},\dots,-\ell_{j-1}^{\mathbf{s}})$.*

In view of the results from Proposition 5.3.7 and Lemma 5.4.1, we want to find ℓ such that on $B_0(\mathbf{s},2r)$,

$$H_W\cdot\nabla\ell_0 + M_0\Big(x,v,i\big(\frac{v}{2}+\ell_0\,\partial_v\ell_0\big)\Big)\big(v+\ell_0\partial_v\ell_0\big)\cdot\partial_v\ell_0 = 0 \tag{5.4.12}$$

and for $j\geq 1$

$$\left[H_W + \begin{pmatrix} 0 \\ M_0\Big(x,v,i\big(\frac{v}{2}+\ell_0\,\partial_v\ell_0\big)\Big)(v+\ell_0\partial_v\ell_0)+\ell_0\,U\end{pmatrix}\right]\cdot\nabla\ell_j \\ +\partial_v\ell_0\cdot U\,\ell_j + R_j(\ell_0,\dots,\ell_{j-1}) = 0 \tag{5.4.13}$$

where U was introduced in (5.4.10). Note that Lemmas 5.4.2 and 5.4.3 ensure that the fact that the $(\ell_j)_{j\geq 0}$ are real valued is compatible with equations (5.4.13).

5.4.1 Solving for $\ell_0^{\mathbf{s}}$

Denote

$$p(x,v,\xi,\eta) = i\xi\cdot v - i\eta\cdot\partial_x V + (-i\,{}^t\eta + {}^tv/2)M_0(x,v,\eta)(i\eta+v/2)$$

the principal symbol of the whole operator P_h and $\tilde p(x,v,\xi,\eta) = -p(x,v,i\xi,i\eta)$ its complexification. After computing the Hamiltonian of $\tilde p$ which vanishes at $(\mathbf{s},0,0,0)$, we find that its linearization at this point is the matrix

$$F = \begin{pmatrix} 0 & \mathrm{Id} & 0 & 0 \\ -\mathrm{Hess}_{\mathbf{s}}V & 0 & 0 & 2M_0(\mathbf{s},0,0) \\ 0 & 0 & 0 & \mathrm{Hess}_{\mathbf{s}}V \\ 0 & \frac{1}{2}M_0(\mathbf{s},0,0) & -\mathrm{Id} & 0 \end{pmatrix}.$$

One can easily check that for any eigenvector (x,v,ξ,η) of F associated to an eigenvalue λ, the vector $(-x,v,\xi,-\eta)$ is an eigenvector associated to $-\lambda$ so the spectrum of F is centrally symmetric with respect to the origin. Moreover, writing

$$F = \begin{pmatrix} 0 & 0 & \mathrm{Id} & 0 \\ 0 & 0 & 0 & \mathrm{Id} \\ \mathrm{Id} & 0 & 0 & 0 \\ 0 & \mathrm{Id} & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & \mathrm{Hess}_{\mathbf{s}}V \\ 0 & \frac{1}{2}M_0(\mathbf{s},0,0) & -\mathrm{Id} & 0 \\ 0 & \mathrm{Id} & 0 & 0 \\ -\mathrm{Hess}_{\mathbf{s}}V & 0 & 0 & 2M_0(\mathbf{s},0,0) \end{pmatrix}$$

and noticing that

$$F\Big(\{v=\eta=0\}\Big)\cap\{v=\eta=0\} = \operatorname{Ker}F\cap\{v=\eta=0\} = \{0\},$$

we see that F satisfies the assumptions of Lemma 5.7.1. Therefore, F has no eigenvalues in $i\mathbb{R}$ so it has $2d$ eigenvalues (counted with algebraic multiplicity) in $\{\operatorname{Re}z>0\}$ while the $2d$ others are in $\{\operatorname{Re}z<0\}$. Therefore we can apply the stable manifold theorem to get that the stable manifolds associated to $H_{\tilde p}$ given in a neighborhood of $(\mathbf{s},0,0,0)$ by

$$\Lambda_\pm = \Big\{(x,v,\xi,\eta)\,;\,\lim_{t\to\mp\infty}\mathrm{e}^{tH_{\tilde p}}(x,v,\xi,\eta) = (\mathbf{s},0,0,0)\Big\}$$

are both of dimension $2d$ and for all $\rho_\pm\in\Lambda_\pm$, we have

$$H_{\tilde p}(\rho_\pm)\in T_{\rho_\pm}\Lambda_\pm \tag{5.4.14}$$

and for $t>0$,

$$\big\|\mathrm{e}^{\mp tH_{\tilde p}}\rho_\pm - (\mathbf{s},0,0,0)\big\| \leq C\mathrm{e}^{-t/C}\|\rho_\pm - (\mathbf{s},0,0,0)\|.$$

Moreover, we have (see for instance [12] Lemmas 3.2 and 3.3) that

$$\tilde{p}(\Lambda_{\pm}) = \{0\} \tag{5.4.15}$$

and $\Lambda_{\pm}$ are Lagrangian manifolds. In order to get some parametrization for those manifolds, we follow the steps of [22], Lemma 8.1.

Lemma 5.4.4. *The tangent spaces $T_{(\mathbf{s},0,0,0)}\Lambda_{\pm}$ that we denote for shortness $T_{\mathbf{s}}\Lambda_{\pm}$ are transverse to both $\{(\mathbf{s},0)\}\times\mathbb{R}^{2d}$ and $\mathbb{R}^{2d}\times\{(0,0)\}$.*

Proof. We provide an adaptation of the proof from [22] as some simplifications appear in our case. Since we are working in the linearized case, we can assume that $\tilde{p}$ coincides with its quadratic approximation at $(\mathbf{s},0,0,0)$ and for commodity we will work with the variable $x_{\mathbf{s}} = x - \mathbf{s}$ instead of x. Note that if a is a quadratic form, its Hamiltonian H_a is then linear and we denote F_a the associated matrix. We then decompose $\tilde{p} = p_2 + p_1 - p_0$ where

$$p_2 = M_0(\mathbf{s},0,0)\eta\cdot\eta, \quad p_1 = v\cdot\xi - \mathrm{Hess}_s V x_{\mathbf{s}}\cdot\eta \quad \text{and} \quad p_0 = \frac{1}{4}M_0(\mathbf{s},0,0)v\cdot v.$$

It is clear that p_2+p_0 is positive semi-definite, moreover, the subspace $\{v=\eta=0\}$ on which p_2+p_0 vanishes satisfies $\{v=\eta=0\}\cap F_{p_1}^{-1}\big(\{v=\eta=0\}\big)=\{0\}$. Thus the quadratic form

$$\tilde{q} = (p_2+p_0) + (p_2+p_0)\circ F_{p_1}$$

is positive definite. Let us denote $L_{\pm} = \Lambda_{\pm}\cap\{x_{\mathbf{s}} = v = 0\}$. To prove that $L_{\pm}=\{0\}$, it is sufficient to establish that $\tilde{q}=0$ on $L_{\pm}$. In order to do so, we will show that $L_{\pm}$ is an F_{p_1}-invariant subspace on which $p_2+p_0=0$. Indeed, it is clear that $p_0=p_1=0$ on $L_{\pm}$ and thanks to (5.4.15) we deduce that p_2 also vanishes on $L_{\pm}$ so in particular $p_2+p_0=0$ on $L_{\pm}$. It also implies that $L_{\pm}$ is included in $\{\eta=0\}$ so $F_{p_2}|_{L_{\pm}}=0$. Besides, we clearly have $F_{p_0}|_{L_{\pm}}=0$ so F_{p_1} coincides on $L_{\pm}$ with $F_{\tilde{p}}$ which leaves $\Lambda_{\pm}$ invariant according to (5.4.14). Since it is easy to see that $\{x_{\mathbf{s}}=v=0\}$ is also invariant under F_{p_1}, we can conclude as announced that $L_{\pm}=\{0\}$. The proof that $\Lambda_{\pm}\cap\{\xi=\eta=0\}=\{0\}$ is similar. □

Since $\Lambda_{\pm}$ are Lagrangian manifolds such that $T_s\Lambda_{\pm}$ are transverse to $\{(\mathbf{s},0)\}\times\mathbb{R}^{2d}$, there exist $\phi_{\pm}\in\mathcal{C}^{\infty}(B_0(\mathbf{s},2r),\mathbb{R})$ vanishing together with their gradients at $(\mathbf{s},0)$ and such that

$$\Lambda_{\pm} = \Big\{\Big((x,v,\nabla\phi_{\pm}(x,v)\Big)\,;(x,v)\in B_0(\mathbf{s},2r)\Big\}.$$

Therefore, $T_{\mathbf{s}}\Lambda_{\pm}$ coincide with the graphs of the matrices $\mathrm{Hess}_{(\mathbf{s},0)}\phi_{\pm}$ which are then invertible according to Lemma 5.4.4. Now we need a result similar to the one of Proposition 8.2 in [22].

Lemma 5.4.5. *The Hessian matrix of $\pm\phi_{\pm}$ at $(\mathbf{s},0)$ is definite positive.*

Proof. The proof is simply an adaptation of the one found in [22]. Here again we will assume that $\tilde{p}$ coincides with its quadratic approximation at $(\mathbf{s},0,0,0)$ and work with the variable $x_{\mathbf{s}}=x-\mathbf{s}$ instead of x. For $\delta\in[0,1]$, let us denote

$$\begin{aligned}\tilde{p}^{\delta} &= (1-\delta)\tilde{p} + \delta\big(\xi^2+\eta^2-(x_{\mathbf{s}}^2+v^2)\big)\\ &= p_2^{\delta} + (1-\delta)p_1 - p_0^{\delta}\end{aligned}$$

where

$$p_2^{\delta} = (1-\delta)p_2 + \delta(\xi^2+\eta^2) \qquad \text{and} \qquad p_0^{\delta} = (1-\delta)p_0 + \delta(x_{\mathbf{s}}^2+v^2).$$

Note in particular that $\tilde{p}^0=\tilde{p}$ and that $\tilde{p}^1 = \big(\xi^2+\eta^2-(x_{\mathbf{s}}^2+v^2)\big)$ corresponds to the well know Schrödinger case (see for instance [12], chapter 3). Besides, we have that

$$F_{\tilde{p}^{\delta}} = \begin{pmatrix} 0 & 0 & \mathrm{Id} & 0\\ 0 & 0 & 0 & \mathrm{Id}\\ \mathrm{Id} & 0 & 0 & 0\\ 0 & \mathrm{Id} & 0 & 0\end{pmatrix}\left[(1-\delta)\begin{pmatrix} 0 & 0 & 0 & \mathrm{Hess}_{\mathbf{s}}V\\ 0 & \frac{1}{2}M_0(\mathbf{s},0,0) & -\mathrm{Id} & 0\\ 0 & \mathrm{Id} & 0 & 0\\ -\mathrm{Hess}_{\mathbf{s}}V & 0 & 0 & 2M_0(\mathbf{s},0,0)\end{pmatrix} + 2\delta\,\mathrm{Id}\right]$$

so Lemma 5.7.1 easily yields that the eigenvalues of $F_{\tilde{p}^\delta}$ cannot cross $i\mathbb{R}$ for some $\delta \in (0,1]$. Moreover, it is clear that for $\delta \in (0,1]$, the quadratic form $p_2^\delta + p_0^\delta$ is positive definite, so the results of Lemma 5.4.4 are true for the $2d$-dimensional Lagrangian planes

$$\Lambda_\pm^\delta = \Big\{(x_{\mathbf{s}}, v, \xi, \eta)\,;\, \lim_{t\to\mp\infty} \mathrm{e}^{tF_{\tilde{p}^\delta}}(x,v,\xi,\eta) = 0\Big\}$$

for all $\delta \in [0,1]$. In particular, there exist $\phi_\pm^\delta \in \mathcal{C}^\infty(B_0(\mathbf{s},2r),\mathbb{R})$ such that

$$T_{\mathbf{s}}\Lambda_\pm^\delta = \Lambda_\pm^\delta = \Big\{\Big(x_{\mathbf{s}}, v, \mathrm{Hess}_{(\mathbf{s},0)}\phi_\pm^\delta \begin{pmatrix} x_{\mathbf{s}} \\ v \end{pmatrix}\Big)\,;(x_{\mathbf{s}},v)\in\mathbb{R}^{2d}\Big\}.$$

Hence the graph of $\mathrm{Hess}_{(\mathbf{s},0)}\phi_\pm^\delta$ is given by $T_{\mathbf{s}}\Lambda_\pm^\delta$ which also corresponds to the sum of the generalized eigenspaces of $F_{\tilde{p}^\delta}$ associated to eigenvalues in $\{\pm \mathrm{Re}\, z < 0\}$ and therefore depends continuously on δ. Besides, by Lemma 5.4.4, $\mathrm{Hess}_{(\mathbf{s},0)}\phi_\pm^\delta$ is invertible for all $\delta \in [0,1]$ and we know from the Schrödinger case that $\pm\mathrm{Hess}_{(\mathbf{s},0)}\phi_\pm^1 > 0$ so necessarily $\pm\mathrm{Hess}_{(\mathbf{s},0)}\phi_\pm > 0$. □

At this point, one can proceed as in [4], Lemma 3.2 to establish the following Lemma.

Lemma 5.4.6. *There exists $\ell_0^{\mathbf{s}} \in \mathcal{C}^\infty(B_0(\mathbf{s},2r),\mathbb{R})$ such that for $(x,v)\in B_0(\mathbf{s},2r)$,*

$$\phi_+(x,v) = W(x,v) - W(\mathbf{s},0) + \frac{\ell_0^{\mathbf{s}}(x,v)^2}{2}.$$

In particular, $\ell_0^{\mathbf{s}}$ vanishes at $(\mathbf{s},0)$. Moreover, $\{\ell_0^{\mathbf{s}} \neq 0\}$ is dense in $B_0(\mathbf{s},2r)$.

This function also appears to solve (5.4.12) as we see in the next Proposition.

Proposition 5.4.7. *The function $\ell_0^{\mathbf{s}}$ from Lemma 5.4.6 is a solution of (5.4.12) in $B_0(\mathbf{s},2r)$. Moreover, the vector $\nabla\ell_0^{\mathbf{s}}(\mathbf{s},0)$ that we denote $\nu^{\mathbf{s}} = \begin{pmatrix}\nu_1^{\mathbf{s}}\\ \nu_2^{\mathbf{s}}\end{pmatrix}$ is not 0 and satisfies $\Phi^{\mathbf{s}}\nu^{\mathbf{s}} = \big(-M_0(\mathbf{s},0,0)\nu_2^{\mathbf{s}}\cdot\nu_2^{\mathbf{s}}\big)\,\nu^{\mathbf{s}}$, where*

$$\Phi^{\mathbf{s}} = \begin{pmatrix} 0 & -\mathrm{Hess}_{\mathbf{s}}V \\ \mathrm{Id} & M_0(\mathbf{s},0,0)\end{pmatrix}.$$

In particular, since $\Phi^{\mathbf{s}}$ is invertible, $\nu_2^{\mathbf{s}} \neq 0$. Finally,

$$\det\Big(\mathrm{Hess}_{(\mathbf{s},0)}\Big(W + \frac{(\ell_0^{\mathbf{s}})^2}{2}\Big)\Big) = 2^{-2d}\big|\det(\mathrm{Hess}_{\mathbf{s}}V)\big|.$$

Proof. The proof is the same as in [4], Lemma 3.3 after matching the notations by setting $\Lambda(\mathbf{s}) = \Phi^{\mathbf{s}}$, $b^0 = H_W$,

$$A^0(\mathbf{s}) = \begin{pmatrix} 0 & 0 \\ 0 & M_0(\mathbf{s},0,0)\end{pmatrix} \quad \text{and} \quad B(\mathbf{s}) = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Hess}_{\mathbf{s}}V & 0\end{pmatrix}.$$

In particular, it is by a Taylor expansion at $(\mathbf{s},0)$ in (5.4.12) that we get

$$\begin{pmatrix} x-\mathbf{s} \\ v \end{pmatrix}\cdot\Big[\begin{pmatrix} 0 & -\mathrm{Hess}_{\mathbf{s}}V \\ \mathrm{Id} & 0\end{pmatrix}\nu^{\mathbf{s}} + \begin{pmatrix} 0 \\ M_0(\mathbf{s},0,0)\nu_2^{\mathbf{s}}\end{pmatrix} + M_0(\mathbf{s},0,0)\nu_2^{\mathbf{s}}\cdot\nu_2^{\mathbf{s}}\,\nu^{\mathbf{s}}\Big] = 0$$

from which we deduce that $\nu^{\mathbf{s}}$ is an eigenvector of $\Phi^{\mathbf{s}}$ associated to the eigenvalue $-M_0(\mathbf{s},0,0)\nu_2^{\mathbf{s}}\cdot\nu_2^{\mathbf{s}}$. □

5.4.2 Solving for $\big(\ell_j^{\mathbf{s}}\big)_{j\geq 1}$

Once again we drop some exponents $\mathbf{s}$ for shortness. Now that ℓ_0 is given by Lemma 5.4.6 and Proposition 5.4.7, we can solve the transport equations (5.4.13) by induction, so we suppose that $\ell_0,\dots,\ell_{j-1}$ are given and we want to find a solution ℓ_j to (5.4.13). Denote

$$\widetilde{U} = H_W + \begin{pmatrix} 0 \\ M_0\big(x,v,i\big(\frac{v}{2}+\ell_0\,\partial_v\ell_0\big)\big)(v+\ell_0\partial_v\ell_0) + \ell_0\,U\end{pmatrix} \in \mathcal{C}^\infty(B_0(\mathbf{s},2r))$$

and

$$\alpha = \partial_v \ell_0 \cdot U \in \mathcal{C}^\infty(B_0(\mathbf{s}, 2r))$$

where U was introduced in (5.4.10). The function ℓ_j must satisfy $(\widetilde{U} \cdot \nabla + \alpha)\ell_j = -R_j(\ell_0, \dots, \ell_{j-1})$ so we are intersted in the operaor $\mathcal{L} = \widetilde{U} \cdot \nabla + \alpha$ that we decompose as $\mathcal{L} = \mathcal{L}_0^{\mathbf{s}} + \mathcal{L}_>$ with

$$\mathcal{L}_0^{\mathbf{s}} = \widetilde{U}_0^{\mathbf{s}} \begin{pmatrix} x - \mathbf{s} \\ v \end{pmatrix} \cdot \nabla + \alpha_0^{\mathbf{s}}$$

where $\widetilde{U}_0^{\mathbf{s}}$ is the differential of $\widetilde{U}$ at $(\mathbf{s}, 0)$ and $\alpha_0^{\mathbf{s}} = \alpha(\mathbf{s}, 0)$, that is

$$\widetilde{U}_0^{\mathbf{s}} = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Hess}_{\mathbf{s}} V + 2M_0(\mathbf{s},0,0)\nu_2^{\mathbf{s}}\, {}^t\nu_1^{\mathbf{s}} & M_0(\mathbf{s},0,0)(\mathrm{Id} + 2\nu_2^{\mathbf{s}}\, {}^t\nu_2^{\mathbf{s}}) \end{pmatrix}$$

and

$$\alpha_0^{\mathbf{s}} = M_0(\mathbf{s},0,0)\nu_2^{\mathbf{s}} \cdot \nu_2^{\mathbf{s}}. \tag{5.4.16}$$

As usual, we will often omitt the exponents $\mathbf{s}$ in the notations. Notice that if we denote $\mathcal{P}^n_{hom}$ the space of homogeneous polynomials of degree n in the variables $(x-\mathbf{s}, v)$, we have $\mathcal{L}_0 \in \mathscr{L}(\mathcal{P}^n_{hom})$ and for $P \in \mathcal{P}^n_{hom}$, $\mathcal{L}_> P(x,v) = O\big((x-\mathbf{s}, v)^{n+1}\big)$ near $(\mathbf{s}, 0)$.

Lemma 5.4.8. *The negative eigenvalue $-\alpha_0^{\mathbf{s}}$ of the matrix $\Phi^{\mathbf{s}}$ from Proposition 5.4.7 is its only one (counting multiplicity) in $\{\mathrm{Re}\, z \leq 0\}$. Moreover, all the eigenvalues of $\widetilde{U}_0^{\mathbf{s}}$ have positive real part and the operator $\mathcal{L}_0^{\mathbf{s}}$ is invertible on $\mathcal{P}^n_{hom}$.*

Proof. It is sufficient to prove the first statement. Indeed, if $-\alpha_0$ is the only eigenvalue of Φ in $\{\mathrm{Re}\, z \leq 0\}$, we can then remark that

$${}^t\widetilde{U}_0 = \Phi + 2\begin{pmatrix} 0 & \nu_1\, {}^t\nu_2 M_0(\mathbf{s},0,0) \\ 0 & \nu_2\, {}^t\nu_2 M_0(\mathbf{s},0,0) \end{pmatrix}$$

and since the last term has its range included in $\mathbb{C}\nu$ and sends ν on $2\alpha_0\nu$, the matrix of ${}^t\widetilde{U}_0$ in a basis $(\nu, b_2, \dots, b_{2d})$ in which Φ becomes triangular is also triangular and has on its diagonal the eigenvalues of Φ except for $-\alpha_0$ which is replaced by $+\alpha_0$. Hence $\mathrm{Spec}(\widetilde{U}_0) = \mathrm{Spec}({}^t\widetilde{U}_0) \subset \{\mathrm{Re}\, z > 0\}$ and we can conclude thanks to Lemma A.1 from [4]. Let us then prove that $-\alpha_0$ is the only eigenvalue (counting multiplicity) of Φ in $\{\mathrm{Re}\, z \leq 0\}$. We proceed as in [4], Lemma 2.6. For $t \in [0,1]$, consider the matrix

$$\Phi_t = 2\,\mathrm{Hess}_{\mathbf{s}} W \begin{pmatrix} (1-t)\mathrm{Id} & -t\mathrm{Id} \\ t\mathrm{Id} & tM_0(\mathbf{s},0,0) + (1-t)\mathrm{Id} \end{pmatrix}$$

which trivially satisfies the assumptions of Lemma 5.7.1 for $t \in [0,1)$. It is also the case of Φ_1 as $\Phi_1(x,0) = (0,x)$. Hence for every $t \in [0,1]$, Φ_t has no eigenvalues in $i\mathbb{R}$ and since these eigenvalues depend continuously on t, we get that

$$\#\big(\mathrm{Spec}\,\Phi_1 \cap \{\mathrm{Re}\, z < 0\}\big) = \#\big(\mathrm{Spec}\,\Phi_0 \cap \{\mathrm{Re}\, z < 0\}\big).$$

But $\Phi_0 = 2\,\mathrm{Hess}_{\mathbf{s}} W$ has exactly one negative eigenvalue (with multiplicity) while all the others are positive since $\mathbf{s} \in \mathcal{U}^{(1)}$, so we have indeed showed that $-\alpha_0$ is the only eigenvalue of $\Phi = \Phi_1$ (counting multiplicity) in $\{\mathrm{Re}\, z \leq 0\}$. ☐

One can then proceed as in [4], section 3.3 (see also [12], chapter 3), i.e use Lemma 5.4.8 to find an approximate solution of (5.4.13) using formal power series and then refine it into an actual solution using again Lemma 5.4.8 as well as the characteristic method. We then get the following result.

Proposition 5.4.9. *For all $j \geq 1$, there exists $\ell_j^{\mathbf{s}} \in \mathcal{C}^\infty(B_0(\mathbf{s}, 2r))$ solving (5.4.13). Moreover, $\ell_j^{\mathbf{s}}$ is real valued in view of Lemmas 5.4.2 and 5.4.3.*

5.5 Computation of the small eigenvalues

Now that we have found $(\ell_j)_{j\geq 0} \subset \mathcal{C}^\infty(B_0(\mathbf{s}, 2r), \mathbb{R})$ solving (5.4.12) and (5.4.13) with ℓ_0 vanishing at $(\mathbf{s}, 0)$, we can use a Borel procedure to construct $\ell \in \mathcal{C}^\infty(\mathbb{R}^{2d}, \mathbb{R})$ supported in $B_0(\mathbf{s}, 3r)$ and satisfying $\ell \sim \sum_{j\geq 0} \ell_j$ on $B_0(\mathbf{s}, 2r)$.

Remark 5.5.1. *The properties a)-c) from (5.3.5) are satisfied by both the functions $\ell^{\mathbf{s},h}$ and $-\ell^{\mathbf{s},h}$. Moreover, by Lemma 5.4.3, $(-\ell_j^{\mathbf{s}})_{j\geq 0}$ also solve (5.4.12) and (5.4.13).*

We are now in position to prove that all the properties from (5.3.5) are satisfied.

Proposition 5.5.2. *We can choose the signs of the functions $(\ell^{\mathbf{s},h})_{\mathbf{j}(\mathbf{m})}$ such that (5.3.5) holds true and the coefficients from the classical expansion of $\ell^{\mathbf{s},h}$ solve (5.4.12) and (5.4.13).*

Proof. Recall that by item b) from Hypothesis 5.3.5, each function $\ell^{\mathbf{s},h}$ corresponds to a unique $\mathbf{m} \in \mathcal{U}^{(0)} \backslash \{\underline{\mathbf{m}}\}$. Thanks to Lemmas 5.4.5 and 5.4.6, it is clear that item d) from (5.3.5) is satisfied by both $\ell^{\mathbf{s},h}$ and $-\ell^{\mathbf{s},h}$. Hence according to Remark 5.5.1, it is sufficient to prove that the signs of $(\ell^{\mathbf{s},h})_{\mathbf{j}(\mathbf{m})}$ can be chosen so that $\theta_{\mathbf{m},h}$ is smooth on a neighborhood of supp $\chi_{\mathbf{m}}$. From (5.3.7), (5.3.8) and (5.3.9) we see that the only parts on which it is not clear that $\theta_{\mathbf{m},h}$ is smooth are

$$F_1 = \bigsqcup_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} \Big(\{|\ell_0^{\mathbf{s}}| \leq 2\gamma\} \cap \partial B_0(\mathbf{s}, r)\Big), \quad F_2 = \bigsqcup_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} \Big(B_0(\mathbf{s}, r) \cap \{|\ell_0^{\mathbf{s}}| = 2\gamma\}\Big)$$

$$\text{and} \qquad F_3 = \partial\Big(E(\mathbf{m}) + B(0, \varepsilon)\Big) \backslash \Big(\bigsqcup_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} \big(B_0(\mathbf{s}, r) \cap \{|\ell_0^{\mathbf{s}}| \leq 2\gamma\}\big)\Big).$$

Here the unions are disjoint for r small enough. Let $\mathbf{s} \in \mathbf{j}(\mathbf{m})$ and $(x, v) \in \overline{B_0(\mathbf{s}, r)} \backslash \{(\mathbf{s}, 0)\}$ such that $\ell_0^{\mathbf{s}}(x, v) = 0$. Using Lemma 5.4.6, we see that if $r > 0$ is small enough,

$$W(x, v) - W(\mathbf{s}, 0) = \phi_+(x, v) > 0 \tag{5.5.1}$$

because $(\mathbf{s}, 0)$ is a non degenerate local minimum of ϕ_+. Hence, $\{\ell_0^{\mathbf{s}} = 0\} \cap B_0(\mathbf{s}, r) \subset \{W \geq \boldsymbol{\sigma}(\mathbf{m})\}$. Now assume by contradiction that for any $r > 0$, the function $\ell_0^{\mathbf{s}}$ takes both positive and negative values on $E(\mathbf{m}) \cap B_0(\mathbf{s}, r)$. Then according to Lemma C.0.1, the two CCs of $U_r \cap \{W < \boldsymbol{\sigma}(\mathbf{m})\}$ are both included in $E(\mathbf{m})$ (the one on which $\ell_0^{\mathbf{s}} > 0$ and the one where $\ell_0^{\mathbf{s}} < 0$). This is a contradiction with the fact that $\mathbf{s} \in \mathtt{V}^{(1)}$. Therefore $\ell_0^{\mathbf{s}}$ has a sign on $E(\mathbf{m}) \cap B_0(\mathbf{s}, r)$ and we can choose it so that $\ell_0^{\mathbf{s}}$ is a positive function on $E(\mathbf{m}) \cap B_0(\mathbf{s}, r)$. By uniform continuity, we can then choose $\varepsilon(\gamma) > 0$ small enough so that

$$\Big(\big(E(\mathbf{m}) + B(0, \varepsilon)\big) \cap B_0(\mathbf{s}, r)\Big) \subseteq \{\ell_0^{\mathbf{s}} \geq -\gamma\}. \tag{5.5.2}$$

Similarly, if we denote $\Omega_{\mathbf{s}}$ the other CC of $\{W < \boldsymbol{\sigma}(\mathbf{m})\}$ which contains $(\mathbf{s}, 0)$ on its boundary, we have since $(\mathbf{s}, 0)$ is not a critical point of $\ell_0^{\mathbf{s}}$ that this function is negative on $\Omega_{\mathbf{s}} \cap B_0(\mathbf{s}, r)$ and

$$\Big(\big(\Omega_{\mathbf{s}} + B(0, \varepsilon)\big) \cap B_0(\mathbf{s}, r)\Big) \subseteq \{\ell_0^{\mathbf{s}} \leq \gamma\}. \tag{5.5.3}$$

Choosing once again $\varepsilon(r)$ small enough, we can even assume that

$$\Big(\overline{E(\mathbf{m}) + B(0, \varepsilon)} \cap \overline{\Omega_{\mathbf{s}} + B(0, \varepsilon)}\Big) \subseteq B_0(\mathbf{s}, r). \tag{5.5.4}$$

We first prove that F_1 does not meet the support of $\chi_{\mathbf{m}}$. Recall that Ω denotes the CC of $\{W \leq \boldsymbol{\sigma}(\mathbf{m})\}$ containing $\mathbf{m}$. For $\mathbf{s} \in \mathbf{j}(\mathbf{m})$, we can deduce from (5.5.1) that if $(x, v) \in \partial B_0(\mathbf{s}, r)$ such that $\ell_0^{\mathbf{s}}(x, v) = 0$, then $(x, v) \notin \Omega$. Hence $|\ell_0^{\mathbf{s}}|$ must attain a positive minimum on $\partial B_0(\mathbf{s}, r) \cap \Omega$, so we can choose $\gamma(r) > 0$ such that $\partial B_0(\mathbf{s}, r) \cap \{|\ell_0^{\mathbf{s}}| \leq 2\gamma\}$ does not intersect Ω. It follows that we can choose $\varepsilon(\gamma) > 0$ such that

$$F_1 \subseteq \big(\mathbb{R}^{2d} \backslash \overline{\Omega + B(0, \varepsilon)}\big) \subseteq \big(\mathbb{R}^{2d} \backslash \operatorname{supp} \chi_{\mathbf{m}}\big).$$

Now we show that $\theta_{\mathbf{m},h}$ is smooth on $F_2 \cap (\Omega + B(0,\varepsilon))$: let $\mathbf{s} \in \mathbf{j}(\mathbf{m})$ and $(x,v) \in B_0(\mathbf{s},r) \cap \{\ell_0^{\mathbf{s}} = 2\gamma\} \cap (\Omega + B(0,\varepsilon))$. According to (5.5.3) and the fact that $\ell^{\mathbf{s},h} = \ell_0^{\mathbf{s}} + O(h)$, there exists a small ball B centered in (x,v) such that

$$B \subset \Big(B_0(\mathbf{s},r) \cap \{\ell^{\mathbf{s},h} > \gamma\} \cap \big(E(\mathbf{m}) + B(0,\varepsilon)\big)\Big).$$

Thus $\theta_{\mathbf{m},h} = 1$ on B and $\theta_{\mathbf{m},h}$ is smooth at (x,v). Similarly, for $(x,v) \in B_0(\mathbf{s},r) \cap \{\ell^{\mathbf{s},h} = -2\gamma\} \cap (\Omega + B(0,\varepsilon))$, we can show that $\theta_{\mathbf{m},h} = 0$ in a neighborhood of (x,v).
It only remains to prove that, as for F_1, the set F_3 does not meet the support of $\chi_{\mathbf{m}}$. First we remark that thanks to (5.5.2), we can forget the absolute value in the definition of F_3 :

$$F_3 = \partial\Big(E(\mathbf{m}) + B(0,\varepsilon)\Big)\backslash\Big(\bigsqcup_{\mathbf{j}(\mathbf{m})} \big(B_0(\mathbf{s},r) \cap \{\ell_0^{\mathbf{s}} \leq 2\gamma\}\big)\Big).$$

If $(x,v) \in F_3 \cap B_0(\mathbf{s},r)$, we have that $\ell_0^{\mathbf{s}}(x,v) > 2\gamma$ so using (5.5.3), we see that (x,v) is outside $\Omega_{\mathbf{s}} + B(0,\varepsilon)$. Since it is not in $(E(\mathbf{m}) + B(0,\varepsilon))$ either, it is outside $\Omega + B(0,\varepsilon)$ which contains the support of $\chi_{\mathbf{m}}$. Now if $(x,v) \in F_3 \backslash \big(\mathbf{j}^W(\mathbf{m}) + B_0(0,r)\big)$, (5.5.4) implies that (x,v) is outside $\cup_{\mathbf{j}(\mathbf{m})}(\Omega_{\mathbf{s}} + B(0,\varepsilon))$ so it is also outside $\Omega + B(0,\varepsilon)$ for ε small enough and the proof is complete. □

Lemma 5.5.3. *Let $\mathbf{m} \in \mathcal{U}^{(0)} \backslash \{\underline{\mathbf{m}}\}$ and denote $\tilde{f}_{\mathbf{m},h} = f_{\mathbf{m},h}/\|f_{\mathbf{m},h}\|$ where $f_{\mathbf{m},h}$ was defined in (5.3.11). With the notation (5.4.16), we have that*

$$\langle P_h \tilde{f}_{\mathbf{m},h}, \tilde{f}_{\mathbf{m},h}\rangle = h e^{-2\frac{S(\mathbf{m})}{h}} \frac{\det(\mathrm{Hess}_{\mathbf{m}} V)^{1/2}}{2\pi} \tilde{B}_h(\mathbf{m}) \in \mathbb{R}$$

with $\tilde{B}_h(\mathbf{m})$ admitting a classical expansion whose first term equals

$$\sum_{\mathbf{s} \in \mathbf{j}(\mathbf{m})} |\det(\mathrm{Hess}_{\mathbf{s}} V)|^{-1/2}\, \alpha_0^{\mathbf{s}}.$$

Proof. Since X_0^h is a skew-adjoint differential operator and $f_{\mathbf{m},h}$ is real valued, we have

$$\langle X_0^h f_{\mathbf{m},h}, f_{\mathbf{m},h}\rangle = 0.$$

Besides, we know from (5.3.16) that

$$b_h f_{\mathbf{m},h} = h(\partial_v \theta)\chi e^{-W_{\mathbf{m}}/h} + O_{L^2}(h^\infty e^{-S(\mathbf{m})/h}) \tag{5.5.5}$$

so we easily deduce from the fact that $(\partial_v\theta)\chi e^{-W_{\mathbf{m}}/h} = O_{L^2}(e^{-S(\mathbf{m})/h})$ and the boundedness of $\mathrm{Op}_h(M^h)$ that

$$\langle Q_h f_{\mathbf{m},h}, f_{\mathbf{m},h}\rangle = h^2\Big\langle \mathrm{Op}_h(M^h)\big((\partial_v\theta)\chi e^{-W_{\mathbf{m}}/h}\big)\,,\,(\partial_v\theta)\chi e^{-W_{\mathbf{m}}/h}\Big\rangle + O\Big(h^\infty e^{-\frac{2S(\mathbf{m})}{h}}\Big).$$

Since we have with the notation (5.3.13)

$$(\partial_v\theta)\chi e^{-W_{\mathbf{m}}/h} = \frac{A_h^{-1}}{2} e^{-\widetilde{W}_{\mathbf{m}}/h}\chi \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} \zeta(\ell^{\mathbf{s}})\partial_v \ell^{\mathbf{s}}\, \mathbf{1}_{B_0(\mathbf{s},r)}$$

and using (5.3.21) with M instead of g, we get that

$$\langle P_h f_{\mathbf{m},h}, f_{\mathbf{m},h}\rangle = \frac{h^2}{4} A_h^{-2} \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} \int_{B_0(\mathbf{s},r)} e^{-2\widetilde{W}_{\mathbf{m}}(x,v)/h}\chi\zeta(\ell^{\mathbf{s}})\tilde{I}^{\mathbf{s}}(x,v)\cdot\partial_v\ell^{\mathbf{s}}\, d(x,v) + O\Big(h^\infty e^{-2\frac{S(\mathbf{m})}{h}}\Big). \tag{5.5.6}$$

where

$$\tilde{I}^{\mathbf{s}}(x,v) = (2\pi h)^{-d}\int_{\mathbb{R}^d}\int_{|v'|\leq r} e^{\frac{i}{h}\eta\cdot(v-v')}\chi(x,v')\zeta\big(\ell^{\mathbf{s}}(x,v')\big) M\Big(x, \frac{v+v'}{2}, \eta + i\psi(x,v,v')\Big)\partial_v\ell^{\mathbf{s}}(x,v')\, dv' d\eta.$$

Mimicking the proof of Proposition 5.7.6, one can show that $\zeta(\ell)$ admits a classical expansion whose first term is $\zeta(\ell_0)$. Besides, since M and ψ also have a classical expansion, we could use the stationnary phase (see for instance [47], Theorem 3.17) as well Proposition 5.7.4 to get an expansion of $\tilde{I}$ similar to the one obtained in (5.4.9). Thus, we get that $\tilde{I}\cdot\partial_v\ell \sim \sum_{k\geq 0} h^k a_k$ where

$$a_0(x,v) = \chi(x,v)\zeta\big(\ell_0(x,v)\big)M_0\Big(x,v,i\big(\frac{v}{2}+\ell_0\,\partial_v\ell_0\big)\Big)\partial_v\ell_0(x,v)\cdot\partial_v\ell_0(x,v).$$

Hence, using the fact that on $B_0(\mathbf{s},r)$,

$$\widetilde{W} - S(\mathbf{m}) = W_{\mathbf{m}} + \frac{\ell_0^2}{2} - S(\mathbf{m}) + \Big(\frac{\ell^2}{2} - \frac{\ell_0^2}{2}\Big),$$

it is clear that

$$(5.5.7)\quad e^{2S(\mathbf{m})/h}\int_{B_0(\mathbf{s},r)} \mathrm{e}^{-2\widetilde{W}(x,v)/h}\chi\zeta(\ell)\tilde{I}(x,v)\cdot\partial_v\ell\,\mathrm{d}(x,v) \sim_h \sum_{k\geq 0} h^k \int_{B_0(\mathbf{s},r)} \mathrm{e}^{-2\frac{W_{\mathbf{m}}(x,v)+\ell_0^2(x,v)/2-S(\mathbf{m})}{h}}\mathrm{e}^{-\frac{(\ell^2-\ell_0^2)(x,v)}{h}}\chi\zeta(\ell)a_k\,\mathrm{d}(x,v).$$

We would like to apply Proposition 5.7.8 so we need to check that the assumptions are satisfied. First, $\mathrm{Hess}_{(\mathbf{s},0)}(W_{\mathbf{m}}+\ell_0^2/2)$ is definite positive by Lemma 5.4.5. Besides, $h^{-1}(\ell^2-\ell_0^2)$ admits a classical expansion whose first term is $2(\ell_1\ell_0)$. Therefore, using the expansion of $\zeta(\ell)$ as well as Proposition 5.7.6, one easily gets that the function

$$\mathrm{e}^{-\frac{(\ell^2-\ell_0^2)}{h}}(\zeta\circ\ell)$$

admits a classical expansion whose first term is $\mathrm{e}^{-2(\ell_1\ell_0)}(\zeta\circ\ell_0)$. Thus, according to Propositions 5.7.8 and 5.4.7, there exists $(b_{k,j})$ such that

$$\frac{|\det(\mathrm{Hess}_{\mathbf{s}}V)|^{1/2}}{(2\pi h)^d}\int_{B_0(\mathbf{s},r)} \mathrm{e}^{-2\frac{W_{\mathbf{m}}(x,v)+\ell_0^2(x,v)/2-S(\mathbf{m})}{h}}\mathrm{e}^{-\frac{(\ell^2-\ell_0^2)(x,v)}{h}}\chi\zeta(\ell)a_k\,\mathrm{d}(x,v)\sim\sum_{j\geq 0}h^j b_{k,j}$$

where $b_{k,0} = a_k(\mathbf{s},0)$. Hence, using (5.5.6), (5.5.7) and Proposition 5.7.4, we deduce that

$$(5.5.8)\qquad 4A_h^2(2\pi)^{-d}h^{-d-2}\mathrm{e}^{2S(\mathbf{m})/h}\langle P_h f_{\mathbf{m},h}, f_{\mathbf{m},h}\rangle \sim \sum_{k\geq 0} h^k c_k$$

with

$$c_0 = \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} |\det(\mathrm{Hess}_{\mathbf{s}}V)|^{-1/2}M_0(\mathbf{s},0,0)\nu_2^{\mathbf{s}}\cdot\nu_2^{\mathbf{s}} = \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} |\det(\mathrm{Hess}_{\mathbf{s}}V)|^{-1/2}\,\alpha_0^{\mathbf{s}}.$$

Similarly, thanks to item a) from Hypothesis 5.3.5, one can use Proposition 5.7.8 as we already did to see that there exists $(\tilde{c}_k)_{k\geq 0}$ such that

$$(5.5.9)\qquad \frac{\det(\mathrm{Hess}_{\mathbf{m}}\mathrm{V})^{1/2}}{(2\pi h)^d}\|f_{\mathbf{m},h}\|^2 \sim \sum_{k\geq 0} h^k\tilde{c}_k$$

with $\tilde{c}_0 = 1$. The conclusion follows from (5.5.8), (5.3.6) and (5.5.9). □

Lemma 5.5.4. *Let $\mathbf{m}\in\mathcal{U}^{(0)}\backslash\{\underline{\mathbf{m}}\}$. Using the notations from Lemma 5.5.3, we have*

i) $\|P_h\tilde{f}_{\mathbf{m},h}\|^2 = O(h^\infty\langle P_h\tilde{f}_{\mathbf{m},h},\tilde{f}_{\mathbf{m},h}\rangle)$

ii) $\|P_h^*\tilde{f}_{\mathbf{m},h}\|^2 = O(h\langle P_h\tilde{f}_{\mathbf{m},h},\tilde{f}_{\mathbf{m},h}\rangle)$.

Proof. To prove $i)$, first remark that thanks to (5.3.15)-(5.3.18) we have

$$(5.5.10)\qquad \int_{\mathbb{R}^{2d}\backslash(\mathbf{j}^W(\mathbf{m})+B_0(0,2r))} |P_h f_{\mathbf{m},h}(x,v)|^2\mathrm{d}(x,v) = O\Big(h^\infty\mathrm{e}^{-2\frac{S(\mathbf{m})}{h}}\Big).$$

Besides, we saw that thanks to Proposition 5.7.8 and Lemma 5.4.5, we have for $\mathbf{s} \in \mathbf{j}(\mathbf{m})$,

$$\int_{B_0(\mathbf{s},2r)} \mathrm{e}^{-2\frac{\widetilde{W}(x,v)}{h}} \mathrm{d}(x,v) = O\Big(h^d \mathrm{e}^{-2\frac{S(\mathbf{m})}{h}}\Big).$$

Moreover, the function ω from Proposition 5.3.7 is $O_{L^\infty(B_0(\mathbf{s},2r))}(h^\infty)$ by Lemma 5.4.1 and the construction of the $(\ell^{\mathbf{s},h})_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}$. Hence, by Proposition 5.3.7,

$$\int_{B_0(\mathbf{s},2r)} |P_h f_{\mathbf{m},h}(x,v)|^2 \mathrm{d}(x,v) = O\Big(h^\infty \mathrm{e}^{-2\frac{S(\mathbf{m})}{h}}\Big). \tag{5.5.11}$$

The conclusion follows from (5.5.10), (5.5.11) as well as (5.5.9) and Lemma 5.5.3. The proof of *ii)* can be obtained similarly with the use of Proposition 5.7.8 and Remark 5.3.8 after noticing that $\overset{*}{\omega}$ also admits a classical expansion whose first term vanishes on $\mathbf{j}^W(\mathbf{m})$. □

From now on, we denote

$$\tilde\lambda_{\mathbf{m},h} = \langle P_h \tilde f_{\mathbf{m},h}, \tilde f_{\mathbf{m},h}\rangle = \langle Q_h \tilde f_{\mathbf{m},h}, \tilde f_{\mathbf{m},h}\rangle \tag{5.5.12}$$

for which we computed a classical expansion in Lemma 5.5.3.

Lemma 5.5.5. *For* $\mathbf{m}$ *and* $\mathbf{m}'$ *two distinct elements of* $\mathcal{U}^{(0)}$*, we have*

i) $\langle P_h \tilde f_{\mathbf{m},h}, \tilde f_{\mathbf{m}',h}\rangle = O\Big(h^\infty \sqrt{\tilde\lambda_{\mathbf{m},h}\tilde\lambda_{\mathbf{m}',h}}\Big)$

ii) There exists $c > 0$ *such that* $\langle \tilde f_{\mathbf{m},h}, \tilde f_{\mathbf{m}',h}\rangle = O(\mathrm{e}^{-c/h})$

Proof. i) : The result is obvious when one of the two minima is $\underline{\mathbf{m}}$. Recall the labeling of the minima that we introduced rigth before Hypothesis 5.3.5 as well as the map π_x from Lemma 5.3.2. Let us first suppose that $\mathbf{m} = \mathbf{m}_{k,j}$ and $\mathbf{m}' = \mathbf{m}_{k,j'}$ with $j \neq j'$ and $k \neq 1$ and denote $E = E(\mathbf{m})$ and $E' = E(\mathbf{m}')$. In particular $\boldsymbol{\sigma}(\mathbf{m}) = \boldsymbol{\sigma}(\mathbf{m}')$. Thanks to (5.3.12) and the fact that P_h is local in x, we have

$$\operatorname{supp} P_h \tilde f_{\mathbf{m},h} \subseteq \big(\pi_x(E)\times\mathbb{R}^d_v\big) + B(0,\varepsilon') \qquad \text{and} \qquad \operatorname{supp} \tilde f_{\mathbf{m}',h} \subseteq \big(E' + B(0,\varepsilon')\big)$$

so up to taking ε' small enough, it is sufficient to show that $\overline{\pi_x(E)\times\mathbb{R}^d_v}$ and $\overline{E'}$ do not intersect. Since our labeling is adapted, E and E' are two distinct CCs of $\{W < \boldsymbol{\sigma}(\mathbf{m})\}$ so by Lemma 5.3.2, $\pi_x(E)\times\mathbb{R}^d_v$ and E' are two disjoint open sets. Thus, using successively Remark 5.3.1 and (5.3.1), we get

$$\begin{aligned}\overline{\pi_x(E)\times\mathbb{R}^d_v} \cap \overline{E'} &= \Big(\partial\big(\pi_x(E)\big)\times\mathbb{R}^d_v\Big)\cap\partial E' \\ &\subseteq \Big(\partial\big(\pi_x(E)\big)\times\{0\}\Big)\cap\partial E' \\ &\subseteq \Big(\partial\big(\pi_x(E)\big)\cap\partial\big(\pi_x(E')\big)\Big)\times\{0\}.\end{aligned}$$

which is empty thanks to Lemma C.0.2 and item b) from Hypothesis 5.3.5.
Let us now treat the case $\mathbf{m} = \mathbf{m}_{k,j}$ and $\mathbf{m}' = \mathbf{m}_{k',j'}$ with $k, k' \geq 2$ and $k \neq k'$. We can suppose that $k < k'$ (i.e $\boldsymbol{\sigma}(\mathbf{m}) > \boldsymbol{\sigma}(\mathbf{m}')$) because we can work with P_h^* instead of P_h if needed. We decompose $P_h \tilde f_{\mathbf{m},h}$ as in (5.3.15) and (5.3.16) and once again we use (5.3.12) to get

$$\operatorname{supp} \tilde f_{\mathbf{m}',h} \subseteq \big(E' + B(0,\varepsilon')\big) \subseteq \Big\{W < \frac{\boldsymbol{\sigma}(\mathbf{m})+\boldsymbol{\sigma}(\mathbf{m}')}{2}\Big\}$$

as well as the fact that P_h is local in x to get a localization of the support of the first term from (5.3.16) :

$$\operatorname{supp}\Big(\mathrm{Op}_h(g)\big((\partial_v\theta_{\mathbf{m}})\chi_{\mathbf{m}}\mathrm{e}^{-W_{\mathbf{m}}/h}\big)\Big) \subseteq \Big(\big(\mathbf{j}(\mathbf{m}) + B(0,r)\big)\times\mathbb{R}^d_v\Big) \subseteq \Big\{W > \frac{\boldsymbol{\sigma}(\mathbf{m})+\boldsymbol{\sigma}(\mathbf{m}')}{2}\Big\}$$

as W increases with the norm of v. Hence, the support of the first term from (5.3.16) does not meet the one of $\tilde f_{\mathbf{m}',h}$. The same goes easily for the first term of (5.3.15). For the second term of (5.3.15), its support is contained in the support of $\nabla\chi_{\mathbf{m}}$ which is itself contained in $\{W \geq \boldsymbol{\sigma}(\mathbf{m})+\tilde\varepsilon\}$ so it clearly does not meet the support of $\tilde{f}_{\mathbf{m}',h}$. It only remains to treat the second term from (5.3.16), i.e $\mathrm{Op}_h(g)\big(\theta_{\mathbf{m}}(\partial_v\chi_{\mathbf{m}})\mathrm{e}^{-W_{\mathbf{m}}/h}\big)$. To this aim, notice that (5.5.5) yields $b_h f_{\mathbf{m}',h} = O_{L^2}(\mathrm{e}^{-S(\mathbf{m}')/h})$ and since by the support properties of $\nabla\chi_{\mathbf{m}}$ we also have $\theta_{\mathbf{m}}(\partial_v\chi_{\mathbf{m}})\mathrm{e}^{-W_{\mathbf{m}}/h} = O_{L^2}(h^\infty \mathrm{e}^{-S(\mathbf{m})/h})$, we get using the Cauchy-Schwarz inequality and the boundedness of $\mathrm{Op}_h(M)$

$$\begin{aligned}\Big\langle \mathrm{Op}_h(g)\big(\theta_{\mathbf{m}}(\partial_v\chi_{\mathbf{m}})\mathrm{e}^{-W_{\mathbf{m}}/h}\big)\,,\, f_{\mathbf{m}',h}\Big\rangle &= \Big\langle \mathrm{Op}_h(M)\big(\theta_{\mathbf{m}}(\partial_v\chi_{\mathbf{m}})\mathrm{e}^{-W_{\mathbf{m}}/h}\big)\,,\, b_h f_{\mathbf{m}',h}\Big\rangle \\ &= O\Big(h^\infty \mathrm{e}^{-\frac{S(\mathbf{m})+S(\mathbf{m}')}{h}}\Big)\end{aligned}$$

which proves the first item.
ii) : Here we can suppose that $V(\mathbf{m}) \geq V(\mathbf{m}')$. Let us first treat the case where $V(\mathbf{m}) = V(\mathbf{m}')$. Then according to item a) from Hypothesis 5.3.5, E and E' are two disjoint open sets. Hence, as we saw earlier, Lemma C.0.2 and item b) from Hypothesis 5.3.5 imply that $\overline{E}\cap\overline{E'} = \emptyset$. The conclusion then follows from (5.3.12).
If $V(\mathbf{m}) > V(\mathbf{m}')$, then item a) from Hypothesis 5.3.5 implies that $(\mathbf{m}, 0)$ is the only global minimum of $W|_{E+B(0,\varepsilon')}$. Therefore using (5.3.12), we can easily compute

$$\langle f_{\mathbf{m},h}, f_{\mathbf{m}',h}\rangle = \int_{E+B(0,\varepsilon')} \theta_{\mathbf{m}}\theta_{\mathbf{m}'}\chi_{\mathbf{m}}\chi_{\mathbf{m}'}\mathrm{e}^{-\frac{2V-V(\mathbf{m})-V(\mathbf{m}')+v^2}{2h}}\mathrm{d}(x,v) = O\Big(\mathrm{e}^{-\frac{V(\mathbf{m})-V(\mathbf{m}')}{2h}}\Big).$$

The conclusion immediately follows from (5.5.9). ☐

Let us consider once again the spectral projection introduced in (5.2.5). We saw in particular that $\Pi_0 = O(1)$.

Lemma 5.5.6. *For any $\mathbf{m}\in\mathcal{U}^{(0)}$, we have*

$$\|(1-\Pi_0)\tilde{f}_{\mathbf{m},h}\| = O\Big(h^\infty\sqrt{\tilde{\lambda}_{\mathbf{m},h}}\Big) \qquad \textit{and} \qquad \|(1-\Pi_0^*)\tilde{f}_{\mathbf{m},h}\| = O\Big(h^{-3/2}\sqrt{\tilde{\lambda}_{\mathbf{m},h}}\Big).$$

Proof. We simply recall the proof from [26] : we write

$$\begin{aligned}(1-\Pi_0)\tilde{f}_{\mathbf{m},h} &= \frac{1}{2i\pi}\int_{|z|=ch^2}\big(z^{-1}-(z-P_h)^{-1}\big)\tilde{f}_{\mathbf{m},h}\mathrm{d}z \\ &= \frac{-1}{2i\pi}\int_{|z|=ch^2} z^{-1}(z-P_h)^{-1}P_h\tilde{f}_{\mathbf{m},h}\mathrm{d}z.\end{aligned}$$

We can then conclude using Lemma 5.5.4 and the resolvent estimate from Theorem 5.1.5. The proof for the adjoint is almost identical. ☐

Lemma 5.5.7. *The family $(\Pi_0\tilde{f}_{\mathbf{m},h})_{\mathbf{m}\in\mathcal{U}^{(0)}}$ is almost orthonormal : there exists $c>0$ such that*

$$\langle \Pi_0\tilde{f}_{\mathbf{m},h}, \Pi_0\tilde{f}_{\mathbf{m}',h}\rangle = \delta_{\mathbf{m},\mathbf{m}'} + O(\mathrm{e}^{-c/h}).$$

In particular, it is a basis of the space $H = \mathrm{Ran}\,\Pi_0$ introduced in (5.2.5).
Moreover, we have

$$\langle P_h\Pi_0\tilde{f}_{\mathbf{m},h}, \Pi_0\tilde{f}_{\mathbf{m}',h}\rangle = \delta_{\mathbf{m},\mathbf{m}'}\tilde{\lambda}_{\mathbf{m},h} + O\Big(h^\infty\sqrt{\tilde{\lambda}_{\mathbf{m},h}\tilde{\lambda}_{\mathbf{m}',h}}\Big).$$

Proof. The proof is the same as the one of Proposition 4.10 in [26]. ☐

Let us re-label the local minima $\mathbf{m}_1,\dots,\mathbf{m}_{n_0}$ so that $(S(\mathbf{m}_j))_{j=1,\dots,n_0}$ is non increasing in j. For shortness, we will now denote

$$\tilde{f}_j = \tilde{f}_{\mathbf{m}_j,h} \qquad \text{and} \qquad \tilde{\lambda}_j = \tilde{\lambda}_{\mathbf{m}_j,h}$$

which still depend on h. Note in particular that according to Lemma 5.5.3, $\tilde{\lambda}_j = O(\tilde{\lambda}_k)$ whenever $1\leq j\leq k\leq n_0$. We also denote $(\tilde{u}_j)_{j=1,\dots,n_0}$ the orthogonalization by the Gram-Schmidt procedure of the family $(\Pi_0\tilde{f}_j)_{j=1,\dots,n_0}$ and

$$u_j = \frac{\tilde{u}_j}{\|\tilde{u}_j\|}.$$

In this setting and with our previous results, we get the following (see [26], Proposition 4.12 for a proof).

Lemma 5.5.8. *For all $1 \leq j,k \leq n_0$, it holds*

$$\langle P_h u_j, u_k\rangle = \delta_{j,k}\tilde\lambda_j + O\Big(h^\infty\sqrt{\tilde\lambda_j\tilde\lambda_k}\Big).$$

In order to compute the small eigenvalues of P_h, let us now consider the restriction $P_h|_H : H \to H$. We denote $\hat u_j = u_{n_0-j+1}$, $\hat\lambda_j = \tilde\lambda_{n_0-j+1}$ and $\mathcal{M}$ the matrix of $P_h|_H$ in the orthonormal basis $(\hat u_1,\dots,\hat u_{n_0})$. Since $\hat u_{n_0} = u_1 = \tilde f_1$, we have

$$\mathcal{M} = \begin{pmatrix}\mathcal{M}' & 0\\ 0 & 0\end{pmatrix} \qquad \text{where} \qquad \mathcal{M}' = \Big(\langle P_h\hat u_j, \hat u_k\rangle\Big)_{1\leq j,k\leq n_0-1}$$

and it is sufficient to study the spectrum of $\mathcal{M}'$. We will also denote $\{\hat S_1 < \dots < \hat S_p\}$ the set $\{S(\mathbf{m}_j)\,;\, 2\leq j\leq n_0\}$ and for $1\leq k\leq p$, E_k the subspace of $L^2(\mathbb{R}^{2d})$ generated by $\{\hat u_r\,;\, S(\mathbf{m}_r) = \hat S_k\}$. Finally, we set $\varpi_k = \mathrm{e}^{-(\hat S_k - \hat S_{k-1})/h}$ for $2\leq k\leq p$ and $\varepsilon_j(\varpi) = \prod_{k=2}^j \varpi_k = \mathrm{e}^{-(\hat S_j-\hat S_1)/h}$ for $2\leq j\leq p$ (with the convention $\varepsilon_1(\varpi) = 1$).

Proposition 5.5.9. *There exists a diagonal matrix $M_h^\#$ admitting a classical expansion whose first term is*

$$M_0^\# = \mathrm{diag}\Bigg(\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m}_{n_0-j+1})}\frac{\det(\mathrm{Hess}_{\mathbf{m}_{n_0-j+1}}V)^{1/2}}{2\pi|\det(\mathrm{Hess}_{\mathbf{s}}V)|^{1/2}}\alpha_0^{\mathbf{s}}\,;\,1\leq j\leq n_0-1\Bigg)$$

such that

$$h^{-1}\mathrm{e}^{2\hat S_1/h}\mathcal{M}' = \Omega(\varpi)\big(M_h^\# + O(h^\infty)\big)\Omega(\varpi)$$

where $\Omega(\varpi) = \mathrm{diag}\big(\varepsilon_1(\varpi)\mathrm{Id}_{E_1},\dots,\varepsilon_p(\varpi)\mathrm{Id}_{E_p}\big)$.

Remark 5.5.10. *In the words of Definition 6.7 from [4], the last Proposition implies that $h^{-1}\mathrm{e}^{2\hat S_1/h}\mathcal{M}'$ is a classical graded symmetric matrix.*

Proof. According to Lemma 5.5.8, we can decompose $\mathcal{M}' = \mathcal{M}'_1 + \mathcal{M}'_2$ with

$$\mathcal{M}'_1 = \mathrm{diag}(\hat\lambda_j\,;\,1\leq j\leq n_0-1) \qquad \text{and} \qquad \mathcal{M}'_2 = \Big(O\Big(h^\infty\sqrt{\hat\lambda_j\hat\lambda_k}\Big)\Big)_{1\leq j,k\leq n_0-1}.$$

We will take $M_h^\# = h^{-1}\mathrm{e}^{2\hat S_1/h}\Omega(\varpi)^{-1}\mathcal{M}'_1\Omega(\varpi)^{-1}$ which is clearly diagonal, so we just need to check that it has the proper classical expansion and that $h^{-1}\mathrm{e}^{2\hat S_1/h}\Omega(\varpi)^{-1}\mathcal{M}'_2\Omega(\varpi)^{-1} = O(h^\infty)$. It is easy to compute

$$h^{-1}\mathrm{e}^{2\hat S_1/h}\Omega(\varpi)^{-1}\mathcal{M}'_1\Omega(\varpi)^{-1} = h^{-1}\mathrm{diag}\Big(\mathrm{e}^{2\hat S_{j'}/h}\hat\lambda_j\,;\,1\leq j\leq n_0-1\Big)$$

where $1\leq j'\leq p$ is such that $\hat S_{j'} = S(\mathbf{m}_{n_0-j+1})$. Hence Lemma 5.5.3 yields

$$h^{-1}\mathrm{e}^{2\hat S_1/h}\Omega(\varpi)^{-1}\mathcal{M}'_1\Omega(\varpi)^{-1} = \mathrm{diag}\Bigg(\frac{\det(\mathrm{Hess}_{\mathbf{m}_{n_0-j+1}}V)^{1/2}}{2\pi}\tilde B_h(\mathbf{m}_{n_0-j+1})\,;\,1\leq j\leq n_0-1\Bigg)$$

where $\tilde B_h(\mathbf{m}_{n_0-j+1})$ was introduced in Lemma 5.5.3 and admits a classical expansion whose first term is

$$\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m}_{n_0-j+1})}|\det(\mathrm{Hess}_{\mathbf{s}}V)|^{-1/2}\,\alpha_0^{\mathbf{s}}$$

so $M_h^\#$ has the desired expansion. Similarly, still using Lemma 5.5.3, one easily gets

$$\Omega(\varpi)^{-1}\mathcal{M}'_2\Omega(\varpi)^{-1} = \Big(O\Big(h^\infty\sqrt{\hat\lambda_j\hat\lambda_k}\,\varepsilon_{j'}(\varpi)^{-1}\varepsilon_{k'}(\varpi)^{-1}\Big)\Big)_{1\leq j,k\leq n_0-1}$$

where $1\leq j'\leq p$ and $1\leq k'\leq p$ are such that $\sqrt{\hat\lambda_j}\,\varepsilon_{j'}(\varpi)^{-1}$ and $\sqrt{\hat\lambda_k}\,\varepsilon_{k'}(\varpi)^{-1}$ are both $O(\sqrt{h}\,\mathrm{e}^{-\hat S_1/h})$ so the proof is complete. ∎

Proof of Theorem 5.1.7. According to Remark 5.5.10, it now suffices to combine the result of Proposition 5.5.9 with Theorem 4 from [4] which gives a description of the spectrum of classical graded almost symmetric matrices. Indeed, using the notations from this reference, we have for $1 \leq j \leq p$ that

$$\mathcal{J} \circ \mathcal{R}_j\Big(M_h^{\#} + O(h^\infty)\Big) = \mathcal{J} \circ \mathcal{R}_j\big(M_h^{\#}\big) + O(h^\infty)$$

and the result comes easily since $M_h^{\#}$ is diagonal. Therefore, we have actually proved that $B_h(\mathbf{m})$ from Theorem 5.1.7 and $\tilde{B}_h(\mathbf{m})$ from Lemma 5.5.3 have the same classical expansion. □

5.6 Return to equilibrium and metastability

The goal of this section is to prove Corollaries 5.1.9 and 5.1.10. We assume that the hypotheses of Theorem 5.1.7 are satisfied and we choose $\mathbf{m}^*$ among the elements of $\mathcal{U}^{(0)} \backslash \{\underline{\mathbf{m}}\}$ for which S is maximal such that the expansion of $\det(\mathrm{Hess}_{\mathbf{m}^*} V)^{1/2} B_h(\mathbf{m}^*)$ is minimal. According to Lemma 5.5.3 and Theorem 5.1.7, one can think of $\lambda_{\mathbf{m}^*,h}$ as the non zero eigenvalue of P_h with the smallest real part modulo $O(h^\infty \mathrm{e}^{-2S(\mathbf{m}^*)/h})$. We will denote $\mathbb{P}_1$ the orthogonal projection on Ker P_h and for shortness λ^* instead of $\lambda_{\mathbf{m}^*,h}$.

Proof of Corollary 5.1.9. We follow the proof of Theorem 1.11 in [26]. We have that

$$\|\mathrm{e}^{-tP_h/h} - \mathbb{P}_1\| \leq \|\mathrm{e}^{-tP_h/h}\Pi_0 - \mathbb{P}_1\| + \|\mathrm{e}^{-tP_h/h}(1-\Pi_0)\|.$$

and thanks to Proposition 5.2.8 and Proposition 2.1 from [20], we easily get

$$\mathrm{e}^{-tP_h/h}(1-\Pi_0) = O(\mathrm{e}^{-cht}).$$

Thus it suffices for the first statement to prove that

$$\|\mathrm{e}^{-tP_h/h}\Pi_0 - \mathbb{P}_1\| \leq C_N \mathrm{e}^{-\mathrm{Re}\,\lambda^*(1-C_N h^N)t/h}.$$

We recall that thanks to the resolvent estimates from Theorem 5.1.5, $\Pi_0 = O(1)$ and since $\mathbb{P}_1$ is an orthogonal projection on Ker P_h, we have that

$$\mathrm{e}^{-tP_h/h}\Pi_0 - \mathbb{P}_1 = \mathrm{e}^{-tP_h/h}(\Pi_0 - \mathbb{P}_1)$$

and $(\Pi_0 - \mathbb{P}_1) = O(1)$. Therefore, it is sufficient to prove that

$$\|\mathrm{e}^{-tP_h/h}|_{\mathrm{Ran}(\Pi_0-\mathbb{P}_1)}\| \leq C_N \mathrm{e}^{-\mathrm{Re}\,\lambda^*(1-C_N h^N)t/h}. \tag{5.6.1}$$

Besides, we saw in Section 5.2 that Ker $P_h = \mathbb{C}\mathcal{M}_h$ where $\mathcal{M}_h$ was defined in (5.1.2) and since the operator Π_0 from (5.2.5) satisfies $\Pi_0^*\mathcal{M}_h = \mathcal{M}_h$, we get that $\mathcal{M}_h^\perp$ is invariant under Π_0 so $\mathrm{Ran}(\Pi_0 - \mathbb{P}_1) = H \cap \mathcal{M}_h^\perp$. Thus, with the notations from Proposition 5.5.9 and according to (5.6.1), it only remains to show that

$$\|\mathrm{e}^{-t\mathcal{M}'/h}\| \leq C_N \mathrm{e}^{-\mathrm{Re}\,\lambda^*(1-C_N h^N)t/h}.$$

This can be done following the steps of [26], proof of Theorem 1.11 as with the notation (5.5.12) we have $\mathrm{Re}\,\lambda^* \leq \tilde{\lambda}_{\mathbf{m}^*,h}(1 + C_N h^N)$. The only difference is that here we have to apply the resolvent estimates given by Theorem 4 from [4] instead of the ones given by Theorem A.4 from [26]. For the last statement, we now asume that for $\mathbf{m} \in \mathcal{U}^{(0)} \backslash \{\mathbf{m}^*\}$, the expansion of $\lambda(\mathbf{m}, h)$ given by Theorem 5.1.7 differs from the one of $\lambda^* = \lambda(\mathbf{m}^*, h)$. In that case, it is clear that λ^* is a simple eigenvalue but it also happens to be a real one. Indeed, using the fact that X_0^h and b_h are differential operators with real coefficients and that M^h is real valued and even in the variable η, we get that λ is an eigenvalue of P_h if and only if $\overline{\lambda}$ is an eigenvalue of P_h. The rest of the proof is then also similar to the end of the proof of Theorem 1.11 from [26]. □

Finally, the proof of Corollary 5.1.10 is a straightforward adaptation of the one of Corollary 1.6 from [4]. (Note that our notations t_k^- and t_k^+ differ from that in [4]).

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5.7 Appendix

5.7.1 Proof of Lemma 5.1.3

Let us begin by showing that there exists a self-adjoint operator A sucht that

$$\varrho(H_0) = b_h^* \circ A \circ b_h. \tag{5.7.1}$$

Since $\varrho(0) = 0$, there exists an analytic function $\tilde{\varrho}$ such that $\varrho(z) = z\tilde{\varrho}(z)$ and $|\tilde{\varrho}(z)| \leq C\langle z\rangle^{-1}$. Using Cauchy's formula, one easily gets that for all $z_0 \in \{\operatorname{Re} z > -\frac{1}{2C}\}$ and f an analytic function on $\{\operatorname{Re} z > -\frac{1}{C}\}$ satisfying $f(z) = O(\langle z\rangle^{-\beta})$ for some $\beta > 0$, we have that

$$f(z_0) = \frac{-1}{2i\pi}\int_{\{\operatorname{Re} z = -\frac{1}{2C}\}} f(z)(z_0 - z)^{-1} \mathrm{d}z. \tag{5.7.2}$$

Working with a Hilbert basis of eigenfunctions of H_0, this identity yields

$$f(H_0) = \frac{-1}{2i\pi}\int_{\{\operatorname{Re} z = -\frac{1}{2C}\}} f(z)(H_0 - z)^{-1} \mathrm{d}z. \tag{5.7.3}$$

Besides, denoting

$$b_h = \begin{pmatrix} b_h^1 \\ \vdots \\ b_h^d \end{pmatrix},$$

we have $b_h H_0 = (b_h^j H_0)_{1\leq j\leq d}$ and using the identity $b_h^j H_0 = b_h^* b_h b_h^j + h b_h^j$, we get $b_h H_0 = H_1 b_h$ where

$$H_1 = \begin{pmatrix} H_0 + h & & \\ & \ddots & \\ & & H_0 + h \end{pmatrix}. \tag{5.7.4}$$

In particular, if u is an eigenfunction of H_0 associated to a positive eigenvalue, the function $b_h u$ is an eigenfunction of H_1 associated to the same eigenvalue and therefore

$$H_0(H_0 - z)^{-1} = b_h^*(H_1 - z)^{-1} b_h. \tag{5.7.5}$$

It follows using (5.7.3) with $f = \tilde{\varrho}$ that (5.7.1) holds with $A = \tilde{\varrho}(H_0 + h) \otimes \operatorname{Id}$:

$$\varrho(H_0) = H_0\tilde{\varrho}(H_0) = b_h^* \circ \tilde{\varrho}(H_0 + h) \otimes \operatorname{Id} \circ b_h.$$

We can improve the integrability in the integral representation of $\tilde{\varrho}(H_0 + h)$ by writing

$$\tilde{\varrho}(z) = \frac{\tilde{\varrho}(z)}{1+z} + \frac{\varrho(z) - \varrho_\infty}{1+z} + \frac{\varrho_\infty}{1+z}$$

which yields always thanks to (5.7.3)

$$\begin{aligned} \tilde{\varrho}(H_0 + h) \otimes \operatorname{Id} = \frac{-1}{2i\pi}\int_{\{\operatorname{Re} z = -\frac{1}{2C}\}} \frac{\tilde{\varrho}(z)}{1+z}(H_1 - z)^{-1} \mathrm{d}z \\ + \frac{-1}{2i\pi}\int_{\{\operatorname{Re} z = -\frac{1}{2C}\}} \frac{\varrho(z) - \varrho_\infty}{1+z}(H_1 - z)^{-1} \mathrm{d}z + \varrho_\infty (H_1 + 1)^{-1}. \end{aligned} \tag{5.7.6}$$

Besides, it is well known (see for instance [12]) that the resolvent $(H_1 - z)^{-1}$ is a pseudo-differential operator and we denote its symbol $R_z(v,\eta)$. Thanks to [10], we even have the explicit expression $R_z(v,\eta) = G_z(v^2/2+2\eta^2)\,\mathrm{Id}$ where G_z is an entire function defined by

$$G_z(\mu) = 2h^{-1}\int_0^1 (1-s)^{-\frac{z}{h}}(1+s)^{\frac{z}{h}+d-2}\mathrm{e}^{-\frac{s}{h}\mu}\mathrm{d}s = 2\int_0^{h^{-1}} (1-h\sigma)^{-\frac{z}{h}}(1+h\sigma)^{\frac{z}{h}+d-2}\mathrm{e}^{-\sigma\mu}\mathrm{d}\sigma.$$

Let us then set in view of (5.7.6)

$$M^h(v,\eta) = \frac{-1}{2i\pi}\int_{\{\mathrm{Re}\,z=-\frac{1}{2C}\}} \frac{\tilde{\varrho}(z)}{1+z}R_z(v,\eta)\mathrm{d}z + \frac{-1}{2i\pi}\int_{\{\mathrm{Re}\,z=-\frac{1}{2C}\}} \frac{\varrho(z)-\varrho_\infty}{1+z}R_z(v,\eta)\mathrm{d}z + \varrho_\infty R_{-1}(v,\eta) \tag{5.7.7}$$

and we now want to show that M^h is a matrix of symbols matching the properties listed in Hypothesis 5.1.2. To this purpose, we need to study more carefully the function R_z for z fixed such that $\mathrm{Re}\,z \leq -1/2C$. We already saw that it is analytic in both variables v and η. Now if we take $(v,\eta)\in\mathbb{R}^d\times\Sigma_\tau$ and put $\mu = v^2/2+2\eta^2$, we get that μ belongs to the sector

$$D_\tau = \{\mu\in\mathbb{C}\,;\,|\mathrm{Im}\,\mu| \leq \mathrm{Re}\,\mu + 4d\tau^2\}.$$

One can then easily adapt Theorem 10 from [10] to show that for $n\in\mathbb{N}$ and $\mu\in D_\tau$, we have

$$\begin{aligned}|\partial_\mu^n G_z(\mu)| &\leq C\int_0^{h^{-1}} \sigma^n (1-h\sigma)^{-\mathrm{Re}\,z/h}(1+h\sigma)^{\mathrm{Re}\,z/h}\mathrm{e}^{-\mathrm{Re}\,\mu\sigma}\mathrm{d}\sigma \\ &\leq C\int_0^{+\infty} \sigma^n \mathrm{e}^{-(\mathrm{Re}\,\mu - 2\mathrm{Re}\,z)\sigma}\mathrm{d}\sigma \leq C_n\langle\mu\rangle^{-(n+1)}\end{aligned} \tag{5.7.8}$$

since $\mathrm{Re}\,\mu - 2\mathrm{Re}\,z > 0$ for τ small enough. From (5.7.8) we can already conclude that $M^h \in \mathcal{M}_d\big(S^0_\tau(\langle(v,\eta)\rangle^{-2})\big)$. Thus $\tilde{\varrho}(H_0+h)\otimes\mathrm{Id} = \mathrm{Op}_h(M^h)$ with M^h sending $\mathbb{R}^{2d}$ in $\mathcal{M}_d(\mathbb{R})$ as H_0 is self-adjoint. Moreover, since R_z is diagonal and even in the variable η, it is also the case of M^h. It only remains to prove that M^h satisfies items b) and d) from Hypothesis 5.1.2. In order to avoid some tedious computations, instead of proving the whole expansion from item b), we only show that M^h admits a principal term M_0 in $\mathcal{M}_d\big(S^0_\tau(\langle(v,\eta)\rangle^{-2})\big)$ from which we will deduce that item d) is satisfied. One easily gets for $\mathrm{Re}\,z\leq -1/2C$ and $\mu\in D_\tau$ fixed by dominated convergence that

$$\lim_{h\to 0} G_z(\mu) = 2\int_0^\infty \mathrm{e}^{\sigma(2z-\mu)}\mathrm{d}\sigma = \frac{1}{\mu/2-z} =: G^0_z(\mu). \tag{5.7.9}$$

We would like to get some estimates of the derivatives $\partial^n_\mu(G_z - G^0_z)$ in $O(h\langle\mu\rangle^{-n-1})$ on D_τ uniformly in $z\in\{\mathrm{Re}\,z\leq -1/2C\}$ in order to apply the formula (5.7.7) to those. We have

$$\begin{aligned}\partial^n_\mu(G_z - G^0_z)(\mu) &= 2\int_0^{h^{-1}} \left[\exp\left(z\left[\frac{1}{h}\ln\left(\frac{1+h\sigma}{1-h\sigma}\right)-2\sigma\right]+(d-2)\ln(1+h\sigma)\right)-1\right](-\sigma)^n\mathrm{e}^{\sigma(2z-\mu)}\mathrm{d}\sigma \\ &\quad - 2\int_{h^{-1}}^\infty (-\sigma)^n e^{\sigma(2z-\mu)}\mathrm{d}\sigma \\ &= 2\int_0^{h^{-1}/2} \left[\exp\left(z\left[\frac{1}{h}\ln\left(\frac{1+h\sigma}{1-h\sigma}\right)-2\sigma\right]+(d-2)\ln(1+h\sigma)\right)-1\right](-\sigma)^n\mathrm{e}^{\sigma(2z-\mu)}\mathrm{d}\sigma \\ &\quad + O\Big(\mathrm{e}^{\frac{\mathrm{Re}\,(2z-\mu)}{Ch}}\Big).\end{aligned} \tag{5.7.10}$$

Let us denote

$$g_{z,h}(\sigma) = \left[\exp\left(z\left[\frac{1}{h}\ln\left(\frac{1+h\sigma}{1-h\sigma}\right)-2\sigma\right]+(d-2)\ln(1+h\sigma)\right)-1\right](-\sigma)^n$$

and observe that for all $0\leq k\leq n$, one has

$$\partial^k_\sigma g_{z,h}(0) = 0 \qquad\text{and}\qquad \partial^k_\sigma g_{z,h}(h^{-1}/2) = O(h^{-n}\langle z\rangle^k). \tag{5.7.11}$$

Besides, on $\sigma \in [0, h^{-1}/2]$, it holds

$$\partial_\sigma^{n+1} g_{z,h}(\sigma) = \sum_{j=1}^{n+1} O\big(h\langle z\rangle^j \langle\sigma\rangle^j \sigma^{j-1}\big). \tag{5.7.12}$$

Now, let us do $n+1$ integrations by parts in the first term from (5.7.10). By (5.7.11), each boundary term is $O(h^{-n}\langle z\rangle^k \langle 2z-\mu\rangle^{-(k+1)} e^{\mathrm{Re}\,(2z-\mu)/Ch})$ while the remaining integral term satisfies

$$\begin{aligned}\left|\frac{2}{(\mu-2z)^{n+1}}\int_0^{h^{-1}/2} \partial_\sigma^{n+1} g_{z,h}(\sigma) e^{\sigma(2z-\mu)} d\sigma\right| &\le C_n h \sum_{j=1}^{n+1} \frac{\langle z\rangle^j}{|2z-\mu|^{n+1}} \int_0^\infty \sigma^{j-1}\langle\sigma\rangle^j e^{\sigma\,\mathrm{Re}(2z-\mu)} d\sigma \\ &\le C_n h \langle\mu\rangle^{-(n+1)}\end{aligned}$$

thanks to (5.7.12). Thus, we have shown that for $n \in \mathbb{N}$, $\mu \in D_\tau$ and $\mathrm{Re}\, z \le -1/2C$,

$$|\partial_\mu^n (G_z - G_z^0)(\mu)| \le C_n h \langle\mu\rangle^{-(n+1)}.$$

Putting $R_z^0(v,\eta) = G_z^0(v^2/2 + 2\eta^2)\,\mathrm{Id}$ and defining $M_0(v,\eta)$ as in (5.7.7) with R_z replaced by R_z^0, we deduce that

$$|\partial^\alpha (M^h - M_0)(v,\eta)| \le C_\alpha h \langle (v,\eta)\rangle^{-2} \qquad \text{on } \mathbb{R}^d \times \Sigma_\tau$$

so item b) from Hypothesis 5.1.2 holds true. Finally, by definition of M_0 and thanks to (5.7.9) and (5.7.2), we have

$$M_0(v,\eta) = \tilde{\varrho}\big(v^2/4 + \eta^2\big)\,\mathrm{Id} \ge \frac{1}{C}\langle (v,\eta)\rangle^{-2}\,\mathrm{Id} \tag{5.7.13}$$

by assumption on ϱ. Therefore item d) from Hypothesis 5.1.2 holds true and the proof is complete.

5.7.2 Linear algebra Lemma

We use the following lemma which is inspired by [4], Lemma 2.6.

Lemma 5.7.1. *Let $M \in \mathcal{M}_{d'}(\mathbb{C})$ such that $M = S(A+T)$ with S hermitian and invertible, A skew-hermitian and T hermitian positive semidefinite. Suppose moreover that*

$$M(\mathrm{Ker}\, T) \cap \mathrm{Ker}\, T = \mathrm{Ker}\, M \cap \mathrm{Ker}\, T = \{0\}.$$

Then M has no spectrum in $i\mathbb{R}$.

Proof. Let $\lambda \in \mathbb{R}$ and $X \in \mathrm{Ker}\,[M - i\lambda]$, we first show that $X \in \mathrm{Ker}\, T$. Since T is hermitian positive semidefinite, it is sufficient to show that $\langle TX, X\rangle = 0$. Using the properties of S, A and T we have

$$\begin{aligned}\langle TX, X\rangle &= \mathrm{Re}\,\big\langle (A+T)X, X\big\rangle \\ &= \mathrm{Re}\,\big\langle S^{-1}S(A+T)X, X\big\rangle \\ &= \mathrm{Re}\,\big(i\lambda\big\langle S^{-1}X, X\big\rangle\big) \\ &= 0\end{aligned}$$

so $X \in \mathrm{Ker}\, T$. Thanks to the assumption, it only remains to prove that $X \in \mathrm{Ker}\, M$. This can be done easily by noticing that

$$MX = i\lambda X \in M(\mathrm{Ker}\, T) \cap \mathrm{Ker}\, T$$

so $MX = 0$ by assumption. ☐

5.7.3 Asymptotic expansions

Let $d' \in \mathbb{N}^*$. Here we use the convention $\sum_{j=0}^{-1} a_j = 0$ for any sequence $(a_j)_{j\geq 0}$ in a vector space. For $K \subseteq \mathbb{R}^{d'}$, the notation $a = O_{\mathcal{C}^\infty(K)}(h^N)$ (respectively $a = O_{L^\infty(K)}(h^N)$) means that for all $\beta \in \mathbb{N}^{d'}$, there exists $C_{\beta,N}$ such that $\|\partial^\beta a\|_{\infty,K} \leq C_{\beta,N} h^N$ (resp. there exists C_N such that $\|a\|_{\infty,K} \leq C_N h^N$). We will also use the notations from Definition A.0.1 and (A.0.1).

Proposition 5.7.2. *Let $m \in \mathbb{N}^*$; $d_1, \dots, d_m \in \mathbb{N}^*$ and for $1 \leq j \leq m$, $K_j \subset \mathbb{R}^{d_j}$ some compact sets. Let a smooth function*

$$\phi_h : \prod_{j=1}^m K_j \to K \subset \Sigma_\tau$$

such that $\phi_h = O_{\mathcal{C}^\infty(\prod_{j=1}^m K_j)}(1)$. Consider $g^h \sim_h \sum_{n\geq 0} h^n g_n$ in $S^0_\tau(1)$ or in $\mathcal{C}^\infty(K)$ if ϕ_h actually takes values in $\mathbb{R}^d$. Then

$$g^h \circ \phi_h \sim_h \sum_{n\geq 0} h^n (g_n \circ \phi_h)$$

in $\mathcal{C}^\infty(\prod_{j=1}^m K_j)$.

Proof. Let $N \in \mathbb{N}$ and denote $r_N = g^h - \sum_{n=0}^{N-1} h^n g_n = O_{S^0_\tau(1)}(h^N)$.

$$\begin{aligned} g^h \circ \phi_h &= \Big(\sum_{n=0}^{N-1} h^n g_n + r_N \Big) \circ \phi_h \\ &= \sum_{n=0}^{N-1} h^n (g_n \circ \phi_h) + r_N \circ \phi_h . \end{aligned}$$

But since all the derivatives of ϕ_h are bounded uniformly in h, and the ones of r_N are $O_{L^\infty(\Sigma_\tau)}(h^N)$, we see that $r_N \circ \phi_h$ is $O_{\mathcal{C}^\infty(\prod_{j=1}^m K_j)}(h^N)$ so we have the announced result. □

Proposition 5.7.3. *Since the matrix M^h from Hypothesis 5.1.2 satisfies $M^h \sim \sum_{n\geq 0} h^n M_n$ in $\mathcal{M}_d\big(S^0_\tau(\langle (v,\eta)\rangle^{-2})\big)$, the vector of symbols g^h defined in Remark 5.3.6 also admits a classical expansion $g^h \sim \sum_{n\geq 0} h^n g_n$ in $\mathcal{M}_{1,d}\big(S^0_\tau(\langle (v,\eta)\rangle^{-1})\big)$, where the (g_n) are given by*

$$g_0(x,v,\eta) = \Big(-i\,{}^t\eta + \frac{{}^t v}{2} \Big) M_0(x,v,\eta)$$

and

$$g_n(x,v,\eta) = \Big(-i\,{}^t\eta + \frac{{}^t v}{2} \Big) M_n(x,v,\eta) - \frac{1}{2}({}^t\nabla_v - \frac{i}{2}{}^t\nabla_\eta) M_{n-1}(x,v,\eta)$$

for $n \geq 1$.

Proof. We have

$$g^h = (-i\,{}^t\eta + {}^t v/2) M^h - \frac{h}{2}({}^t\nabla_v - \frac{i}{2}{}^t\nabla_\eta) M^h$$

and the last term clearly admits the expansion

$$-\sum_{n\geq 1} h^n \frac{1}{2}({}^t\nabla_v - \frac{i}{2}{}^t\nabla_\eta) M_{n-1}$$

in $S^0_\tau(\langle (v,\eta)\rangle^{-2})$. For the first term of g^h, it suffices to notice that for any $N \in \mathbb{N}$,

$$\Big(-i\,{}^t\eta + \frac{{}^t v}{2} \Big)\, O_{\mathcal{M}_d\left(S^0_\tau(\langle (v,\eta)\rangle^{-2})\right)}(h^N) = O_{\mathcal{M}_{1,d}\left(S^0_\tau(\langle (v,\eta)\rangle^{-1})\right)}(h^N).$$

□

Proposition 5.7.4. *Let K a compact set in $\mathbb{R}^{d'}$ and $a \sim_h \sum_{n\geq 0} h^n a_n$ in $\mathcal{C}^\infty(K)$ such that for all $n \geq 0$, $a_n \sim_h \sum_{j\geq 0} h^j a_{n,j}$ in $\mathcal{C}^\infty(K)$. Then*

$$a \sim_h \sum_{n\geq 0} h^n \sum_{j=0}^{n} a_{j,n-j} \quad \text{in } \mathcal{C}^\infty(K).$$

Proof. It suffices to write for $N \in \mathbb{N}$

$$\begin{aligned} a &= \sum_{n=0}^{N-1} h^n \Big(\sum_{j=0}^{N-1-n} h^j a_{n,j} + O_{\mathcal{C}^\infty(K)}(h^{N-n}) \Big) + O_{\mathcal{C}^\infty(K)}(h^N) \\ &= \sum_{n=0}^{N-1} h^n \sum_{j=0}^{n} a_{j,n-j} + O_{\mathcal{C}^\infty(K)}(h^N). \end{aligned}$$

□

Proposition 5.7.5. *Let K a compact set in $\mathbb{R}^{d'}$ and $a \in \mathcal{C}^\infty(K)$ such that for all $\beta \in \mathbb{N}^{d'}$, there exists $a_{\beta,j} \in \mathcal{C}^\infty(K)$ such that $\partial^\beta a \sim \sum_{j\geq 0} h^j a_{\beta,j}$ in $L^\infty(K)$. Then $a_{\beta,j} = \partial^\beta a_{0,j}$, i.e*

$$a \sim \sum_{j\geq 0} h^j a_{0,j} \quad \text{in } \mathcal{C}^\infty(K).$$

Proof. For simplicity, we take $d' = 1$. Let us denote $a_j = a_{0,j}$. By induction, it is sufficient to prove the result for $\beta = 1$, i.e prove that $a_{1,j} = a'_j$. Here again, it suffices to prove the case $j = 0$ which we can then apply to the function $h^{-1}(a - a_0)$ and so on. Let x in the interior of K and $t \in \mathbb{R}^*$ in a neighborhood of 0. We look at the differential fraction

$$\begin{aligned} \frac{a_0(x+t) - a_0(x)}{t} &= \frac{a(x+t) - a(x)}{t} + \frac{O(h)}{t} \\ &= a'(x) + t\int_0^1 (1-s)a''(x+st)\mathrm{d}s + \frac{O(h)}{t} \\ &= a_{1,0}(x) + O(h) + t\int_0^1 (1-s)a''(x+st)\mathrm{d}s + \frac{O(h)}{t} \\ &\xrightarrow[h\to 0]{} a_{1,0}(x) + t\int_0^1 (1-s)a_{2,0}(x+st)\mathrm{d}s. \end{aligned}$$

Taking now the limit $t \to 0$, we get $a'_0(x) = a_{1,0}(x)$ which was the desired result. □

Proposition 5.7.6. *Recall the notation (5.4.11) and let $K \subset \mathbb{R}^{d'}$ a compact set, $\Psi : K \to D(0,\tau)^d$ a smooth function such that $\Psi \sim \sum_{j\geq 0} h^j \Psi_j$ in $\mathcal{C}^\infty(K)$ and b an analytic function on Σ_τ. Then*

$$b \circ \Psi \sim \sum_{j\geq 0} h^j b_j \tag{5.7.14}$$

in $\mathcal{C}^\infty(K)$, with

$$b_0 = b \circ \Psi_0 \quad \text{and for } j \geq 1, \quad b_j = \sum_{|\beta|=1}^{j} \frac{\partial^\beta b \circ \Psi_0}{\beta!} \sum_{s\in S_{\beta,j}} \prod_{k\in K_\beta} \Big(\sum_{a\in A_{\beta,s,k}} \prod_{l=1}^{\beta_k} (\Psi_{a_l})_k \Big),$$

where $K_\beta = supp\ \beta = \{k \in [\![1,d]\!] \,;\, \beta_k \neq 0\}$, $S_{\beta,j} = \{s \in \mathbb{N}^d \,;\, \operatorname{supp} s = K_\beta,\ |s| = j$ and $s \geq \beta\}$ and $A_{\beta,s,k} = \{a \in (\mathbb{N}^*)^{\beta_k} \,;\, |a| = s_k\}$.

Proof. We first prove that (5.7.14) holds in $L^\infty(K)$. Doing a Taylor expansion of b, we have for $N \in \mathbb{N}^*$ that

$$\begin{aligned} b\circ\Psi &= b\circ\Psi_0 + \sum_{|\beta|=1}^{N-1} \frac{\partial^\beta b\circ\Psi_0}{\beta!}(\Psi-\Psi_0)^\beta + O\Big((\Psi-\Psi_0)^N\Big) \\ &= b\circ\Psi_0 + \sum_{|\beta|=1}^{N-1} \frac{\partial^\beta b\circ\Psi_0}{\beta!}(\Psi-\Psi_0)^\beta + O_{L^\infty(K)}(h^N) \end{aligned} \tag{5.7.15}$$

since $\Psi-\Psi_0 = O_{\mathcal{C}^\infty(K)}(h)$. Now one can see that

$$(\Psi-\Psi_0)^\beta \sim \sum_{j\geq|\beta|} h^j \sum_{s\in S_{\beta,j}} \prod_{k\in K_\beta} \Big(\sum_{a\in A_{\beta,s,k}} \prod_{l=1}^{\beta_k} (\Psi_{a_l})_k \Big)$$

so (5.7.15) gives

$$\begin{aligned} b\circ\Psi &= b\circ\Psi_0 + \sum_{|\beta|=1}^{N-1} \frac{\partial^\beta b\circ\Psi_0}{\beta!}\left[\sum_{j=|\beta|}^{N-1} h^j \sum_{s\in S_{\beta,j}} \prod_{k\in K_\beta} \Big(\sum_{a\in A_{\beta,s,k}} \prod_{l=1}^{\beta_k} (\Psi_{a_l})_k \Big) + O_{\mathcal{C}^\infty(K)}(h^N)\right] + O_{L^\infty(K)}(h^N) \\ &= b\circ\Psi_0 + \sum_{j=1}^{N-1} h^j \sum_{|\beta|=1}^{j} \frac{\partial^\beta b\circ\Psi_0}{\beta!} \sum_{s\in S_{\beta,j}} \prod_{k\in K_\beta} \Big(\sum_{a\in A_{\beta,s,k}} \prod_{l=1}^{\beta_k} (\Psi_{a_l})_k \Big) + O_{L^\infty(K)}(h^N) \end{aligned}$$

which proves that (5.7.14) holds in $L^\infty(K)$.
Besides, the derivatives of $b\circ\Psi$ are linear combinations of products of some derivatives of Ψ with some $\partial^\gamma b\circ\Psi$ where γ is a integer multi-index. Hence the expansion of Ψ in $\mathcal{C}^\infty(K)$ and the result that we just proved applied to $\partial^\gamma b\circ\Psi$ instead of $b\circ\Psi$ yield that for all $\beta\in\mathbb{N}^{d'}$, $\partial^\beta(b\circ\Psi)$ admits a classical expansion in $L^\infty(K)$ whose coefficients are smooth. Therefore, Proposition 5.7.5 enables us to conclude that (5.7.14) holds in $\mathcal{C}^\infty(K)$. ☐

Corollary 5.7.7. *Using the notations from the proof of Lemma 5.4.1, we have*

$$g_n\Big(x, \frac{v+v'}{2}, \eta + i\psi(x,v,v')\Big) \sim \sum_{j\geq 0} h^j g_{n,j}(x,v,v',\eta) \quad \text{on } B_0(\mathbf{s},2r)\times B_\infty(0,2r)$$

with

$$g_{n,0}(x,v,v',\eta) = g_n\Big(x, \frac{v+v'}{2}, \eta + i\psi_0(x,v,v')\Big)$$

and for $j\geq 1$

$$g_{n,j}(x,v,v',\eta) = iD_\eta g_n\Big(x, \frac{v+v'}{2}, \eta + i\psi_0(x,v,v')\Big)\big(\psi_j(x,v,v')\big) + R_j^1(\ell_0,\dots,\ell_{j-1})$$

where $R_j^1 : \big(\mathcal{C}^\infty(B_0(\mathbf{s},2r))\big)^j \to \mathcal{C}^\infty(B_0(\mathbf{s},2r))$.

Proof. Since $\psi(\mathbf{s},0,0) = O(h)$, we can suppose that r was chosen small enough so that $(x,v,v',\eta)\mapsto \eta + i\psi(x,v,v')$ sends $B_0(\mathbf{s},2r)\times B_\infty(0,2r)$ in $D(0,\tau)^d$ Hence we can use Proposition 5.7.6 to get that

$$g_n\Big(x, \frac{v+v'}{2}, \eta + i\psi(x,v,v')\Big) \sim \sum_{j\geq 0} h^j g_{n,j}(x,v,v',\eta) \quad \text{on } B_0(\mathbf{s},2r)\times B_\infty(0,2r)$$

with

$$g_{n,0}(x,v,v',\eta) = g_n\Big(x, \frac{v+v'}{2}, \eta + i\psi_0(x,v,v')\Big)$$

and for $j\geq 1$

$$g_{n,j}(x,v,v',\eta) = \sum_{|\beta|=1}^{j} \frac{i^{|\beta|}}{\beta!}\partial_\eta^\beta g_n\Big(x, \frac{v+v'}{2}, \eta + i\psi_0(x,v,v')\Big) \sum_{s\in S_{\beta,j}} \prod_{k\in K_\beta} \Big(\sum_{a\in A_{\beta,s,k}} \prod_{l=1}^{\beta_k} (\psi_{a_l})_k \Big) \tag{5.7.16}$$

where $K_\beta = \operatorname{supp} \beta = \{k \in [\![1, d]\!] \,;\, \beta_k \neq 0\}$, $S_{\beta,j} = \{s \in \mathbb{N}^d \,;\, \operatorname{supp} s = K_\beta,\, |s| = j \text{ and } s \geq \beta\}$ and $A_{\beta,s,k} = \{a \in (\mathbb{N}^*)^{\beta_k} \,;\, |a| = s_k\}$. Now, we see thanks to (5.7.16) that the terms of $g_{n,j}(x, v, v', \eta)$ for which $|\beta| = 1$ yield

$$iD_\eta g_n\Big(x, \frac{v+v'}{2}, \eta + i\psi_0(x, v, v')\Big)\big(\psi_j(x, v, v')\big)$$

while the terms for which $|\beta| > 1$ only feature the functions $\ell_0, \dots, \ell_{j-1}$. □

Finally, we state the version of Laplace's method for integral approximation that we use in this paper.

Proposition 5.7.8. *Let $x_0 \in \mathbb{R}^{d'}$, K a compact neighborhood of x_0 and $\varphi \in \mathcal{C}^\infty(K)$ such that x_0 is a non degenerate minimum of φ and its only global minimum on K. Let also $a_h \sim \sum_{j\geq 0} h^j a_j$ in $\mathcal{C}^\infty(K)$ and denote $H \in \mathcal{M}_{d'}(\mathbb{R})$ the Hessian of φ at x_0. The integral*

$$\frac{\det(H)^{1/2}}{(2\pi h)^{d'/2}} \int_K a_h(x) \mathrm{e}^{-\frac{\varphi(x)-\varphi(x_0)}{h}} \mathrm{d}x$$

admits a classical expansion whose first term is given by $a_0(x_0)$.

5.7.4 Proof of Lemma 5.4.3

According to the proof of Corollary 5.7.7 and the end of the proof of Lemma 5.4.1 from which we keep the notations, we have the following expression for R_j :

$$\begin{aligned}
R_j(\ell_0, \dots, \ell_{j-1})(x,v) = &\sum_{\substack{n_1+n_2+n_3+n_4=j \\ n_3, n_4 \neq j}} \frac{1}{i^{n_1} n_1!} (\partial_{v'} \cdot \partial_\eta)^{n_1} \Big(g_{n_2,n_3}(x, v, v', \eta) \partial_v \ell_{n_4}(x, v')\Big)\Big|_{\substack{v'=v \\ \eta=0}} \\
&+ \sum_{|\beta|=2}^{j} \frac{i^{|\beta|}}{\beta!} \partial_\eta^\beta g_0\Big(x, \frac{v+v'}{2}, i\big(\frac{v}{2} + \ell_0(x,v)\, \partial_v \ell_0(x,v)\big)\Big) \\
&\times \sum_{s \in S_{\beta,j}} \prod_{k \in K_\beta} \Big(\sum_{a \in A_{\beta,s,k}} \prod_{l=1}^{\beta_k} \big(\psi_{a_l}(x,v,v)\big)_k\Big) \partial_v \ell_0(x,v) \\
&+ iD_\eta g_0\big(x, v, i(v/2 + \ell_0(x,v)\, \partial_v \ell_0(x,v))\big) \sum_{k=1}^{j-1} \big(\ell_k \partial_v \ell_{j-k}\big)(x,v)\, \partial_v \ell_0(x,v).
\end{aligned} \tag{5.7.17}$$

Using Lemma 5.4.2 and (5.7.16), it is clear that the last two terms of $R_j(\ell_0, \cdots, \ell_{j-1})$ given by (5.7.17) and the terms of the first sum for which $n_1 = 0$ are real valued. For the rest of the first term, we start by noticing that one can establish by induction that for $n_1 \geq 1$,

$$(\partial_{v'} \cdot \partial_\eta)^{n_1} = \sum_{p \in [\![1,d]\!]^{n_1}} \partial_{v'}^{\gamma(p)} \partial_\eta^{\gamma(p)} \tag{5.7.18}$$

where using the notation (5.3.20), we define $\gamma(p) = \sum_{k=1}^{n_1} e_{p_k}$ (note that $|\gamma(p)| = n_1$). Besides, we have for $0 \leq n_2 \leq j$ and $p \in [\![1, d]\!]^{n_1}$

$$\partial_\eta^{\gamma(p)} g_{n_2,0}(x, v, v', 0) = \partial_\eta^{\gamma(p)} g_{n_2}\Big(x, \frac{v+v'}{2}, i\psi_0(x, v, v')\Big) \in i^{n_1}\mathbb{R}^d \tag{5.7.19}$$

according to Lemma 5.4.2 and in the case $j \geq 2$, for $1 \leq n_3 \leq j-1$

$$\begin{aligned}
&\partial_\eta^{\gamma(p)} g_{n_2,n_3}(x, v, v', 0) \\
&= \sum_{|\beta|=1}^{n_3} \frac{i^{|\beta|}}{\beta!} \partial_\eta^{\beta+\gamma(p)} g_{n_2}\Big(x, \frac{v+v'}{2}, i\psi_0(x, v, v')\Big) \sum_{s \in S_{\beta,n_3}} \prod_{k \in K_\beta} \Big(\sum_{a \in A_{\beta,s,k}} \prod_{l=1}^{\beta_k} (\psi_{a_l})_k\Big) \in i^{n_1}\mathbb{R}^d
\end{aligned} \tag{5.7.20}$$

where we used (5.7.16) and Lemma 5.4.2 once again. The combination of (5.7.18), (5.7.19) and (5.7.20) enables us to conclude that the term

$$\sum_{\substack{n_1+n_2+n_3+n_4=j \\ n_1\neq 0;\, n_3,n_4\neq j}} \frac{1}{i^{n_1} n_1!} \big(\partial_{v'}\cdot\partial_\eta\big)^{n_1} \Big(g_{n_2,n_3}(x,v,v',\eta)\partial_v \ell_{n_4}(x,v')\Big)\Big|_{\substack{v'=v\\ \eta=0}}$$

from (5.7.17) is also real so $R_j(\ell_0^{\mathbf{s}},\dots,\ell_{j-1}^{\mathbf{s}})$ is real valued. For the last statement, it suffices to use the formula (5.7.17) after noticing that ψ (and hence the (g_{n_2,n_3})) remain unchanged when ℓ is replaced by $-\ell$.

Chapitre 6

Small spectrum of a non local factorized semiclassical operator associated to a general confining potential

On fournit dans ce chapitre des démonstrations (en anglais) issues de [35] des résultats présentés au chapitre 3. Tout au long de ce travail, on se place sous l'Hypothèse 3.1.3.

6.1 General labeling of the potential minima

We once again consider the labeling procedure described in Appendix C and we start by recalling a few notations. First,

$$\mathcal{U}^{(k)} \text{ is the set of critical points of } W \text{ of index } k.$$

We also denote $\sigma_2 > \cdots > \sigma_N$ where $N \geq 2$ the different separating saddle values of W with the convention $\sigma_1 = +\infty$. For $\sigma \in \mathbb{R} \cup \{+\infty\}$, we call $\mathcal{C}_\sigma$ the set of all the connected components (CC) of $\{W < \sigma\}$. We denote for $k \in [\![1, N]\!]$

$$\mathtt{U}_k^{(0)} = \left\{\mathbf{m}_{k',j}\,;\, 1 \leq k' \leq k\,,\, j \in \mathbb{N}^*\right\} \cap \left\{W < \sigma_k\right\}$$

and

$$T_k : \mathtt{U}_k^{(0)} \to \mathcal{C}_{\sigma_k}$$

the bijective map sending $\mathbf{m} \in \mathtt{U}_k^{(0)}$ on the element of $\mathcal{C}_{\sigma_k}$ to which it belongs.
We now introduce some material from [31]. Let us denote

$$\underline{\mathbf{m}} = \mathbf{m}_{1,1} \qquad \text{and} \qquad \underline{\mathcal{U}}^{(0)} = \mathcal{U}^{(0)} \backslash \{\underline{\mathbf{m}}\}$$

and define for $\mathbf{m} = \mathbf{m}_{k,j} \in \underline{\mathcal{U}}^{(0)}$

$$\widehat{\mathbf{m}} = T_{k-1}^{-1}\big(E_-(\mathbf{m})\big)$$

where $E_-(\mathbf{m})$ is the element of $\mathcal{C}_{\sigma_{k-1}}$ containing $\mathbf{m}$. Since $\widehat{\mathbf{m}}$ and $\mathbf{m}$ both belong to $E_-(\mathbf{m})$, we have $W(\widehat{\mathbf{m}}) \leq W(\mathbf{m})$ and $\widehat{\mathbf{m}} \in \mathtt{U}_k^{(0)}$.

Definition 6.1.1.
- *A minimum $\mathbf{m} \in \underline{\mathcal{U}}^{(0)}$ is said to be of type I if $W(\widehat{\mathbf{m}}) < W(\mathbf{m})$. Otherwise (i.e when $W(\widehat{\mathbf{m}}) = W(\mathbf{m})$), we say that $\mathbf{m}$ is of type II.*
- *We define an equivalence relation $\mathcal{R}$ on $\underline{\mathcal{U}}^{(0)}$ by $\mathbf{m}\mathcal{R}\mathbf{m}'$ if and only if the two following conditions are satisfied :*

i) $\boldsymbol{\sigma}(\mathbf{m}) = \boldsymbol{\sigma}(\mathbf{m}') = \sigma_k$

ii) *There exist some minima* $\mathbf{m}^1, \dots, \mathbf{m}^K$ *such that* $\mathbf{m}^1 = \mathbf{m}$, $\mathbf{m}^K = \mathbf{m}'$ *and for all* $1 \le n \le K-1$, *we have* $\overline{T_k(\mathbf{m}^n)} \cap \overline{T_k(\mathbf{m}^{n+1})} \neq \emptyset$ *with*

$$\mathbf{m}^n \in \{\mathbf{m}_{k,j}\,;\, j \in \mathbb{N}^*\} \cup \{\widehat{\mathbf{m}}_{k,j}\,;\, j \in \mathbb{N}^* \text{ and } \mathbf{m}_{k,j} \text{ is of type II}\}.$$

We denote the associated equivalence classes $(\mathcal{U}_\alpha^{(0)})_{\alpha\in\mathcal{A}}$, where $\mathcal{A}$ is a finite set. It is shown in [31] (Proposition 2.6) that for $\mathbf{m}\mathcal{R}\mathbf{m}'$, we have $\boldsymbol{\sigma}(\mathbf{m}) = \boldsymbol{\sigma}(\mathbf{m}')$ and $\widehat{\mathbf{m}} = \widehat{\mathbf{m}}'$. Finally, for $\mathbf{m} \in \mathcal{U}_\alpha^{(0)}$, we put

$$\boldsymbol{\sigma}(\alpha) = \boldsymbol{\sigma}(\mathbf{m}) \qquad \text{and} \qquad \widehat{\mathcal{U}}_\alpha^{(0)} = \mathcal{U}_\alpha^{(0)} \cup \{\widehat{\mathbf{m}}\}$$

which does not depend on the choice of $\mathbf{m} \in \mathcal{U}_\alpha^{(0)}$, as well as

$$E^\alpha(\mathbf{m}) = T_{k(\alpha)}(\mathbf{m}) \qquad \text{and} \qquad \mathbf{j}^\alpha(\mathbf{m}) = \partial E^\alpha(\mathbf{m}) \cap \mathtt{V}^{(1)}$$

for $\mathbf{m} \in \widehat{\mathcal{U}}_\alpha^{(0)}$, where $k(\alpha)$ is such that $\boldsymbol{\sigma}(\alpha) = \sigma_{k(\alpha)}$. When $\mathbf{m} \in \mathcal{U}_\alpha^{(0)}$, we have $E^\alpha(\mathbf{m}) = E(\mathbf{m})$ but for $\widehat{\mathbf{m}} \in \widehat{\mathcal{U}}_\alpha^{(0)} \cap \mathcal{U}_{\alpha'}^{(0)}$, we have $E(\widehat{\mathbf{m}}) = E^{\alpha'}(\widehat{\mathbf{m}}) \neq E^\alpha(\widehat{\mathbf{m}})$.

6.2 Gaussian quasimodes

Throughout the paper, for $d' \in \mathbb{N}^*$, $\Omega \subseteq \mathbb{R}^{d'}$ and $a \in \mathcal{C}^\infty(\Omega)$ a function depending on h and such that for all $\beta \in \mathbb{N}^{d'}$ we have $\partial^\beta a = O_{L^\infty}(1)$, we will say that $a \in \mathcal{C}^\infty(\Omega)$ admits a classical expansion on Ω and denote $a \sim \sum_{j\ge 0} h^j a_j$, where $(a_j)_{j\ge0} \subset \mathcal{C}^\infty(\Omega)$ are independent of h, provided that for all $\beta \in \mathbb{N}^{d'}$ and $N \in \mathbb{N}$, there exists $C_{\beta,N}$ such that

$$\Big\|\partial^\beta\Big(a - \sum_{j=0}^{N-1} h^j a_j\Big)\Big\|_{\infty,\Omega} \le C_{\beta,N} h^N.$$

It implies in particular that $\partial^\beta a_j = O_{L^\infty}(1)$. From now on, the letter r will denote a small universal positive constant whose value may decrease as we progress in this paper (one can think of r as $1/C$).

Let $\alpha \in \mathcal{A}$ and $\mathbf{m} \in \widehat{\mathcal{U}}_\alpha^{(0)}$. For each $\mathbf{s} \in \mathbf{j}^\alpha(\mathbf{m})$ we introduce a function $\ell^{\mathbf{s},\mathbf{m}}$ that will appear in our quasimodes. Note that thanks to Lemma C.0.1, there are at most two functions $\ell^{\mathbf{s},\mathbf{m}}$ and $\ell^{\mathbf{s},\mathbf{m}'}$ associated to a saddle point $\mathbf{s} \in \mathtt{V}^{(1)}$. Our goal will be to find some functions $\ell^{\mathbf{s},\mathbf{m}}$ such that our quasimodes are the most accurate possible. In order to begin the computations that will yield the equations that the function $\ell^{\mathbf{s},\mathbf{m}}$ should satisfy, we will for the moment assume that it satisfies the following :

(6.2.1)

a) $\ell^{\mathbf{s},\mathbf{m}}$ is a smooth real valued function on $\mathbb{R}^d$ whose support is contained in $B(\mathbf{s}, 3r)$

b) $\ell^{\mathbf{s},\mathbf{m}}$ admits a classical expansion $\ell^{\mathbf{s},\mathbf{m}} \sim \sum h^j \ell_j^{\mathbf{s},\mathbf{m}}$ on $B(\mathbf{s}, 2r)$

c) $\ell_0^{\mathbf{s},\mathbf{m}}$ vanishes at $\mathbf{s}$

d) $\mathbf{s}$ is a local minimum of the function $W + (\ell_0^{\mathbf{s},\mathbf{m}})^2/2$ which is non degenerate

e) the functions $\theta_{\mathbf{m},h}^\alpha$ (which depends on $\ell^{\mathbf{s},\mathbf{m}}$) and χ_α that we will introduce in (6.2.3)-(6.2.5) are such that $\theta_{\mathbf{m},h}^\alpha$ is smooth on a neighborhood of supp χ_α.

Once we will have found the desired function $\ell^{\mathbf{s},\mathbf{m}}$, we will see in Proposition 6.5.8 that these assumptions are actually satisfied. Denote $\zeta \in \mathcal{C}_c^\infty(\mathbb{R}, [0,1])$ an even cut-off function supported in $[-\gamma, \gamma]$ that is equal to 1 on $[-\gamma/2, \gamma/2]$ where $\gamma > 0$ is a parameter to be fixed later and

$$A_h = \frac{1}{2}\int_{\mathbb{R}} \zeta(s) e^{-\frac{s^2}{2h}} ds = \int_0^\gamma \zeta(s) e^{-\frac{s^2}{2h}} ds = \frac{\sqrt{\pi h}}{\sqrt{2}}(1 + O(e^{-c/h})) \qquad \text{for some } c > 0. \tag{6.2.2}$$

We now define for each $\mathbf{m} \in \widehat{\mathcal{U}}_\alpha^{(0)}$ a function $\theta_{\mathbf{m},h}^\alpha$ as follows : if $x \in B(\mathbf{s}, r) \cap \{|\ell_0^{\mathbf{s},\mathbf{m}}| \le 2\gamma\}$ for some $\mathbf{s} \in \mathbf{j}^\alpha(\mathbf{m})$,

$$\theta_{\mathbf{m},h}^\alpha(x) = \frac{1}{2}\Big(1 + A_h^{-1}\int_0^{\ell^{\mathbf{s},\mathbf{m}}(x)} \zeta(s) e^{-s^2/2h} ds\Big) \tag{6.2.3}$$

whereas we set

$$\theta^{\alpha}_{\mathbf{m},h} = 1 \quad \text{on } \Big(E^{\alpha}(\mathbf{m}) + B(0,\varepsilon)\Big)\setminus\Big(\bigsqcup_{\mathbf{s}\in\mathbf{j}^{\alpha}(\mathbf{m})} \big(B(\mathbf{s},r)\cap\{|\ell_0^{\mathbf{s},\mathbf{m}}| \le 2\gamma\}\big)\Big)$$

with $\varepsilon(r) > 0$ to be fixed later and

$$\theta_{\mathbf{m},h} = 0 \quad \text{everywhere else.}$$

Note that $\theta^{\alpha}_{\mathbf{m},h}$ takes values in $[0,1]$ and that we have

$$\operatorname{supp}\theta^{\alpha}_{\mathbf{m},h} \subseteq E^{\alpha}(\mathbf{m}) + B(0,\varepsilon') \tag{6.2.4}$$

where $\varepsilon' = \max(\varepsilon, r)$.
Denote now Ω_α the CC of $\{W \le \boldsymbol{\sigma}(\alpha)\}$ containing $\widehat{\mathcal{U}}^{(0)}_{\alpha}$. The CCs of $\{W \le \boldsymbol{\sigma}(\alpha)\}$ are separated so for $\varepsilon > 0$ small enough, there exists $\tilde{\varepsilon} > 0$ such that

$$\min\big\{W(x)\,;\, \mathrm{d}\big(x,\Omega_\alpha\big) = \varepsilon\big\} = \boldsymbol{\sigma}(\alpha) + 2\tilde{\varepsilon}.$$

Thus the distance between $\{W \le \boldsymbol{\sigma}(\alpha) + \tilde{\varepsilon}\} \cap \big(\Omega_\alpha + B(0,\varepsilon)\big)$ and $\partial\big(\Omega_\alpha + B(0,\varepsilon)\big)$ is positive and we can consider a cut-off function

$$\chi_\alpha \in \mathcal{C}^{\infty}_{c}(\mathbb{R}^d, [0,1]) \tag{6.2.5}$$

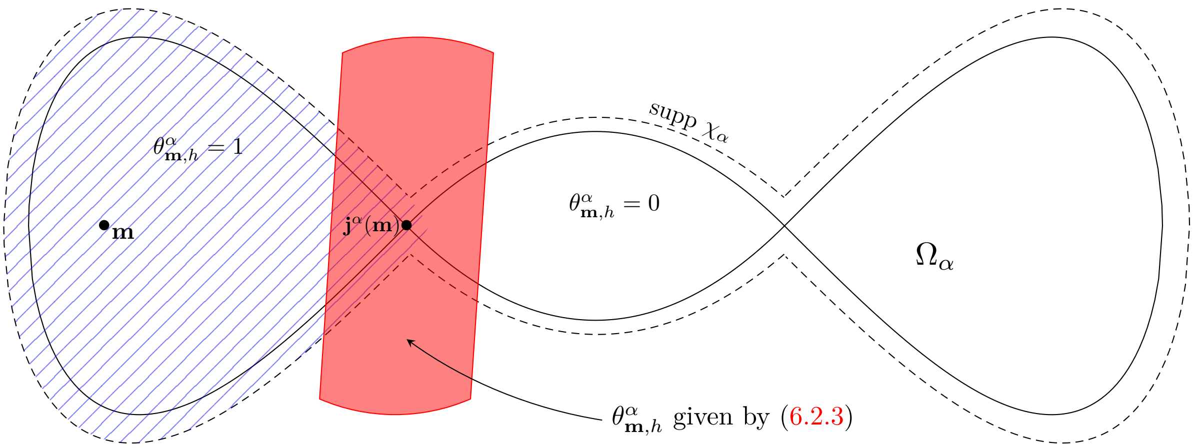
such that

$$\chi_\alpha = 1 \text{ on } \{W \le \boldsymbol{\sigma}(\alpha) + \tilde{\varepsilon}\} \cap \big(\Omega_\alpha + B(0,\varepsilon)\big)$$

and

$$\operatorname{supp}\chi_\alpha \subset \big(\Omega_\alpha + B(0,\varepsilon)\big).$$

To sum up, we have the following picture :



We also denote

$$W_{\mathbf{m}}(x) = W(x) - W(\mathbf{m})$$

and it is clear that

$$W_{\mathbf{m}} \ge S(\mathbf{m}) + \tilde{\varepsilon} \qquad \text{on the support of } \nabla\chi_\alpha \text{ as soon as } \mathbf{m} \in \mathcal{U}^{(0)}_{\alpha}. \tag{6.2.6}$$

The global quasimode associated to the eigenvalue 0 is

$$f_{\underline{\mathbf{m}},h}(x) = h^{-d/4} c_h(\underline{\mathbf{m}}) \mathrm{e}^{-W_{\underline{\mathbf{m}}}(x)/h} \in \operatorname{Ker} P_h \tag{6.2.7}$$

which is an exact one, while for $\mathbf{m} \in \mathcal{U}^{(0)}_{\alpha}$, our quasimodes will be linear combinations of the functions

$$h^{-d/4} c^{\alpha}_{h}(\tilde{\mathbf{m}}) \chi_\alpha(x) \theta^{\alpha}_{\tilde{\mathbf{m}},h}(x) \mathrm{e}^{-W_{\tilde{\mathbf{m}}}(x)/h}$$

where $\tilde{\mathbf{m}} \in \widehat{\mathcal{U}}_\alpha^{(0)}$. Here $c_h^\alpha(\mathbf{m})$ and $c_h(\underline{\mathbf{m}})$ are normalization factors ensuring that

$$\|h^{-d/4} c_h^\alpha(\tilde{\mathbf{m}}) \chi_\alpha(x) \theta^\alpha_{\tilde{\mathbf{m}},h}(x) \mathrm{e}^{-W_{\tilde{\mathbf{m}}}(x)/h}\| = 1 \quad \text{and} \quad \|h^{-d/4} c_h(\underline{\mathbf{m}}) \mathrm{e}^{-W_{\underline{\mathbf{m}}}(x)/h}\| = 1. \tag{6.2.8}$$

In particular, we have that for all $\mathbf{m} \in \widehat{\mathcal{U}}_\alpha^{(0)}$, the constant $c_h^\alpha(\mathbf{m})$ (resp. the constant $c_h(\underline{\mathbf{m}})$) admits an asymptotic expansion whose first term is

$$\pi^{-d/4}\Big(\sum_{\tilde{\mathbf{m}} \in H^\alpha(\mathbf{m})} \det \mathcal{W}_{\tilde{\mathbf{m}}}^{-1/2}\Big)^{-1/2} \qquad \text{resp.} \quad \pi^{-d/4}\Big(\sum_{\tilde{\mathbf{m}} \in H(\underline{\mathbf{m}})} \det \mathcal{W}_{\tilde{\mathbf{m}}}^{-1/2}\Big)^{-1/2} \tag{6.2.9}$$

with $H^\alpha(\mathbf{m}) = \{\tilde{\mathbf{m}} \in \mathcal{U}^{(0)} \cap E^\alpha(\mathbf{m})\,;\, W(\tilde{\mathbf{m}}) = W(\mathbf{m})\}$, $H(\underline{\mathbf{m}}) = \{\tilde{\mathbf{m}} \in \mathcal{U}^{(0)}\,;\, W(\tilde{\mathbf{m}}) = W(\underline{\mathbf{m}})\}$ and

$$\mathcal{W}_x \text{ is the Hessian of } W \text{ at } x. \tag{6.2.10}$$

Let us introduce the coefficients that we will use to define our quasimodes, in the spirit of [31]. We denote $\mathcal{F}_\alpha$ the finite-dimensional vector space of functions from $\widehat{\mathcal{U}}_\alpha^{(0)}$ into $\mathbb{R}$ endowed with the natural euclidian structure

$$\langle \varphi, \varphi' \rangle_{\mathcal{F}_\alpha} = \sum_{\tilde{\mathbf{m}} \in \widehat{\mathcal{U}}_\alpha^{(0)}} \varphi(\tilde{\mathbf{m}}) \varphi'(\tilde{\mathbf{m}}).$$

Denoting also

$$\mathcal{U}_\alpha^{(0),I} \text{ the elements of } \mathcal{U}_\alpha^{(0)} \text{ of type I}$$

and using (6.2.9), the following is established in [31], Lemma 3.6 and below.

Lemma 6.2.1. *Recall the notation (6.2.10). One can construct an h-dependent orthonormal family $(\varphi^\alpha_{\mathbf{m}})_{\mathbf{m} \in \widehat{\mathcal{U}}_\alpha^{(0)}} \subset \mathcal{F}_\alpha$ such that*

a) $\varphi^\alpha_{\widehat{\mathbf{m}}}(\mathbf{m}) = \widehat{c}_h c_h^\alpha(\mathbf{m})^{-1} \mathbf{1}_{\widehat{\mathcal{U}}_\alpha^{(0)} \setminus \mathcal{U}_\alpha^{(0),I}}(\mathbf{m})$ where $\widehat{c}_h$ is a normalization constant such that $\|\varphi^\alpha_{\widehat{\mathbf{m}}}\|_{\mathcal{F}_\alpha} = 1$.

b) If $\{\mathbf{m}, \tilde{\mathbf{m}}\} \cap \mathcal{U}_\alpha^{(0),I} \neq \emptyset$, then $\varphi^\alpha_{\mathbf{m}}(\tilde{\mathbf{m}}) = \delta_{\mathbf{m},\tilde{\mathbf{m}}}$.

c) If $\varphi^\alpha_{\mathbf{m}}(\tilde{\mathbf{m}}) \neq 0$, then $W(\mathbf{m}) = W(\tilde{\mathbf{m}})$.

d) Each $\varphi^\alpha_{\mathbf{m}}$ admits an asymptotic expansion. The leading term of $\varphi^\alpha_{\widehat{\mathbf{m}}}$ is given by

$$\varphi^{\alpha,0}_{\widehat{\mathbf{m}}}(\mathbf{m}) = \widehat{c}_0 c_0^\alpha(\mathbf{m})^{-1} \mathbf{1}_{\widehat{\mathcal{U}}_\alpha^{(0)} \setminus \mathcal{U}_\alpha^{(0),I}}(\mathbf{m}) = \left(\frac{\sum_{\tilde{\mathbf{m}} \in H^\alpha(\mathbf{m})} \det \mathcal{W}_{\tilde{\mathbf{m}}}^{-1/2}}{\sum_{\mathbf{m}' \in \widehat{\mathcal{U}}_\alpha^{(0)} \setminus \mathcal{U}^{(0),I}} \sum_{\tilde{\mathbf{m}} \in H^\alpha(\mathbf{m}')} \det \mathcal{W}_{\tilde{\mathbf{m}}}^{-1/2}}\right)^{1/2} \mathbf{1}_{\widehat{\mathcal{U}}_\alpha^{(0)} \setminus \mathcal{U}_\alpha^{(0),I}}(\mathbf{m}).$$

Finally, for all $\mathbf{m} \in \mathcal{U}_\alpha^{(0)}$, the leading term of $\varphi^\alpha_{\mathbf{m}}$ can be computed explicitly and is orthogonal to the one of $\varphi^\alpha_{\widehat{\mathbf{m}}}$.

For $\mathbf{m} \in \mathcal{U}_\alpha^{(0)}$, our quasimodes will be the functions

$$f_{\mathbf{m},h}(x) = h^{-d/4} \chi_\alpha(x) \Big(\sum_{\tilde{\mathbf{m}} \in \widehat{\mathcal{U}}_\alpha^{(0)}} \varphi^\alpha_{\mathbf{m}}(\tilde{\mathbf{m}}) c_h^\alpha(\tilde{\mathbf{m}}) \theta^\alpha_{\tilde{\mathbf{m}},h}(x)\Big) \mathrm{e}^{-W_{\mathbf{m}}(x)/h}. \tag{6.2.11}$$

Note that $f_{\mathbf{m},h}$ belongs to $\mathcal{C}_c^\infty(\mathbb{R}^d)$ thanks to item e) from Hypothesis 6.2.1 and that

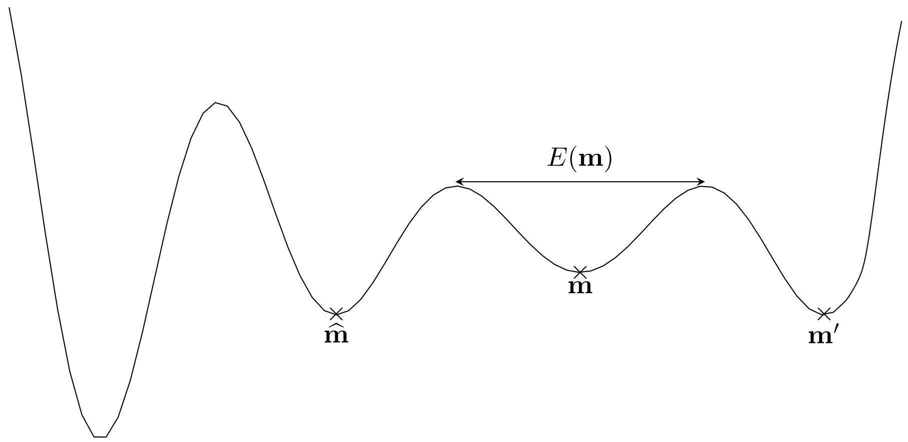
$$\operatorname{supp} f_{\mathbf{m},h} \subseteq E_-(\mathbf{m}) \tag{6.2.12}$$

thanks to (6.2.4).

6.3 Orthogonality

The goal of this section is to show that the family of quasimodes that we introduced in (6.2.11) and (6.2.7) is almost orthonormal. This result was already established in [4] in the case where W has no type II minimum (see Remark 6.3 from [4]). Therefore, we will consider $\mathbf{m} \in \mathcal{U}_{\alpha}^{(0)}$, $\mathbf{m}' \in \mathcal{U}_{\alpha'}^{(0)}$ and we will study here the orthogonality of the quasimodes $f_{\mathbf{m},h}$ and $f_{\mathbf{m}',h}$ with $\mathbf{m}$ or $\mathbf{m}'$ (or both) a type II minimum. We follow the spirit of [31] (Proposition 3.10) and adapt it to the gaussian quasimodes framework.

• The case where $\mathbf{m}\mathcal{R}\mathbf{m}'$ and one of them is of type I, say $\mathbf{m}$ (in particular $\mathbf{m} \neq \mathbf{m}'$ because of our assumption).



In that case, item b) from Lemma 6.2.1 implies $\varphi_{\mathbf{m}}(\tilde{\mathbf{m}}) = \delta_{\mathbf{m},\tilde{\mathbf{m}}}$ for all $\tilde{\mathbf{m}} \in \widehat{\mathcal{U}}_{\alpha}^{(0)}$ and $\varphi_{\mathbf{m}'}(\mathbf{m}) = 0$ so by (6.2.4), we have

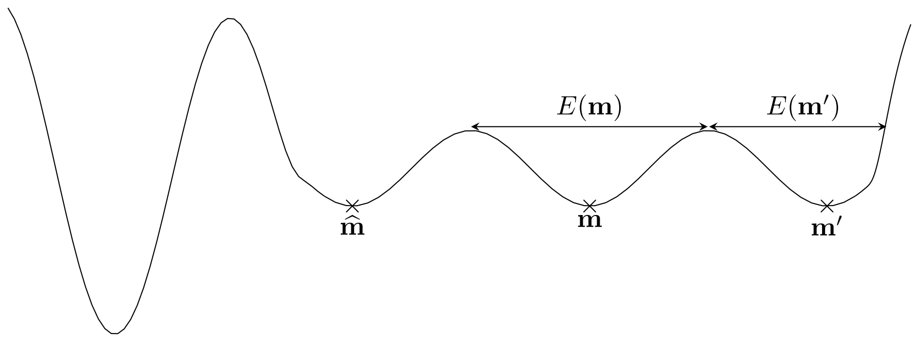
$$\operatorname{supp} f_{\mathbf{m},h} \subseteq E(\mathbf{m}) + B(0,\varepsilon') \qquad \text{and} \qquad \operatorname{supp} f_{\mathbf{m}',h} \subseteq \big(\mathbb{R}^d \backslash E(\mathbf{m})\big) + B(0,\varepsilon')$$

and hence

$$\operatorname{supp} f_{\mathbf{m},h} \cap \operatorname{supp} f_{\mathbf{m}',h} \subseteq \big\{2W - W(\mathbf{m}) - W(\mathbf{m}') \geq c > 0\big\}.$$

Consequently, $\langle f_{\mathbf{m},h}, f_{\mathbf{m}',h}\rangle = O(\mathrm{e}^{-c/h})$.

• The case where $\mathbf{m}\mathcal{R}\mathbf{m}'$ (i.e $\alpha = \alpha'$) and both minima are of type II.



In that case, start by noticing that because of (6.2.4), if $\tilde{\mathbf{m}}$, $\tilde{\mathbf{m}}' \in \widehat{\mathcal{U}}_{\alpha}^{(0)}$ with $\tilde{\mathbf{m}} \neq \tilde{\mathbf{m}}'$, we have

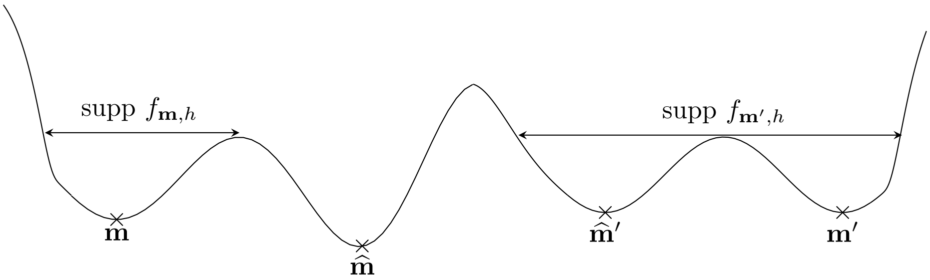
$$\operatorname{supp} \theta^{\alpha}_{\tilde{\mathbf{m}},h} \cap \operatorname{supp} \theta^{\alpha}_{\tilde{\mathbf{m}}',h} \subseteq \{W_{\tilde{\mathbf{m}}} \geq c > 0\}.$$

Therefore, using the definitions of our quasimodes (6.2.11) as well as item c) from Lemma 6.2.1 and (6.2.8), we compute

$$\begin{aligned}\langle f_{\mathbf{m},h}, f_{\mathbf{m}',h}\rangle + O(\mathrm{e}^{-c/h}) &= h^{-d/2} \sum_{\tilde{\mathbf{m}} \in \widehat{\mathcal{U}}_{\alpha}^{(0)}} \varphi^{\alpha}_{\mathbf{m}}(\tilde{\mathbf{m}})\varphi^{\alpha}_{\mathbf{m}'}(\tilde{\mathbf{m}}) c^{\alpha}_{h}(\tilde{\mathbf{m}})^2 \int_{\mathbb{R}^d} \chi^2_{\alpha}(\theta^{\alpha}_{\tilde{\mathbf{m}},h})^2 \mathrm{e}^{-2W_{\tilde{\mathbf{m}}}/h} \mathrm{d}x \\ &= \langle \varphi^{\alpha}_{\mathbf{m}}, \varphi^{\alpha}_{\mathbf{m}'}\rangle_{\mathcal{F}_{\alpha}} \\ &= \delta_{\mathbf{m},\mathbf{m}'}\end{aligned}$$

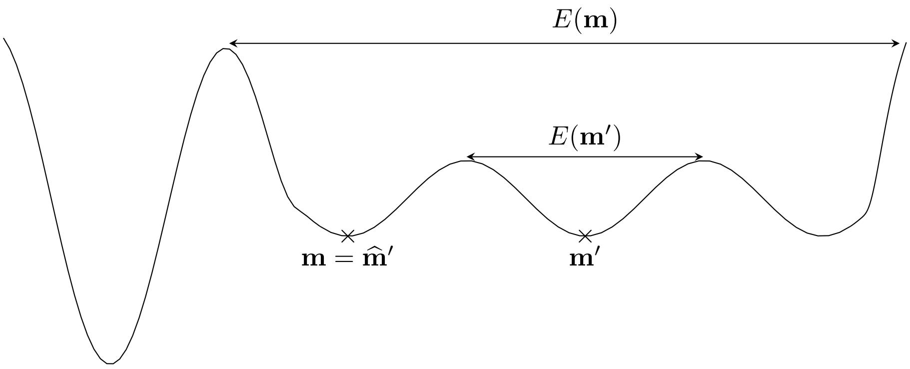
by Lemma 6.2.1.

- The case where $\mathbf{m}\not\mathcal{R}\mathbf{m}'$ and $\boldsymbol{\sigma}(\mathbf{m}) = \boldsymbol{\sigma}(\mathbf{m}')$.



In that case, it follows from the support properties of our quasimodes and the result of Proposition 3.8 from [31] that ε' can be chosen small enough so that the supports of $f_{\mathbf{m},h}$ and $f_{\mathbf{m}',h}$ do not intersect. Thus, $\langle f_{\mathbf{m},h}, f_{\mathbf{m}',h}\rangle = 0$.

- The case where $\mathbf{m}\not\mathcal{R}\mathbf{m}'$, $\boldsymbol{\sigma}(\mathbf{m}) > \boldsymbol{\sigma}(\mathbf{m}')$ and $W(\mathbf{m}) = W(\mathbf{m}')$.



In that situation, thanks to (6.2.11), (6.2.12) and (6.2.4) we can suppose that

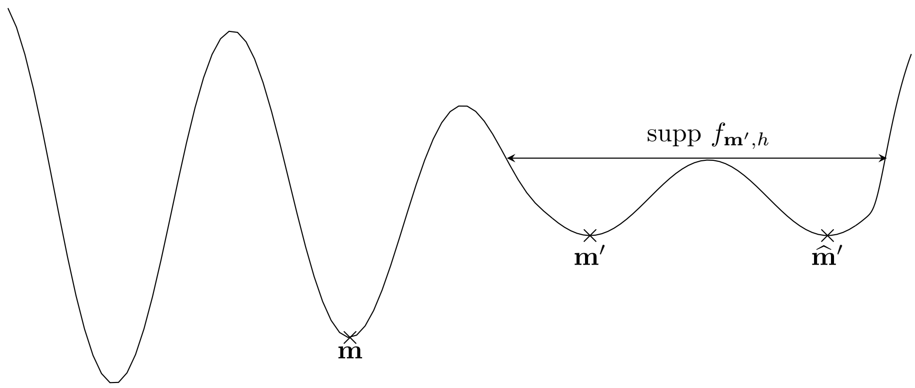
$$f_{\mathbf{m},h} = \tilde{c}_h(\mathbf{m})h^{-d/4}\mathrm{e}^{-W_{\mathbf{m}}/h} \quad \text{on the support of } f_{\mathbf{m}',h}, \text{ with } \tilde{c}_h(\mathbf{m}) = O(1) \tag{6.3.1}$$

(otherwise the supports of $f_{\mathbf{m},h}$ and $f_{\mathbf{m}',h}$ are disjoint and the result is obvious) and consequently $\mathbf{m}'$ is of type II. Hence, using a standard Laplace method, we can write

$$\begin{aligned}
\langle f_{\mathbf{m},h}, f_{\mathbf{m}',h}\rangle &= h^{-d/2}\tilde{c}_h(\mathbf{m}) \sum_{\tilde{\mathbf{m}}'\in\widehat{\mathcal{U}}^{(0)}_{\alpha'}} \varphi^{\alpha'}_{\mathbf{m}'}(\tilde{\mathbf{m}}')c_h^{\alpha'}(\tilde{\mathbf{m}}') \int_{\mathbb{R}^d} \chi_{\alpha'}\theta^{\alpha'}_{\tilde{\mathbf{m}}',h}\,\mathrm{e}^{-2W_{\mathbf{m}}/h}\mathrm{d}x \\
&= h^{-d/2}\tilde{c}_h(\mathbf{m}) \sum_{\tilde{\mathbf{m}}'\in\widehat{\mathcal{U}}^{(0)}_{\alpha'}} \varphi^{\alpha'}_{\mathbf{m}'}(\tilde{\mathbf{m}}')c_h^{\alpha'}(\tilde{\mathbf{m}}') \int_{\mathbb{R}^d} \chi^2_{\alpha'}(\theta^{\alpha'}_{\tilde{\mathbf{m}}',h})^2\,\mathrm{e}^{-2W_{\mathbf{m}}/h}\mathrm{d}x + O(\mathrm{e}^{-c/h}) \\
&= \tilde{c}_h(\mathbf{m}) \sum_{\tilde{\mathbf{m}}'\in\widehat{\mathcal{U}}^{(0)}_{\alpha'}} \varphi^{\alpha'}_{\mathbf{m}'}(\tilde{\mathbf{m}}')c_h^{\alpha'}(\tilde{\mathbf{m}}')^{-1} + O(\mathrm{e}^{-c/h}) \qquad \text{by (6.2.8)} \\
&= \tilde{c}_h(\mathbf{m}) \sum_{\tilde{\mathbf{m}}'\in\widehat{\mathcal{U}}^{(0)}_{\alpha'}\setminus\mathcal{U}^{(0),I}_{\alpha'}} \varphi^{\alpha'}_{\mathbf{m}'}(\tilde{\mathbf{m}}')c_h^{\alpha'}(\tilde{\mathbf{m}}')^{-1} + O(\mathrm{e}^{-c/h}) \qquad \text{by item b) from Lemma 6.2.1} \\
&= \frac{\tilde{c}_h(\mathbf{m})}{\widehat{c}_h}\langle \varphi^{\alpha'}_{\mathbf{m}'}, \varphi^{\alpha'}_{\widehat{\mathbf{m}}'}\rangle_{\mathcal{F}_{\alpha'}} + O(\mathrm{e}^{-c/h}) \qquad \text{by item a) from Lemma 6.2.1} \\
&= O(\mathrm{e}^{-c/h})
\end{aligned}$$

where we also used the orthogonality of the family $(\varphi_{\mathbf{m}})_{\mathbf{m}\in\widehat{\mathcal{U}}^{(0)}_{\alpha'}}$.

- The case where $\mathbf{m}\not\mathcal{R}\mathbf{m}'$, $\boldsymbol{\sigma}(\mathbf{m}) > \boldsymbol{\sigma}(\mathbf{m}')$ and $W(\mathbf{m}) \neq W(\mathbf{m}')$.



Here again we can suppose that (6.3.1) holds true so $W(\mathbf{m}) < W(\mathbf{m}')$. By item c) from Lemma 6.2.1, we have $W \geq W(\mathbf{m}')$ on the support of $f_{\mathbf{m}',h}$. Therefore, $W_{\mathbf{m}} \geq c > 0$ on the support of $f_{\mathbf{m}',h}$ and since $f_{\mathbf{m}',h} = O_{L^\infty}(h^{-d/4})$, we get $\langle f_{\mathbf{m},h}, f_{\mathbf{m}',h}\rangle = O(\mathrm{e}^{-c/h})$.

As a result of the above discussion, we obtain the following statement.

Proposition 6.3.1. *The family of quasimodes $(f_{\mathbf{m},h})_{\mathbf{m}\in\mathcal{U}^{(0)}}$ introduced in (6.2.11) and (6.2.7) is almost orthonormal :*

$$\langle f_{\mathbf{m},h}, f_{\mathbf{m}',h}\rangle = \delta_{\mathbf{m},\mathbf{m}'} + O(\mathrm{e}^{-c/h}).$$

6.4 Action of the operator P_h

Let us fix $\mathbf{m} \in \mathcal{U}_\alpha^{(0)}$. For $\tilde{\mathbf{m}} \in \widehat{\mathcal{U}}_\alpha^{(0)}$ we will denote

(6.4.1)
$$\widetilde{W}_{\tilde{\mathbf{m}},h} = W_{\mathbf{m}} + \sum_{\mathbf{s}\in\mathbf{j}^\alpha(\tilde{\mathbf{m}})} (\ell^{\mathbf{s},\tilde{\mathbf{m}}})^2/2$$

and

(6.4.2)
$$\psi^{\tilde{\mathbf{m}},h}(x,y) = \int_0^1 \nabla\widetilde{W}_{\tilde{\mathbf{m}},h}(y+t(x-y))\mathrm{d}t.$$

Remark 6.4.1. *Using Hypothesis 3.1.3 and symbolic calculus, one gets $d_W^* Op_h(q^h) = Op_h(g^h)$, with $g^h \sim \sum_n h^n g_n$ in $\mathcal{M}_{1,d}\big(S_1^0(\langle(x,\xi)\rangle^{-1})\big)$ given by*

(6.4.3)
$$g_0(x,\xi) = \Big(-i\,\xi^t + \nabla W^t\Big)q_0(x,\xi)$$

and

(6.4.4)
$$g_n(x,\xi) = \Big(-i\,\xi^t + \nabla W^t\Big)q_n(x,\xi) - \partial_x^t q_{n-1}(x,\xi) + \sum_{k=0}^n i^k \sum_{\substack{\beta\in\mathbb{N}^d;\\ |\beta|=k}} c_{k,\beta}(x)\partial_\xi^\beta(q_{n-k})(x,\xi)$$

for some smooth functions $c_{k,\beta}$ taking values in $\mathbb{R}^d$.

Proposition 6.4.2. *Let $f_{\mathbf{m},h}$ be the quasimode defined in (6.2.11). With the notations introduced in (6.2.2) and (6.4.1), one has*

$$P_h f_{\mathbf{m},h} = \frac{h^{1-d/4}}{2} A_h^{-1} \sum_{\tilde{\mathbf{m}}\in\widehat{\mathcal{U}}_\alpha^{(0)}} \varphi_{\mathbf{m}}^\alpha(\tilde{\mathbf{m}}) c_h^\alpha(\tilde{\mathbf{m}}) \omega^{\tilde{\mathbf{m}},\alpha}\, \mathrm{e}^{\frac{-\widetilde{W}_{\tilde{\mathbf{m}},h}}{h}} \mathbf{1}_{\mathbf{j}^\alpha(\tilde{\mathbf{m}})+B(0,2r)} + O_{L^2}\Big(h^\infty \mathrm{e}^{-\frac{S(\mathbf{m})}{h}}\Big)$$

where $\omega^{\tilde{\mathbf{m}},\alpha}$ is a function bounded uniformly in h and defined for $x \in \mathbf{j}^\alpha(\tilde{\mathbf{m}}) + B(0,2r)$ by

$$\omega^{\tilde{\mathbf{m}},\alpha}(x) = \sum_{\mathbf{s}\in\mathbf{j}^\alpha(\tilde{\mathbf{m}})} (2\pi h)^{-d} \int_{\mathbb{R}^d}\int_{|y-\mathbf{s}|\leq 2r} \mathrm{e}^{\frac{i}{h}\xi\cdot(x-y)} g^h\Big(\frac{x+y}{2}, \xi + i\psi^{\tilde{\mathbf{m}},h}(x,y)\Big)\nabla\ell^{\mathbf{s},\tilde{\mathbf{m}}}(y)\,\mathrm{d}y\mathrm{d}\xi.$$

Proof. In order to lighten the notations, we will drop some of the exponents and indexes $\mathbf{m}$, $\mathbf{s}$, α and h in the proof. Let $\tilde{\mathbf{m}} \in \widehat{\mathcal{U}}_\alpha^{(0)}$. By (6.2.1), we have on the support of χ that $\theta_{\tilde{\mathbf{m}}}^\alpha$ is smooth and since it is constant outside of $B(\mathbf{s}, r)$, we have

$$\nabla\theta_{\tilde{\mathbf{m}}}^\alpha = \frac{A_h^{-1}}{2} \sum_{\mathbf{s}\in\mathbf{j}^\alpha(\tilde{\mathbf{m}})} \mathrm{e}^{-(\ell^{\mathbf{s},\tilde{\mathbf{m}}})^2/2h}\zeta(\ell^{\mathbf{s},\tilde{\mathbf{m}}})\nabla\ell^{\mathbf{s},\tilde{\mathbf{m}}}\,\mathbf{1}_{B(\mathbf{s},r)}. \tag{6.4.5}$$

We can then use Remark 6.4.1 to write

$$\begin{aligned} P_h(f) &= h^{1-d/4} \sum_{\tilde{\mathbf{m}}\in\widehat{\mathcal{U}}_\alpha^{(0)}} \varphi_{\mathbf{m}}^\alpha(\tilde{\mathbf{m}}) c_h^\alpha(\tilde{\mathbf{m}}) \mathrm{Op}_h(g)\big(\nabla\theta_{\tilde{\mathbf{m}}}^\alpha \chi \mathrm{e}^{-W_{\mathbf{m}}/h} + \nabla\chi\theta_{\tilde{\mathbf{m}}}^\alpha \mathrm{e}^{-W_{\mathbf{m}}/h}\big) \\ &= \frac{h^{1-d/4}}{2}A_h^{-1} \sum_{\tilde{\mathbf{m}}\in\widehat{\mathcal{U}}_\alpha^{(0)}} \varphi_{\mathbf{m}}^\alpha(\tilde{\mathbf{m}}) c_h^\alpha(\tilde{\mathbf{m}}) \sum_{\mathbf{s}\in\mathbf{j}^\alpha(\tilde{\mathbf{m}})} \mathrm{Op}_h(g)\Big(\zeta(\ell^{\mathbf{s},\tilde{\mathbf{m}}})\chi \mathrm{e}^{\frac{-\widetilde{W}_{\tilde{\mathbf{m}},h}}{h}}\nabla\ell^{\mathbf{s},\tilde{\mathbf{m}}}\,\mathbf{1}_{B(\mathbf{s},r)}\Big) + O\Big(h\mathrm{e}^{-\frac{S(\mathbf{m})+\tilde{\varepsilon}}{h}}\Big) \end{aligned} \tag{6.4.6}$$

where we used (6.2.6) and the fact that $\mathrm{Op}_h(g)$ is bounded uniformly in h since $g \in S(\langle(v,\xi)\rangle^{-1})$. Now we have for $\mathbf{s} \in \mathbf{j}^\alpha(\tilde{\mathbf{m}})$

$$\begin{aligned} (2\pi h)^d \mathrm{Op}_h(g)\Big(\zeta(\ell^{\mathbf{s},\tilde{\mathbf{m}}})\chi \mathrm{e}^{\frac{-\widetilde{W}_{\tilde{\mathbf{m}},h}}{h}}\nabla\ell^{\mathbf{s},\tilde{\mathbf{m}}}\,\mathbf{1}_{B(\mathbf{s},r)}\Big)(x) &= \int_{\mathbb{R}^d}\int_{|y-\mathbf{s}|\leq r} \mathrm{e}^{\frac{i}{h}\xi\cdot(x-y)} g\Big(\frac{x+y}{2},\xi\Big) \\ &\quad \times \chi(y)\zeta\big(\ell^{\mathbf{s},\tilde{\mathbf{m}}}(y)\big)\mathrm{e}^{-\widetilde{W}_{\tilde{\mathbf{m}}}(y)/h}\nabla\ell^{\mathbf{s},\tilde{\mathbf{m}}}(y)\,\mathrm{d}y\mathrm{d}\xi. \end{aligned} \tag{6.4.7}$$

Let us now treat separately the cases $|x-\mathbf{s}| \geq 2r$ and $|x-\mathbf{s}| < 2r$.
When $|x-\mathbf{s}| \geq 2r$, we have $|x-y| \geq r$ so we can apply the non stationnary phase to the integral in ξ to get that for all $N \geq 1$, there exists $C_N > 0$ such that

$$\Big|\int_{\mathbb{R}^d}\int_{|y-\mathbf{s}|\leq r} \mathrm{e}^{\frac{i}{h}\xi\cdot(x-y)} g\Big(\frac{x+y}{2},\xi\Big)\chi(y)\zeta\big(\ell^{\mathbf{s},\tilde{\mathbf{m}}}(y)\big)\mathrm{e}^{-\widetilde{W}_{\tilde{\mathbf{m}}}(y)/h}\nabla\ell^{\mathbf{s},\tilde{\mathbf{m}}}(y)\,\mathrm{d}y\mathrm{d}\xi\Big| \leq C_N h^N |x-\mathbf{s}|^{-N}\mathrm{e}^{-\frac{S(\mathbf{m})}{h}}$$

where we used item d) from (6.2.1), the fact that $W_{\mathbf{m}}(\mathbf{s}) + (\ell_0^{\mathbf{s},\tilde{\mathbf{m}}})^2(\mathbf{s})/2 = S(\mathbf{m})$ and the estimate $|x-y| \geq |x-\mathbf{s}|/2$. Hence we have shown that

$$P_h f\,\mathbf{1}_{\big\{\mathrm{dist}\big(\cdot,\cup_{\tilde{\mathbf{m}}\in\widehat{\mathcal{U}}_\alpha^{(0)}}\mathbf{j}^\alpha(\tilde{\mathbf{m}})\big)\geq 2r\big\}} = O\Big(h^\infty \mathrm{e}^{-\frac{S(\mathbf{m})}{h}}\Big). \tag{6.4.8}$$

Now for the case $|x-\mathbf{s}| < 2r$, let us denote $J_1^{\mathbf{s},\tilde{\mathbf{m}}}(x)$ the RHS of (6.4.7). Proceeding as in [33] in order to take the $\mathrm{e}^{-\widetilde{W}_{\tilde{\mathbf{m}}}(y)/h}$ in front of the oscillatory integral, we get that

$$J_1^{\mathbf{s},\tilde{\mathbf{m}}}(x) = \mathrm{e}^{-\widetilde{W}_{\tilde{\mathbf{m}}}(x)/h} J_2^{\mathbf{s},\tilde{\mathbf{m}}}(x) \tag{6.4.9}$$

where

$$J_2^{\mathbf{s},\tilde{\mathbf{m}}}(x) = \int_{\mathbb{R}^d}\int_{|y-\mathbf{s}|\leq r} \mathrm{e}^{\frac{i}{h}\big(\xi - i\psi(x,y)\big)\cdot\big(x-y\big)} g\Big(\frac{x+y}{2},\xi\Big)\chi(y)\zeta\big(\ell^{\mathbf{s},\tilde{\mathbf{m}}}(y)\big)\nabla\ell^{\mathbf{s},\tilde{\mathbf{m}}}(y)\,\mathrm{d}y\mathrm{d}\xi$$

and ψ is the function defined in (6.4.2). Applying the Cauchy formula as in [36] (proof of Proposition 3.13), one gets $J_2^{\mathbf{s},\tilde{\mathbf{m}}} = J_3^{\mathbf{s},\tilde{\mathbf{m}}}$ where

$$J_3^{\mathbf{s},\tilde{\mathbf{m}}}(x) = \int_{\mathbb{R}^d}\int_{|y-\mathbf{s}|\leq r} \mathrm{e}^{\frac{i}{h}\xi\cdot(x-y)} g\Big(\frac{x+y}{2},\xi + i\psi(x,y)\Big)\chi(y)\zeta\big(\ell^{\mathbf{s},\tilde{\mathbf{m}}}(y)\big)\nabla\ell^{\mathbf{s},\tilde{\mathbf{m}}}(y)\,\mathrm{d}y\mathrm{d}\xi$$

Combined with (6.4.7) and (6.4.9), this yields for $|x-\mathbf{s}| < 2r$

$$(2\pi h)^d \mathrm{Op}_h(g)\Big(\zeta(\ell^{\mathbf{s},\tilde{\mathbf{m}}})\chi \mathrm{e}^{\frac{-\widetilde{W}_{\tilde{\mathbf{m}},h}}{h}}\nabla\ell^{\mathbf{s},\tilde{\mathbf{m}}}\,\mathbf{1}_{B(\mathbf{s},r)}\Big)(x) = \mathrm{e}^{\frac{-\widetilde{W}_{\tilde{\mathbf{m}},h}(x)}{h}} J_3^{\mathbf{s},\tilde{\mathbf{m}}}(x). \tag{6.4.10}$$

Therefore, setting on $\mathbf{j}^\alpha(\tilde{\mathbf{m}}) + B(0, 2r)$

$$\tilde{\omega}^{\tilde{\mathbf{m}},\alpha} = (2\pi h)^{-d} \sum_{\mathbf{s}\in\mathbf{j}^\alpha(\tilde{\mathbf{m}})} J_3^{\mathbf{s},\tilde{\mathbf{m}}}(x),$$

we have according to (6.4.6), (6.4.8) and (6.4.10)

$$P_h f = \frac{h^{1-d/4}}{2} A_h^{-1} \sum_{\tilde{\mathbf{m}}\in\widehat{\mathcal{U}}_\alpha^{(0)}} \varphi_{\mathbf{m}}^\alpha(\tilde{\mathbf{m}}) c_h^\alpha(\tilde{\mathbf{m}}) \tilde{\omega}^{\tilde{\mathbf{m}},\alpha}\, \mathrm{e}^{\frac{-\widetilde{W}_{\tilde{\mathbf{m}},h}}{h}} \mathbf{1}_{\mathbf{j}^\alpha(\tilde{\mathbf{m}})+B(0,2r)} + O\Big(h^\infty \mathrm{e}^{-\frac{S(\mathbf{m})}{h}}\Big).$$

Hence it is sufficient to check that on $\mathbf{j}^\alpha(\tilde{\mathbf{m}}) + B(0, 2r)$

$$\big(\tilde{\omega}^{\tilde{\mathbf{m}},\alpha} - \omega^{\tilde{\mathbf{m}},\alpha}\big)\mathrm{e}^{\frac{-\widetilde{W}_{\tilde{\mathbf{m}},h}}{h}} = O\Big(h^\infty \mathrm{e}^{-\frac{S(\mathbf{m})}{h}}\Big).$$

This can be done easily using again the non stationary phase with x in an h-independent neighborhood of $\mathbf{s}$ on which $\chi\zeta(\ell) - 1$ vanishes since item d) from (6.2.1) implies that

$$\mathrm{e}^{\frac{-\widetilde{W}_{\tilde{\mathbf{m}},h}}{h}} = O(\mathrm{e}^{-(S(\mathbf{m})+\delta)/h})$$

outside of this neighborhood for some $\delta > 0$. □

6.5 Choice of $\ell^{\mathbf{s},\tilde{\mathbf{m}}}$

From now on, we also fix $\tilde{\mathbf{m}} \in \widehat{\mathcal{U}}_\alpha^{(0)}$ and $\mathbf{s} \in \mathbf{j}^\alpha(\tilde{\mathbf{m}})$. We write for shortness $\ell^{\mathbf{s}}$ instead of $\ell^{\mathbf{s},\tilde{\mathbf{m}}}$.

Lemma 6.5.1. *The function $\omega^{\tilde{\mathbf{m}},\alpha}$ admits the classical expansion $\omega^{\tilde{\mathbf{m}},\alpha} \sim \sum_{j\geq 0} h^j \omega_j^{\tilde{\mathbf{m}},\alpha}$ on $B(\mathbf{s}, 2r)$ where*

$$\omega_0^{\tilde{\mathbf{m}},\alpha} = q_0\Big(x, i\big(\nabla W + \ell_0^{\mathbf{s}}\, \nabla \ell_0^{\mathbf{s}}\big)\Big)\big(2\nabla W + \ell_0^{\mathbf{s}} \nabla \ell_0^{\mathbf{s}}\big) \cdot \nabla \ell_0^{\mathbf{s}}$$

and for $j \geq 1$,

(6.5.1)
$$\begin{aligned}\omega_j^{\tilde{\mathbf{m}},\alpha} = {}& 2\, q_0\Big(x, i\big(\nabla W + \ell_0^{\mathbf{s}}\, \nabla \ell_0^{\mathbf{s}}\big)\Big)\big(\nabla W + \ell_0^{\mathbf{s}} \nabla \ell_0^{\mathbf{s}}\big) \cdot \nabla \ell_j^{\mathbf{s}} \\ &+ i\, \ell_0^{\mathbf{s}}\big(2\, \nabla W^t + \ell_0^{\mathbf{s}}\, (\nabla \ell_0^{\mathbf{s}})^t\big) D_\xi q_0\big(x, i(\nabla W + \ell_0^{\mathbf{s}} \nabla \ell_0^{\mathbf{s}})\big)\big(\nabla \ell_j^{\mathbf{s}}\big)\, \nabla \ell_0^{\mathbf{s}} \\ &+ q_0\Big(x, i\big(\nabla W + \ell_0^{\mathbf{s}}\, \nabla \ell_0^{\mathbf{s}}\big)\Big)\nabla \ell_0^{\mathbf{s}} \cdot \nabla \ell_0^{\mathbf{s}}\, \ell_j^{\mathbf{s}} \\ &+ i\big(2\, \nabla W^t + \ell_0^{\mathbf{s}}\, (\nabla \ell_0^{\mathbf{s}})^t\big) D_\xi q_0\big(x, i(\nabla W + \ell_0^{\mathbf{s}} \nabla \ell_0^{\mathbf{s}})\big)\big(\nabla \ell_0^{\mathbf{s}}\big)\, \nabla \ell_0^{\mathbf{s}}\, \ell_j^{\mathbf{s}} \\ &+ R_j(\ell_0^{\mathbf{s}}, \dots, \ell_{j-1}^{\mathbf{s}})\end{aligned}$$

where $R_j : \big(\mathcal{C}^\infty(B(\mathbf{s}, 2r))\big)^j \to \mathcal{C}^\infty(B(\mathbf{s}, 2r))$ and D_ξ denotes the partial differential with respect to the variable ξ.

Proof. Once again, we drop some of the exponents and indexes $\tilde{\mathbf{m}}$, $\mathbf{s}$, α and h in the proof. Denote $B_\infty(\mathbf{s}, 2r) = \{(y, \xi) \in \mathbb{R}^{2d}\,;\, \max(|y-\mathbf{s}|, |\xi|) < 2r\}$. We need to get an expansion of $g(x/2+y/2, \xi+i\psi(x, y))$ that we will then be able to combine with the stationnary phase to get an expansion of ω. Let us start with an expansion of ψ : the expansion of ℓ yields

$$\nabla\widetilde{W} - \nabla W \sim \sum_{j\geq 0} h^j \sum_{k=0}^{j} \ell_k \nabla \ell_{j-k} \quad \text{on } B(\mathbf{s}, 2r)$$

so using (6.4.2), we get

$$\psi \sim \sum_{j\geq 0} h^j \psi_j \quad \text{on } B(\mathbf{s}, 2r) \times \{|y| \leq 2r\}$$

where

$$\psi_0(x,y) = \int_0^1 \big(\nabla W + \ell_0 \nabla \ell_0\big)(y + t(x-y))\mathrm{d}t \tag{6.5.2}$$

and for $j \geq 1$,

$$\psi_j(x,y) = \int_0^1 \sum_{k=0}^{j} \big(\ell_k \nabla \ell_{j-k}\big)(y + t(x-y))\mathrm{d}t. \tag{6.5.3}$$

Proceeding as in [36] (proof of Lemma 4.1), we then get thanks to Remark 6.4.1 that

$$g\Big(\frac{x+y}{2}, \xi + i\psi(x,y)\Big) \sim \sum_{j\geq 0} h^j \sum_{n=0}^{j} g_{n,j-n}(x,y,\xi) \tag{6.5.4}$$

on $B(\mathbf{s},2r) \times B_\infty(\mathbf{s},2r)$; with

$$g_{n,0}(x,y,\xi) = g_n\Big(\frac{x+y}{2}, \xi + i\psi_0(x,y)\Big) \tag{6.5.5}$$

and for $j \geq 1$

$$g_{n,j}(x,y,\xi) = iD_\xi g_n\Big(\frac{x+y}{2}, \xi + i\psi_0(x,y)\Big)\big(\psi_j(x,y)\big) + R_j^1(\ell_0, \dots, \ell_{j-1}) \tag{6.5.6}$$

where $R_j^1 : \big(\mathcal{C}^\infty(B(\mathbf{s},2r))\big)^j \to \mathcal{C}^\infty(B(\mathbf{s},2r))$. Thus, using the expansion (6.5.4) that we just got, the one of $\nabla \ell$, and the one for an oscillatory integral given by the stationnary phase (see for instance [47], Theorem 3.17) as well as Proposition C.3 from [36], we finally get

$$\omega \sim \sum_{j\geq 0} h^j \omega_j \quad \text{on } B(\mathbf{s},2r),$$

where

$$\omega_j(x) = \sum_{n_1+n_2+n_3+n_4=j} \frac{1}{i^{n_1} n_1!} \big(\partial_y \cdot \partial_\xi\big)^{n_1} \Big(g_{n_2,n_3}(x,y,\xi) \nabla \ell_{n_4}(y)\Big)\Big|_{\substack{y=x\\ \xi=0}}.$$

We can already use (6.5.5) to deduce the expression of ω_0 by noticing that according to (6.5.2), $\psi_0(x,x) = \nabla W + \ell_0 \nabla \ell_0$. For $j \geq 1$, the terms of ω_j in which the function ℓ_j appears are obviously the one given by $n_4 = j$, but also the one given by $n_3 = j$ according to (6.5.6). Indeed, in that case, we have using (6.5.3) that

$$\begin{aligned} g_{0,j}(x,x,0) = i\ell_0 D_\xi g_0\big(x, i(\nabla W + \ell_0 \nabla \ell_0)\big)\big(\nabla \ell_j\big) \\ + iD_\xi g_0\big(x, i(\nabla W + \ell_0 \nabla \ell_0)\big)\big(\nabla \ell_0\big)\, \ell_j + R_j^2(\ell_0, \dots, \ell_{j-1}) \end{aligned}$$

where $R_j^2 : \big(\mathcal{C}^\infty(B(\mathbf{s},2r))\big)^j \to \mathcal{C}^\infty(B(\mathbf{s},2r))$. We can now conclude as for any $X \in \mathbb{R}^d$,

$$\begin{aligned} D_\xi g_0\big(x, i(\nabla W + \ell_0 \nabla \ell_0)\big)(X) = -i\, X^t q_0\big(x, i(\nabla W + \ell_0 \nabla \ell_0)\big) \\ + \big(2\, \nabla W^t + \ell_0\, (\nabla \ell_0)^t\big) D_\xi q_0\big(x, i(\nabla W + \ell_0 \nabla \ell_0)\big)(X) \end{aligned}$$

according to (6.4.3). □

Denote $(q^n_{m,p})_{m,p}$ the entries of the matrix q_n from Hypothesis 3.1.1. Since we have for $X \in \mathbb{R}^d$

$$D_\xi q_0\big(x, i(\nabla W + \ell_0 \nabla \ell_0)\big)(X) = \Big(\partial_\xi q^0_{m,p}\big(x, i(\nabla W + \ell_0 \nabla \ell_0)\big) \cdot X\Big)_{1\leq m,p\leq d}$$

we get by putting

$$\begin{aligned} U(x) = q_0\Big(x, i\big(\nabla W + \ell_0\, \nabla \ell_0\big)\Big) \nabla \ell_0 \\ + \sum_{1\leq m,p\leq d} \big(2\partial x_m W + \ell_0 \partial_{x_m} \ell_0\big) i\partial_\xi q^0_{m,p}\Big(x, i\big(\nabla W + \ell_0\, \nabla \ell_0\big)\Big)\partial_{x_p} \ell_0 \end{aligned} \tag{6.5.7}$$

that equation (6.5.1) reads

$$\omega_j = \Big(q_0\Big(x, i(\nabla W + \ell_0\,\nabla\ell_0)\Big)(2\nabla W + \ell_0\nabla\ell_0) + \ell_0\,U\Big)\cdot\nabla\ell_j + U\cdot\nabla\ell_0\,\ell_j + R_j(\ell_0,\dots,\ell_{j-1}).$$

Lemma 6.5.2. *Let $x,\ y \in B(\mathbf{s}, 2r)$. For any $n\in\mathbb{N}$, $\beta\in\mathbb{N}^d$ and $1\le m,p\le d$, we have*

$$\partial_\xi^\beta q_{m,p}^n\Big(\frac{x+y}{2}, i\psi_0^{\tilde{\mathbf{m}},h}(x,y)\Big) \in i^{|\beta|}\mathbb{R}$$

and

$$\partial_\xi^\beta g_n\Big(\frac{x+y}{2}, i\psi_0^{\tilde{\mathbf{m}},h}(x,y)\Big) \in i^{|\beta|}\mathbb{R}^d.$$

In particular, U defined in (6.5.7) sends $B(\mathbf{s},2r)$ in $\mathbb{R}^d$.

Proof. Since ℓ_0 vanishes at $\mathbf{s}$, we can suppose that r is such that $i\psi_0(x,y)$ is in

$$D(0,1)^d = \{z\in\mathbb{C}\,;\,|z|<1\}^d$$

so by analyticity and using the parity of $q_{m,p}^n$, we have

$$\partial_\xi^\beta q_{m,p}^n\Big(\frac{x+y}{2}, i\psi_0(x,y)\Big) = \sum_{\substack{\gamma\in\mathbb{N}^d;\\ |\gamma|+|\beta|\in 2\mathbb{N}}} i^{|\gamma|}\frac{\partial_\xi^{\gamma+\beta}q_{m,p}^n\big(\frac{x+y}{2},0\big)}{\gamma!}\,\psi_0(x,y)^\gamma \in i^{|\beta|}\mathbb{R}.$$

The result for g_n follows easily using (6.4.3) and (6.4.4). □

Using this Lemma, we also get the following result (see [36] Appendix D for a proof).

Lemma 6.5.3. *The term $R_j(\ell_0^{\mathbf{s},\tilde{\mathbf{m}}},\dots,\ell_{j-1}^{\mathbf{s},\tilde{\mathbf{m}}})$ from Lemma 6.5.1 is real valued. Moreover, it satisfies*

$$R_j(\ell_0^{\mathbf{s},\tilde{\mathbf{m}}},\dots,\ell_{j-1}^{\mathbf{s},\tilde{\mathbf{m}}}) = -R_j(-\ell_0^{\mathbf{s},\tilde{\mathbf{m}}},\dots,-\ell_{j-1}^{\mathbf{s},\tilde{\mathbf{m}}}).$$

In view of the results from Proposition 6.4.2 and Lemma 6.5.1, we want to find $\ell^{\mathbf{s},\tilde{\mathbf{m}}}$ such that on $B(\mathbf{s},2r)$,

$$q_0\Big(x, i(\nabla W + \ell_0\,\nabla\ell_0)\Big)(2\nabla W + \ell_0\nabla\ell_0)\cdot\nabla\ell_0 = 0 \tag{6.5.8}$$

and for $j\ge 1$

$$\Big(q_0\Big(x, i(\nabla W + \ell_0\,\nabla\ell_0)\Big)(2\nabla W + \ell_0\nabla\ell_0) + \ell_0\,U\Big)\cdot\nabla\ell_j + U\cdot\nabla\ell_0\,\ell_j + R_j(\ell_0,\dots,\ell_{j-1}) = 0 \tag{6.5.9}$$

where U was introduced in (6.5.7). Note that Lemmas 6.5.2 and 6.5.3 ensure that the fact that the $(\ell_j)_{j\ge 0}$ are real valued is compatible with equations (6.5.9).

6.5.1 Solving for $\ell_0^{\mathbf{s},\tilde{\mathbf{m}}}$

Denote

$$p(x,\xi) = (-i\,\xi^t + \nabla W^t)q_0(x,\xi)(i\xi + \nabla W) = q_0(x,\xi)\,\xi\cdot\xi + q_0(x,\xi)\,\nabla W\cdot\nabla W$$

the principal symbol of the whole operator P_h and $\tilde{p}(x,\xi) = -p(x,i\xi)$ its complexification. Notice that the quadratic approximation of $\tilde{p}$ at $(\mathbf{s},0)$ coincides with the one of the complexification of the symbol of the Schrödinger operator $-h^2\Delta + |\nabla W|^2$. Hence, we get all the results from [12], chapter 3. In particular, denoting

$$\Lambda_\pm = \Big\{(x,\xi)\,;\,\lim_{t\to\mp\infty} \mathrm{e}^{tH_{\tilde{p}}}(x,\xi) = (\mathbf{s},0)\Big\}$$

the stable manifolds associated to the Hamiltonian of $\tilde{p}$ near $(\mathbf{s},0)$, we obtain the following.

Lemma 6.5.4. *There exist $\phi_\pm \in \mathcal{C}^\infty(B(\mathbf{s}, 2r), \mathbb{R})$ vanishing together with their gradients at $\mathbf{s}$ and such that*

$$\Lambda_\pm = \Big\{ \big(x, \nabla\phi_\pm(x)\big)\,;\, x \in B(\mathbf{s}, 2r) \Big\}.$$

Moreover, the Hessian matrix of $\pm\phi_\pm$ at $\mathbf{s}$ is definite positive.

At this point, one can proceed as in [4], Lemmas 3.2 and 3.3 to establish the following Proposition after matching the notations by setting $\Lambda(\mathbf{s}) = 2\mathcal{W}_s$, $b^0 = 0$, $A^0(\mathbf{s}) = \mathrm{Id}$ and $B(\mathbf{s}) = 0$.

Proposition 6.5.5. *Recall the notation (6.2.10). There exists $\ell_0^{\mathbf{s},\tilde{\mathbf{m}}} \in \mathcal{C}^\infty(B(\mathbf{s}, 2r), \mathbb{R})$ such that*

- *For $x \in B(\mathbf{s}, 2r)$,*

$$\phi_+(x) = W(x) - W(\mathbf{s}) + \frac{\ell_0^{\mathbf{s},\tilde{\mathbf{m}}}(x)^2}{2}.$$

  In particular, $\ell_0^{\mathbf{s},\tilde{\mathbf{m}}}$ vanishes at $\mathbf{s}$.

- *The function $\ell_0^{\mathbf{s},\tilde{\mathbf{m}}}$ is a solution of (6.5.8) in $B(\mathbf{s}, 2r)$.*
- *The vector $\nabla\ell_0^{\mathbf{s},\tilde{\mathbf{m}}}(\mathbf{s})$ that we denote $\nu^{\mathbf{s},\tilde{\mathbf{m}}}$ is not 0 and satisfies*

$$2\mathcal{W}_s\nu^{\mathbf{s},\tilde{\mathbf{m}}} = -\big|\nu^{\mathbf{s},\tilde{\mathbf{m}}}\big|^2\,\nu^{\mathbf{s},\tilde{\mathbf{m}}}.$$

- *Finally,*

$$\det\left(\mathrm{Hess}_{\mathbf{s}}\left(W + \frac{(\ell_0^{\mathbf{s},\tilde{\mathbf{m}}})^2}{2}\right)\right) = \big|\det \mathcal{W}_{\mathbf{s}}\big|.$$

6.5.2 Solving for $(\ell_j^{\mathbf{s},\tilde{\mathbf{m}}})_{j\geq 1}$

Once again we drop some exponents $\mathbf{s}$ and $\tilde{\mathbf{m}}$ for shortness. Now that ℓ_0 is given by Proposition 6.5.5, we can solve the transport equations (6.5.9) by induction, so we suppose that $\ell_0, \ldots, \ell_{j-1}$ are given and we want to find a solution ℓ_j to (6.5.9). Denote

$$\widetilde{U} = q_0\Big(x, i\big(\nabla W + \ell_0\,\nabla\ell_0\big)\Big)(2\nabla W + \ell_0\nabla\ell_0) + \ell_0\, U \in \mathcal{C}^\infty(B(\mathbf{s}, 2r))$$

and

$$\tau = \nabla\ell_0 \cdot U \in \mathcal{C}^\infty(B(\mathbf{s}, 2r))$$

where U was introduced in (6.5.7). The function ℓ_j must satisfy $(\widetilde{U}\cdot\nabla + \tau)\ell_j = -R_j(\ell_0, \ldots, \ell_{j-1})$ so we are intersted in the operaor $\mathcal{L} = \widetilde{U}\cdot\nabla + \tau$ that we decompose as $\mathcal{L} = \mathcal{L}_0^{\mathbf{s}} + \mathcal{L}_>$ with

$$\mathcal{L}_0^{\mathbf{s}} = \widetilde{U}_0^{\mathbf{s}}(x - \mathbf{s})\cdot\nabla + \tau_0^{\mathbf{s}}$$

where $\widetilde{U}_0^{\mathbf{s}}$ is the differential of $\widetilde{U}$ at $\mathbf{s}$ and $\tau_0^{\mathbf{s}} = \tau(\mathbf{s})$, that is with (6.2.10)

$$\widetilde{U}_0^{\mathbf{s}} = 2\big(\mathcal{W}_s + \nu\nu^t\big) \qquad \text{and} \qquad \tau_0^{\mathbf{s}} = \big|\nu^{\mathbf{s},\tilde{\mathbf{m}}}\big|^2.$$

As usual, we will often omit the exponents $\mathbf{s}$ in the notations. Notice that if we denote $\mathcal{P}_{hom}^n$ the space of homogeneous polynomials of degree n in the variables $(x - \mathbf{s})$, we have $\mathcal{L}_0 \in \mathscr{L}(\mathcal{P}_{hom}^n)$ and for $P \in \mathcal{P}_{hom}^n$, $\mathcal{L}_> P(x) = O\big((x-\mathbf{s})^{n+1}\big)$ near $\mathbf{s}$. Using Proposition 6.5.5, it is easy to check that the spectrum of $\widetilde{U}_0^{\mathbf{s}}$ is exactly the spectrum of $2\mathcal{W}_s$ except that the negative eigenvalue $-\tau_0^{\mathbf{s}}$ is replaced by $\tau_0^{\mathbf{s}}$. We can then apply Lemma A.1 from [4] to get that $\mathcal{L}_0^{\mathbf{s}}$ is invertible on $\mathcal{P}_{hom}^n$. Thanks to this fact, one can proceed as in [4], section 3.3 (see also [12], chapter 3), i.e find an approximate solution of (6.5.9) using formal power series and then refine it into an actual solution using the characteristic method. This gives the following result.

Proposition 6.5.6. *For all $j \geq 1$, there exists $\ell_j^{\mathbf{s},\tilde{\mathbf{m}}} \in \mathcal{C}^\infty(B(\mathbf{s}, 2r))$ solving (6.5.9). Moreover, $\ell_j^{\mathbf{s},\tilde{\mathbf{m}}}$ is real valued in view of Lemmas 6.5.2 and 6.5.3.*

6.5.3 Construction of $\ell^{\mathbf{s},\tilde{\mathbf{m}}}$

Now that we have found $(\ell_j)_{j\geq 0} \subset \mathcal{C}^\infty(B(\mathbf{s},2r),\mathbb{R})$ solving (6.5.8) and (6.5.9) with ℓ_0 vanishing at $\mathbf{s}$, we can use a Borel procedure to construct $\ell \in \mathcal{C}^\infty(\mathbb{R}^d,\mathbb{R})$ supported in $B(\mathbf{s},3r)$ and satisfying $\ell \sim \sum_{j\geq 0} h^j \ell_j$ on $B(\mathbf{s},2r)$.

Remark 6.5.7. *The properties a)-d) from (6.2.1) are satisfied by both the functions $\ell^{\mathbf{s},\tilde{\mathbf{m}}}$ and $-\ell^{\mathbf{s},\tilde{\mathbf{m}}}$. Moreover, by Lemma 6.5.3, $(-\ell_j^{\mathbf{s},\tilde{\mathbf{m}}})_{j\geq 0}$ also solve (6.5.8) and (6.5.9).*

At this point, a straightforward adaptation of the proof of Proposition 5.2 from [36] yields the following result which states that all the properties from (6.2.1) are satisfied.

Proposition 6.5.8. *We can choose the signs of the functions $(\ell^{\mathbf{s},\tilde{\mathbf{m}}})_{\mathbf{s}\in\mathbf{j}^\alpha(\tilde{\mathbf{m}})}$ such that (6.2.1) holds true and the coefficients from the classical expansion of $\ell^{\mathbf{s},\tilde{\mathbf{m}}}$ solve (6.5.8) and (6.5.9).*

We end this section with the following observation from [4] (Lemma 6.4).

Lemma 6.5.9. *If $\mathbf{s} \in \mathbf{j}^\alpha(\mathbf{m}) \cap \mathbf{j}^{\alpha'}(\mathbf{m}')$ with $\mathbf{m} \neq \mathbf{m}'$, we can suppose (up to a modification by $O(h^\infty)$) that*

$$\ell^{\mathbf{s},\mathbf{m}} = -\ell^{\mathbf{s},\mathbf{m}'}$$

and consequently,

$$\theta^\alpha_{\mathbf{m},h} = 1 - \theta^{\alpha'}_{\mathbf{m}',h} \qquad \text{on } B(\mathbf{s},r).$$

6.6 Interaction between two wells

Let α, $\alpha' \in \mathcal{A}$ as well as $\mathbf{m} \in \mathcal{U}^{(0)}_\alpha$ and $\mathbf{m}' \in \mathcal{U}^{(0)}_{\alpha'}$.

Lemma 6.6.1. *For all $\tilde{\mathbf{m}} \in \widehat{\mathcal{U}}^{(0)}_\alpha$ and $\tilde{\mathbf{m}}' \in \widehat{\mathcal{U}}^{(0)}_{\alpha'}$ such that $\mathbf{j}^\alpha(\tilde{\mathbf{m}}) \cap \mathbf{j}^{\alpha'}(\tilde{\mathbf{m}}') \neq \emptyset$, the following holds :*

$$\alpha = \alpha' \qquad \text{or} \qquad \varphi^\alpha_{\mathbf{m}}(\tilde{\mathbf{m}})\varphi^{\alpha'}_{\mathbf{m}'}(\tilde{\mathbf{m}}') = 0.$$

Proof. First, notice that since $\widehat{\mathcal{U}}^{(0)}_\alpha \subset E_-(\mathbf{m})$ and $\widehat{\mathcal{U}}^{(0)}_{\alpha'} \subset E_-(\mathbf{m}')$, our hypothesis implies that $E_-(\mathbf{m}) = E_-(\mathbf{m}')$. In particular, $\widehat{\mathbf{m}}' = \widehat{\mathbf{m}}$. If $\widehat{\mathbf{m}} \notin \{\tilde{\mathbf{m}},\, \tilde{\mathbf{m}}'\}$, we easily have $\tilde{\mathbf{m}}\mathcal{R}\tilde{\mathbf{m}}'$ and $\alpha = \alpha'$. Let us now suppose that

$$\widehat{\mathbf{m}} \in \{\tilde{\mathbf{m}},\, \tilde{\mathbf{m}}'\}. \tag{6.6.1}$$

According to Lemma 3.4 from [36], $\mathbf{m}$ and $\mathbf{m}'$ are in the same CC of $\{W \leq \boldsymbol{\sigma}(\mathbf{m})\}$ that we denote $E_\leqslant$. By uniqueness of $\widehat{\mathbf{m}}$ in $E_-(\mathbf{m})$, each CC of $\{W < \boldsymbol{\sigma}(\mathbf{m})\} \cap E_\leqslant$ contains exactly one element from $\boldsymbol{\sigma}^{-1}(\{\boldsymbol{\sigma}(\mathbf{m})\}) \cup \{\widehat{\mathbf{m}}\}$. If $\mathbf{m}$ or $\mathbf{m}'$ is of type II, we get by definition of $\mathcal{R}$ that $\alpha = \alpha'$. Otherwise, (6.6.1) combined with item b) from Lemma 6.2.1 yield $\varphi^\alpha_{\mathbf{m}}(\tilde{\mathbf{m}})\varphi^{\alpha'}_{\mathbf{m}'}(\tilde{\mathbf{m}}') = 0$. ☐

With the notations from Section 6.2, let us denote for $\tilde{\mathbf{m}} \in \widehat{\mathcal{U}}^{(0)}_\alpha$ and $\tilde{\mathbf{m}}' \in \widehat{\mathcal{U}}^{(0)}_{\alpha'}$

$$\mathcal{N}^{\alpha,\alpha'}_{\tilde{\mathbf{m}},\tilde{\mathbf{m}}'} = h^{-d/2} c^\alpha_h(\tilde{\mathbf{m}}) c^{\alpha'}_h(\tilde{\mathbf{m}}') \mathrm{e}^{\frac{W(\tilde{\mathbf{m}})+W(\tilde{\mathbf{m}}')}{h}} \Big\langle P_h\big(\chi_\alpha \theta^\alpha_{\tilde{\mathbf{m}},h} \mathrm{e}^{-W/h}\big), \chi_{\alpha'}\theta^{\alpha'}_{\tilde{\mathbf{m}}',h}\mathrm{e}^{-W/h}\Big\rangle.$$

When $\alpha = \alpha'$, we denote for shortness $\mathcal{N}^{\alpha,\alpha}_{\tilde{\mathbf{m}},\tilde{\mathbf{m}}'} = \mathcal{N}^{\alpha}_{\tilde{\mathbf{m}},\tilde{\mathbf{m}}'}$.

Lemma 6.6.2. *Let $\tilde{\mathbf{m}} \in \widehat{\mathcal{U}}^{(0)}_\alpha$ and $\tilde{\mathbf{m}}' \in \widehat{\mathcal{U}}^{(0)}_{\alpha'}$.*

- *If $\mathbf{j}^\alpha(\tilde{\mathbf{m}}) \cap \mathbf{j}^{\alpha'}(\tilde{\mathbf{m}}') = \emptyset$, we have*

$$\mathcal{N}^{\alpha,\alpha'}_{\tilde{\mathbf{m}},\tilde{\mathbf{m}}'} = O\Big(h^\infty \mathrm{e}^{-\frac{S(\tilde{\mathbf{m}})+S(\tilde{\mathbf{m}}')}{h}}\Big).$$

- *When $\alpha = \alpha'$, we have*

$$\mathcal{N}^{\alpha}_{\tilde{\mathbf{m}},\tilde{\mathbf{m}}'} = h\mathrm{e}^{-\frac{S(\tilde{\mathbf{m}})+S(\tilde{\mathbf{m}}')}{h}} N^\alpha_{\tilde{\mathbf{m}},\tilde{\mathbf{m}}'}$$

  with $N^\alpha_{\tilde{\mathbf{m}},\tilde{\mathbf{m}}'}$ admitting an asymptotic expansion whose first term is

$$N^{\alpha,0}_{\tilde{\mathbf{m}},\tilde{\mathbf{m}}'} = \frac{(-1)^{1-\delta_{\tilde{\mathbf{m}},\tilde{\mathbf{m}}'}}}{2\pi}\Big(\sum_{\mathbf{m}\in H^\alpha(\tilde{\mathbf{m}})} \det \mathcal{W}_{\mathbf{m}}^{-1/2}\Big)^{-1/2}\Big(\sum_{\mathbf{m}'\in H^\alpha(\tilde{\mathbf{m}}')} \det \mathcal{W}_{\mathbf{m}'}^{-1/2}\Big)^{-1/2}\sum_{\mathbf{s}\in \mathbf{j}^\alpha(\tilde{\mathbf{m}})\cap \mathbf{j}^\alpha(\tilde{\mathbf{m}}')} |\det \mathcal{W}_{\mathbf{s}}|^{-1/2}\tau_0^{\mathbf{s}}$$

where we recall the notation (6.2.10) and that $\tau_0^{\mathbf{s}}$ is the negative eigenvalue of $2\mathcal{W}_{\mathbf{s}}$.

Proof. We will use the following localizations and estimates obtained thanks to (6.2.6) and (6.4.5) :

$$d_W\left(\chi_\alpha\theta^\alpha_{\tilde{\mathbf{m}},h}\mathrm{e}^{-W/h}\right)\mathbf{1}_{\left\{\mathrm{dist}\left(\cdot,\mathbf{j}^\alpha(\widehat{\mathcal{U}}^{(0)}_\alpha)\right)\geq r\right\}} = O\Big(h^\infty \mathrm{e}^{-\frac{\boldsymbol{\sigma}(\mathbf{m})}{h}}\Big); \tag{6.6.2}$$

$$d_W\left(\chi_\alpha\theta^\alpha_{\tilde{\mathbf{m}},h}\mathrm{e}^{-W/h}\right) = O\Big(h^{\frac{2+d}{4}} \mathrm{e}^{-\frac{\boldsymbol{\sigma}(\mathbf{m})}{h}}\Big) \tag{6.6.3}$$

and by the non stationnary phase applied as for (6.4.8)

$$\mathrm{Op}_h(q_h)\Big(d_W\left(\chi_\alpha\theta^\alpha_{\tilde{\mathbf{m}},h}\mathrm{e}^{-W/h}\right)\Big)\mathbf{1}_{\left\{\mathrm{dist}\left(\cdot,\mathbf{j}^\alpha(\widehat{\mathcal{U}}^{(0)}_\alpha)\right)\geq 2r\right\}} = O\Big(h^\infty \mathrm{e}^{-\frac{\boldsymbol{\sigma}(\mathbf{m})}{h}}\Big). \tag{6.6.4}$$

Using the factorized structure of P_h, the boundedness of $\mathrm{Op}_h(q_h)$ as well as (6.6.2), (6.6.3) and (6.6.4), we can write that

$$\begin{aligned}&\Big\langle P_h\big(\chi_\alpha\theta^\alpha_{\tilde{\mathbf{m}},h}\mathrm{e}^{-W/h}\big),\chi_{\alpha'}\theta^{\alpha'}_{\tilde{\mathbf{m}}',h}\mathrm{e}^{-W/h}\Big\rangle + O\big(h^\infty\mathrm{e}^{-\frac{\boldsymbol{\sigma}(\mathbf{m})+\boldsymbol{\sigma}(\mathbf{m}')}{h}}\big)\\ &\qquad= \sum_{\mathbf{s}\in \mathbf{j}^\alpha(\tilde{\mathbf{m}})\cap \mathbf{j}^{\alpha'}(\tilde{\mathbf{m}}')}\Big\langle \mathrm{Op}_h(q_h)\Big(d_W\big(\chi_\alpha\theta^\alpha_{\tilde{\mathbf{m}},h}\mathrm{e}^{-W/h}\big)\Big),d_W\big(\chi_{\alpha'}\theta^{\alpha'}_{\tilde{\mathbf{m}}',h}\mathrm{e}^{-W/h}\big)\Big\rangle_{\mathbf{s}}\end{aligned} \tag{6.6.5}$$

where $\langle\cdot,\cdot\rangle_{\mathbf{s}}$ denotes the inner product on $L^2(B(\mathbf{s},r))$. This already proves the first statement. Now when $\alpha=\alpha'$, thanks to the fact that $\mathrm{e}^{-W/h}\in \mathrm{Ker}\, d_W$ and by Lemma 6.5.9, we have for $\mathbf{s}\in\mathbf{j}^\alpha(\tilde{\mathbf{m}})\cap\mathbf{j}^\alpha(\tilde{\mathbf{m}}')$

$$\begin{aligned}d_W\big(\chi_\alpha\theta^\alpha_{\tilde{\mathbf{m}},h}\mathrm{e}^{-W/h}\big) &= d_W\big((\chi_\alpha\theta^\alpha_{\tilde{\mathbf{m}},h}-1)\mathrm{e}^{-W/h}\big) = d_W\big(\chi_\alpha(\theta^\alpha_{\tilde{\mathbf{m}},h}-1)\mathrm{e}^{-W/h}\big) + O\big(h^\infty\mathrm{e}^{-\frac{\boldsymbol{\sigma}(\mathbf{m})}{h}}\big)\\ &= (-1)^{1-\delta_{\tilde{\mathbf{m}},\tilde{\mathbf{m}}'}} d_W\big(\chi_\alpha\theta^\alpha_{\tilde{\mathbf{m}}',h}\mathrm{e}^{-W/h}\big) + O\big(h^\infty\mathrm{e}^{-\frac{\boldsymbol{\sigma}(\mathbf{m})}{h}}\big)\qquad \text{on } B(\mathbf{s},r).\end{aligned}$$

Thus, (6.6.5) becomes

$$\begin{aligned}&\Big\langle P_h\big(\chi_\alpha\theta^\alpha_{\tilde{\mathbf{m}},h}\mathrm{e}^{-W/h}\big),\chi_{\alpha}\theta^{\alpha}_{\tilde{\mathbf{m}}',h}\mathrm{e}^{-W/h}\Big\rangle + O\big(h^\infty\mathrm{e}^{-\frac{\boldsymbol{\sigma}(\mathbf{m})+\boldsymbol{\sigma}(\mathbf{m}')}{h}}\big)\\ &\qquad= (-1)^{1-\delta_{\tilde{\mathbf{m}},\tilde{\mathbf{m}}'}}\sum_{\mathbf{s}\in \mathbf{j}^\alpha(\tilde{\mathbf{m}})\cap \mathbf{j}^{\alpha}(\tilde{\mathbf{m}}')}\Big\langle \mathrm{Op}_h(q_h)\Big(d_W\big(\chi_\alpha\theta^\alpha_{\tilde{\mathbf{m}},h}\mathrm{e}^{-W/h}\big)\Big),d_W\big(\chi_{\alpha}\theta^{\alpha}_{\tilde{\mathbf{m}},h}\mathrm{e}^{-W/h}\big)\Big\rangle_{\mathbf{s}}.\end{aligned}$$

We can now work as in [36], proof of Lemma 5.3, to get that

$$\Big\langle \mathrm{Op}_h(q_h)\Big(d_W\big(\chi_\alpha\theta^\alpha_{\tilde{\mathbf{m}},h}\mathrm{e}^{-W/h}\big)\Big),d_W\big(\chi_{\alpha}\theta^{\alpha}_{\tilde{\mathbf{m}},h}\mathrm{e}^{-W/h}\big)\Big\rangle_{\mathbf{s}} = \frac{h}{2\pi}(\pi h)^{d/2}\mathrm{e}^{-\frac{2W(\mathbf{s})}{h}}\vartheta^{\mathbf{s},h}$$

with $\vartheta^{\mathbf{s},h}$ admitting a classical expansion whose first term is $|\det\mathcal{W}_{\mathbf{s}}|^{-1/2}\,\tau_0^{\mathbf{s}}$. Combining this with (6.2.9), we get the announced result. ☐

6.7 Interaction between two quasimodes

By (6.2.11), we have for $\mathbf{m}\in\mathcal{U}^{(0)}_\alpha$ and $\mathbf{m}'\in\mathcal{U}^{(0)}_{\alpha'}$

$$\begin{aligned}\langle P_h f_{\mathbf{m},h}, f_{\mathbf{m}',h}\rangle &= h^{-d/2}\sum_{\tilde{\mathbf{m}}\in\widehat{\mathcal{U}}^{(0)}_\alpha}\sum_{\tilde{\mathbf{m}}'\in\widehat{\mathcal{U}}^{(0)}_{\alpha'}}\varphi^\alpha_{\mathbf{m}}(\tilde{\mathbf{m}})\varphi^{\alpha'}_{\mathbf{m}'}(\tilde{\mathbf{m}}')c^\alpha_h(\tilde{\mathbf{m}})c^{\alpha'}_h(\tilde{\mathbf{m}}')\Big\langle P_h\big(\chi_\alpha\theta^\alpha_{\tilde{\mathbf{m}},h}\mathrm{e}^{-W_{\mathbf{m}}/h}\big),\chi_{\alpha'}\theta^{\alpha'}_{\tilde{\mathbf{m}}',h}\mathrm{e}^{-W_{\mathbf{m}'}/h}\Big\rangle\\ &= \sum_{\tilde{\mathbf{m}}\in\widehat{\mathcal{U}}^{(0)}_\alpha}\sum_{\tilde{\mathbf{m}}'\in\widehat{\mathcal{U}}^{(0)}_{\alpha'}}\varphi^\alpha_{\mathbf{m}}(\tilde{\mathbf{m}})\varphi^{\alpha'}_{\mathbf{m}'}(\tilde{\mathbf{m}}')\mathcal{N}^{\alpha,\alpha'}_{\tilde{\mathbf{m}},\tilde{\mathbf{m}}'}.\end{aligned}$$

According to Lemma 6.6.2, the leading terms in the previous sum are the ones for which $\tilde{\mathbf{m}}$, $\tilde{\mathbf{m}}'$ are such that $\mathbf{j}^{\alpha}(\tilde{\mathbf{m}}) \cap \mathbf{j}^{\alpha'}(\tilde{\mathbf{m}}') \neq \emptyset$. Combined with Lemma 6.6.1, we get

$$\langle P_h f_{\mathbf{m},h}, f_{\mathbf{m}',h}\rangle = \delta_{\alpha,\alpha'} \sum_{\tilde{\mathbf{m}},\tilde{\mathbf{m}}' \in \widehat{\mathcal{U}}_{\alpha}^{(0)}} \varphi_{\mathbf{m}}^{\alpha}(\tilde{\mathbf{m}}) \varphi_{\mathbf{m}'}^{\alpha}(\tilde{\mathbf{m}}') \mathcal{N}_{\tilde{\mathbf{m}},\tilde{\mathbf{m}}'}^{\alpha} + O\big(h^{\infty} \mathrm{e}^{-\frac{S(\mathbf{m})+S(\mathbf{m}')}{h}}\big). \tag{6.7.1}$$

We now want to go from quasimodes to actual eigenfunctions. This is where the optimization on the choice of the functions $\ell^{\mathbf{s},\mathbf{m}}$ will enable us to have the correct error terms. Here we briefly remind the procedure and give the main arguments. We refer to [26] for more details. First, combining Proposition 6.4.2, item d) from (6.2.1), Proposition 6.5.8 and a standard Laplace method, we obtain the following fundamental estimate.

Lemma 6.7.1. *Let $\mathbf{m} \in \underline{\mathcal{U}}^{(0)}$. We have*

$$\|P_h f_{\mathbf{m},h}\| = O\big(h^{\infty} \mathrm{e}^{-\frac{S(\mathbf{m})}{h}}\big).$$

Now, considering the orthogonal projector on the generalized eigenspace associated to the small eigenvalues of P_h given by

$$\Pi_0 = \frac{1}{2i\pi} \int_{|z|=ch} (z - P_h)^{-1} \mathrm{d}z \tag{6.7.2}$$

and writing

$$(1 - \Pi_0) f_{\mathbf{m},h} = \frac{-1}{2i\pi} \int_{|z|=ch} z^{-1} (z - P_h)^{-1} P_h f_{\mathbf{m},h} \mathrm{d}z,$$

Lemma 6.7.1 together with the resolvent estimate (3.1.4) give that for any $\mathbf{m} \in \mathcal{U}^{(0)}$, we have

$$\|(1 - \Pi_0) f_{\mathbf{m},h}\| = O\big(h^{\infty} \mathrm{e}^{-\frac{S(\mathbf{m})}{h}}\big).$$

Proceeding as in Proposition 4.10 from [26], one can then establish the following thanks to Proposition 6.3.1.

Lemma 6.7.2. *The family $(\Pi_0 f_{\mathbf{m},h})_{\mathbf{m} \in \mathcal{U}^{(0)}}$ is almost orthonormal : there exists $c > 0$ such that*

$$\langle \Pi_0 f_{\mathbf{m},h}, \Pi_0 f_{\mathbf{m}',h}\rangle = \delta_{\mathbf{m},\mathbf{m}'} + O(\mathrm{e}^{-c/h}).$$

In particular, it is a basis of the space $H = \operatorname{Ran} \Pi_0$ introduced in (6.7.2).
Moreover, we have

$$\langle P_h \Pi_0 f_{\mathbf{m},h}, \Pi_0 f_{\mathbf{m}',h}\rangle = \langle P_h f_{\mathbf{m},h}, f_{\mathbf{m}',h}\rangle + O\big(h^{\infty} \mathrm{e}^{-\frac{S(\mathbf{m})+S(\mathbf{m}')}{h}}\big).$$

Let us re-label the local minima $\mathbf{m}_1, \ldots, \mathbf{m}_{n_0}$ so that $(S(\mathbf{m}_j))_{j=1,\ldots,n_0}$ is non increasing in j. For shortness, we will now denote

$$f_j = f_{\mathbf{m}_j,h}$$

which still depends on h. We also denote $(\tilde{u}_j)_{j=1,\ldots,n_0}$ the orthogonalization by the Gram-Schmidt procedure of the family $(\Pi_0 f_j)_{j=1,\ldots,n_0}$ and

$$u_j = \frac{\tilde{u}_j}{\|\tilde{u}_j\|}.$$

In this setting and with our previous results, we get the following (see [26], Proposition 4.12 for a proof).

Lemma 6.7.3. *For all $1 \leq j, k \leq n_0$, it holds*

$$\langle P_h u_j, u_k\rangle = \langle P_h f_j, f_k\rangle + O\big(h^{\infty} \mathrm{e}^{-\frac{S(\mathbf{m}_j)+S(\mathbf{m}_k)}{h}}\big).$$

In order to compute the small eigenvalues of P_h, let us now consider the restriction $P_h|_H : H \to H$. We denote $\hat{u}_j = u_{n_0-j+1}$ and $\mathcal{M}$ the matrix of $P_h|_H$ in the orthonormal basis $(\hat{u}_1, \dots, \hat{u}_{n_0})$. Since $\hat{u}_{n_0} = u_1 = f_1$, we have

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}' & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{where} \qquad \mathcal{M}' = \Big(\langle P_h \hat{u}_j, \hat{u}_k\rangle\Big)_{1\leq j,k\leq n_0-1}$$

and it is sufficient to study the spectrum of $\mathcal{M}'$. We will also denote $\{\hat{S}_1 < \cdots < \hat{S}_p\}$ the set $\{S(\mathbf{m}_j)\,;\, 2 \leq j \leq n_0\}$ and for $1 \leq k \leq p$, E_k the subspace of $L^2(\mathbb{R}^d)$ generated by $\{\hat{u}_r\,;\, S(\mathbf{m}_r) = \hat{S}_k\}$. Finally, we set $\varpi_k = \mathrm{e}^{-(\hat{S}_k - \hat{S}_{k-1})/h}$ for $2 \leq k \leq p$ and $\varepsilon_j(\varpi) = \prod_{k=2}^{j} \varpi_k = \mathrm{e}^{-(\hat{S}_j - \hat{S}_1)/h}$ for $2 \leq j \leq p$ (with the convention $\varepsilon_1(\varpi) = 1$).

For a given class $\alpha \in \mathcal{A}$, let us denote $n_\alpha = |\mathcal{U}_\alpha^{(0)}|$ and also label its elements $\mathbf{m}_1^\alpha, \dots, \mathbf{m}_{n_\alpha}^\alpha$ so that $(S(\mathbf{m}_j^\alpha))_{j=1,\dots,n_\alpha}$ is non decreasing in j. We also set $\mathbf{m}_{n_\alpha+1}^\alpha = \widehat{\mathbf{m}}$ for some $\mathbf{m} \in \mathcal{U}_\alpha^{(0)}$. We will consider the matrix

$$M_h^\alpha = \big(N^\alpha \varphi_{\mathbf{m}_j^\alpha}^\alpha \cdot \varphi_{\mathbf{m}_k^\alpha}^\alpha\big)_{1\leq j,k\leq n_\alpha} = \mathcal{T}_\alpha^* N^\alpha \mathcal{T}_\alpha$$

where N^α is the matrix introduced in Lemma 6.6.2 and

$$\mathcal{T}_\alpha = \big(\varphi_{\mathbf{m}_k^\alpha}^\alpha(\mathbf{m}_j^\alpha)\big)_{\substack{1\leq j\leq n_\alpha+1 \\ 1\leq k\leq n_\alpha}}.$$

Before we can state our main result, we need to introduce some material from [4]. For the finite dimensional vector space $E = E_1 \oplus \cdots \oplus E_p$, and $j \in \{1, \dots, p\}$, let us write a general matrix $M \in \mathcal{M}(E)$ by blocks

$$(6.7.3)\quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : (E_1 \oplus \cdots \oplus E_{j-1}) \oplus (E_j \oplus \cdots \oplus E_p) \longrightarrow (E_1 \oplus \cdots \oplus E_{j-1}) \oplus (E_j \oplus \cdots \oplus E_p).$$

If $A \in \mathcal{M}(E_1 \oplus \cdots \oplus E_{j-1})$ is invertible, the Schur matrix of M (with respect to the vector space $E_1 \oplus \cdots \oplus E_{j-1}$) is the matrix on $E_j \oplus \cdots \oplus E_p$ defined by

$$\mathcal{R}_j(M) = D - CA^{-1}B,$$

where by convention $\mathcal{R}_1(M) = M$. By the Schur complement method, M is invertible if and only if $\mathcal{R}_j(M)$ is invertible. We will also denote by $\mathcal{J} : \mathcal{M}(\oplus_{k=j,\dots,p} E_k) \to \mathcal{M}(E_j)$ the restriction map to the first vector space E_j of $\oplus_{k=j,\dots,p} E_k$. More precisely, with the notations of (6.7.3), we will write $\mathcal{J}(M) = A$ when $j = 1$. Of course, the map $\mathcal{J}$ depends on $j \in \{1, \dots, p\}$, but we omit this dependence since the set on which $\mathcal{J}$ is acting will be obvious in the sequel. We will also use the convention

$$\mathrm{Spec}\big(\mathcal{J} \circ \mathcal{R}_j(M_h^\alpha)\big) = \emptyset \quad \text{if} \quad \hat{S}_j \notin \{S(\mathbf{m}_k^\alpha)\,;\, k = 1, \dots, n_\alpha\}.$$

Theorem 6.7.4. *With the notations introduced above, we have*

$$\mathrm{Spec}(\mathcal{M}') \subset h\mathrm{e}^{-2\hat{S}_1/h} \bigcup_{\alpha\in\mathcal{A}} \bigcup_{j=1}^{p} \varepsilon_j(\varpi)^2 \Big(\mathrm{Spec}\big(\mathcal{J} \circ \mathcal{R}_j(M_h^\alpha)\big) + D\big(0, O(h^\infty)\big)\Big)$$

with M_h^α admitting an asymptotic expansion whose first term is $\mathcal{T}_0^* N^{\alpha,0} \mathcal{T}_0$ where $N^{\alpha,0}$ is defined in Lemma 6.6.2 and $\mathcal{T}_0$ is the leading term of the matrix $\mathcal{T}_\alpha$ given by Lemma 6.2.1.

Proof. Consider the symmetric matrix $M_h^\# \in \mathcal{M}_{n_0-1}(\mathbb{R})$ defined by

$$(M_h^\#)_{j,k} = \begin{cases} N^\alpha \varphi_{\mathbf{m}_{n_0-j+1}}^\alpha \cdot \varphi_{\mathbf{m}_{n_0-k+1}}^\alpha & \text{if } \mathbf{m}_{n_0-j+1}\,,\ \mathbf{m}_{n_0-k+1} \in \mathcal{U}_\alpha^{(0)} \\ 0 & \text{otherwise} \end{cases}$$

and notice that in view of Lemma 6.7.3 and (6.7.1), we have

$$h^{-1}\mathrm{e}^{2\hat{S}_1/h}\mathcal{M}' = \Omega(\varpi)\big(M_h^\# + O(h^\infty)\big)\Omega(\varpi)$$

where $\Omega(\varpi) = \mathrm{diag}\big(\varepsilon_1(\varpi)\mathrm{Id}_{E_1}, \dots, \varepsilon_p(\varpi)\mathrm{Id}_{E_p}\big)$. Clearly, M_h^α is the restriction to α of the matrix $M_h^\#$ which is permutation similar to the block-diagonal matrix $\mathrm{diag}\big(M_h^\alpha\,;\,\alpha \in \mathcal{A}\big)$. In particular, $M_h^\#$ admits an asymptotic expansion thanks to Lemmas 6.2.1 and 6.6.2. Moreover, it is positive definite as each M_h^α appears to be definite positive. Indeed, $\mathcal{T}_0$ is clearly injective as the family $(\varphi_{\mathbf{m}}^\alpha)_{\mathbf{m}\in\mathcal{U}_\alpha^{(0)}}$ is orthonormal and it is shown in [4], Proposition 6.8, that $N^{\alpha,0} = L_\alpha^* L_\alpha$ where L_α is an injective matrix, so M_0^α is positive definite. In the words of Definition 6.7 from [4], we obtain that $h^{-1}\mathrm{e}^{2\hat{S}_1/h}\mathcal{M}'$ is a classical graded symmetric matrix so we can apply Theorem 4 from [4] to get

$$\mathrm{Spec}(\mathcal{M}') \subset h\mathrm{e}^{-2\hat{S}_1/h} \bigcup_{j=1}^{p} \varepsilon_j(\varpi)^2 \Big(\mathrm{Spec}\big(\mathcal{J} \circ \mathcal{R}_j(M_h^\#)\big) + D\big(0, O(h^\infty)\big)\Big).$$

We can then conclude as

$$\mathrm{Spec}\big(\mathcal{J} \circ \mathcal{R}_j(M_h^\#)\big) = \bigcup_{\alpha\in\mathcal{A}} \mathrm{Spec}\big(\mathcal{J} \circ \mathcal{R}_j(M_h^\alpha)\big)$$

(see [4], Theorem 6 and above for details). □

Chapitre 7

Spectral asymptotics and Metastability for the Linear Relaxation Boltzmann equation

On fournit dans ce chapitre des démonstrations (en anglais) issues de [34] des résultats présentés au chapitre 4.

7.1 Introduction

7.1.1 Motivations

We are interested in the linear Boltzmann equation :

$$\begin{cases} h\partial_t u + v\cdot h\partial_x u - \partial_x V\cdot h\partial_v u + Q_{\mathcal{H}}(h,u) = 0 \\ u_{|t=0} = u_0 \end{cases} \tag{7.1.1}$$

in a semiclassical framework (i.e in the limit $h\to 0$), where h is a *semiclassical parameter* and corresponds to the temperature of the system. Here we denoted for shortness ∂_x and ∂_v the partial gradients with respect to x and v. This equation is used to model the evolution of a system of charged particles in a gas on which acts an electrical force associated to the real valued potential V that only depends on the space variable x. The operator $Q_{\mathcal{H}}$ is called *collision operator* and models the interactions between the particles. Here the unknown is the function $u:\mathbb{R}_+\to L^1(\mathbb{R}^{2d})$ giving the probability density of the system of particles at time $t\in\mathbb{R}_+$, position $x\in\mathbb{R}^d$ and velocity $v\in\mathbb{R}^d$. For our purpose, we introduce the square roots of the usual Maxwellian distributions

$$\mu_h(v) = \frac{\mathrm{e}^{-\frac{v^2}{4h}}}{(2\pi h)^{d/4}} \qquad \text{and} \qquad \mathcal{M}_h = \mathrm{e}^{-\frac{V}{2h}}\mu_h. \tag{7.1.2}$$

This paper is devoted to the study of the linear BGK model for which the collision operator is

$$Q_{\mathcal{H}}(h,u) = h\Big(u - \int_{v'\in\mathbb{R}^d} u(x,v')\,\mathrm{d}v'\mu_h^2\Big) \tag{7.1.3}$$

and corresponds to a simple relaxation towards the Maxwellian. Denoting $Q^*_{\mathcal{H}}(h,\cdot)$ the formal adjoint of $Q_{\mathcal{H}}(h,\cdot)$, one can easily compute

$$Q_{\mathcal{H}}(h,\mathcal{M}_h^2) = 0 \qquad \text{and} \qquad Q^*_{\mathcal{H}}(h,1) = 0 \tag{7.1.4}$$

so in particular $\mathcal{M}_h^2$ is a stable state of (7.1.1) and $Q_{\mathcal{H}}$ features the local conservation of mass. In order to do a perturbative study of the time independent operator associated to (7.1.1) near $\mathcal{M}_h^2$, we introduce the natural Hilbert space

$$\mathcal{H} = \big\{u\in\mathcal{D}'\,;\,\mathcal{M}_h^{-1}u\in L^2(\mathbb{R}^{2d})\big\}.$$

It is clear from the Cauchy Schwarz inequality that $\mathcal{H}$ is indeed a subset of $L^1(\mathbb{R}^{2d})$ provided that $\mathrm{e}^{-\frac{V}{2h}} \in L^2(\mathbb{R}^d_x)$. In view of (7.1.4) and the definition of $\mathcal{H}$, it is more convenient to work with the new unknown

$$f = \mathcal{M}_h^{-1} u \, : \mathbb{R}_+ \to L^2(\mathbb{R}^{2d})$$

for which the new equation becomes

$$\begin{cases} h\partial_t f + v \cdot h\partial_x f - \partial_x V \cdot h\partial_v f + Q_h(f) = 0 \\ f_{|t=0} = f_0 \end{cases} \tag{7.1.5}$$

where

$$Q_h = \mathcal{M}_h^{-1} \circ Q_{\mathcal{H}}(h, \cdot) \circ \mathcal{M}_h. \tag{7.1.6}$$

Denoting with the notation (7.1.2),

$$\Pi_h : L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^{2d})$$

the orthogonal projection on $\mu_h \, L^2(\mathbb{R}^d_x)$, we have by (7.1.3) and (7.1.6)

$$Q_h = h(\mathrm{Id} - \Pi_h). \tag{7.1.7}$$

Our study will be focused on the spectral properties of the new time independent operator

$$\begin{aligned} P_h &= v \cdot h\partial_x - \partial_x V \cdot h\partial_v + h(\mathrm{Id} - \Pi_h) \\ &= X_0^h + Q_h \end{aligned}$$

where the notation X_0^h will stand for the operator $v \cdot h\partial_x - \partial_x V \cdot h\partial_v$, but also for the vector field $(x, v) \mapsto h(v, -\partial_x V(x))$.

This type of questions has recently known some major progress on the impulse of microlocal methods. The operator P_h was already studied in 2016 in [39] where the use of hypocoercive techniques enabled to get some resolvent estimates and establish a rough localization of its small spectrum which consists of exponentially small eigenvalues in correspondance with the minima of the potential V. This type of result is similar to the one obtained for example for the Witten Laplacian by Helffer and Sjöstrand in [19] in the 1980's. Such a localization already leads to return to equilibrium and metastability results which can be improved as the description of the small spectrum becomes more precise. For example, sharp asymptotics of the small eigenvalues of the Witten Laplacian were obtained later in the 2000's in [6] and [18] and later again for Kramers-Fokker-Planck type operators by Hérau et al. in [22]. In these papers, the idea was to exhibit a supersymmetric structure for the operator and then study both the derivative acting from 0-forms into 1-forms and its adjoint with the help of basic quasimodes. However, these methods do not apply to the Boltzmann equation as in that case the matrix appearing in the modification of the inner product does not obey good estimates with respect to the semiclassical parameter h (see for instance [38] for the case of the *mild relaxation* collision operator).

This is why our goal in this paper will be to give precise spectral asymptotics for the operator P_h through a more recent approach which consists in directly constructing a family of accurate *gaussian quasimodes* for our operator in the spirit of [4, 26] for Fokker-Planck type differential operators and [36] for the mild relaxation Boltzmann equation. Here the first difficulty is that like in [36], the operator that we consider is non local and hence it is harder to compute its action on the constructed quasimodes. This will be overcome thanks to the factorization result stated in Proposition 7.2.2. The second and main difficulty is that unlike in [36], the bad microlocal properties of Q_h are such that its action on a gaussian quasimode as used in [4, 26, 36] does not yield a precise exponential, but rather a superposition of exponentials (see Lemma 7.2.4) wich will lead to the introduction of some new quasimodes given by a superposition of "usual" gaussian quasimodes. The result that we manage to establish is similar to the one from [18] for the Witten Laplacian as well as the ones from [22, 23] with recent improvements by Bony et al. in [4] for the Fokker-Planck equation.

7.1.2 Setting and main results

For $d' \in \mathbb{N}^*$ and $Z \in \mathbb{C}^{d'}$, we use the standard notation $\langle Z\rangle = (1+|Z|^2)^{1/2}$. Let us introduce a few notations of semiclassical microlocal analysis which will be used in all this paper. These are mainly extracted from [47], chapter 4. For our purpose, it is sufficient to consider pseudo-differential operators acting only in the variable v. We will denote $\eta \in \mathbb{R}^d$ the dual variable of v and use the semiclassical Fourier transform

$$\mathcal{F}_h(f)(\eta) = \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{i}{h}v\cdot\eta} f(v)\,\mathrm{d}v.$$

We consider the space of semiclassical symbols

$$S^\kappa\big(\langle(v,\eta)\rangle^k\big) = \big\{a_h \in \mathcal{C}^\infty(\mathbb{R}^{2d})\,;\, \forall \alpha \in \mathbb{N}^{2d}, \exists\, C_\alpha > 0 \textit{ such that } |\partial^\alpha a_h(v,\eta)| \leq C_\alpha h^{-\kappa|\alpha|}\langle(v,\eta)\rangle^k\big\}$$

where $k \in \mathbb{R}$ and $\kappa \in [0,1/2]$. Given a symbol $a_h \in S^\kappa(\langle(v,\eta)\rangle^k)$, we define the associated semiclassical pseudo-differential operator for the Weyl quantization acting on functions $u \in \mathcal{S}(\mathbb{R}^d)$ by

$$\mathrm{Op}_h(a_h)u(v) = (2\pi h)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{e}^{\frac{i}{h}(v-v')\cdot\eta} a_h\Big(\frac{v+v'}{2},\eta\Big)u(v')\,\mathrm{d}v'\mathrm{d}\eta$$

where the integrals may have to be interpreted as oscillating integrals. We will denote $\Psi^\kappa(\langle(v,\eta)\rangle^k)$ the set of such operators. Note that the operator $\mathrm{Op}_h(a_h)$ admits the distributional kernel

$$K_h(v,v') = \mathcal{F}_h^{-1}\bigg(a_h\Big(\frac{v+v'}{2},\cdot\Big)\bigg)(v-v').$$

Conversely, if an operator $\mathrm{Op}_h(a_h) \in \Psi^\kappa(\langle(v,\eta)\rangle^k)$ admits the distributional kernel $K_h(v,v')$, then its symbol is given by

$$a_h(v,\eta) = \mathcal{F}_h\Big(\big(K_h \circ A\big)(v,\cdot)\Big)(\eta) \tag{7.1.8}$$

where A denotes the change of variables

$$A(v,v') = (v+v'/2, v-v'/2).$$

We will also make a few confining assumptions on the function V, assuring for instance that the bottom spectrum of the associated Witten Laplacian is discrete. In particular, our potential will satisfy Assumption 2 from [26] and Hypothesis 1.1 from [39].

Hypothesis 7.1.1. *The potential V is a smooth Morse function depending only on the space variable $x \in \mathbb{R}^d$ with values in $\mathbb{R}$ which is bounded from below and such that*

$$|\partial_x V(x)| \geq \frac{1}{C} \qquad \textit{for } |x| > C.$$

Moreover, for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \geq 2$, there exists C_α such that

$$|\partial_x^\alpha V| \leq C_\alpha.$$

In particular, for every $0 \leq k \leq d$, the set of critical points of index k of V that we denote $\mathtt{U}^{(k)}$ is finite and we set

$$n_0 = \#\mathtt{U}^{(0)}. \tag{7.1.9}$$

Finally, we will suppose that $n_0 \geq 2$.

The last assumption comes from the fact that when $n_0 = 1$, the so-called *small spectrum* of the operator P_h (i.e its eigenvalues with exponentially small modulus) is trivial, so there is nothing to study. It is shown in [30], Lemma 3.14 that for a function V satisfying Hypothesis 7.1.1, we have $V(x) \geq |x|/C$ outside of a compact. In particular, under Hypothesis 7.1.1, it holds $e^{-V/2h} \in L^2(\mathbb{R}^d_x)$. Moreover, in our setting, X_0^h is a smooth vector field whose differential is bounded on $\mathbb{R}^{2d}$, so the operator X_0^h endowed with the domain

$$D = \{u \in L^2(\mathbb{R}^{2d})\,;\, X_0^h u \in L^2(\mathbb{R}^{2d})\}$$

is skew-adjoint on $L^2(\mathbb{R}^{2d})$ and the set $\mathcal{S}(\mathbb{R}^{2d})$ is a core for this operator. Since moreover the collision operator Q_h defined in (7.1.7) is bounded and self-adjoint, we have $(P_h, D)^* = (-X_0^h + Q_h, D)$ and (P_h, D) is m-accretive on $L^2(\mathbb{R}^{2d})$.

For an operator such as P_h, which is not for instance self-adjoint with compact resolvent, we do not have any information a priori on its spectrum (except here that it is contained in $\{z \in \mathbb{C}\,;\, \mathrm{Re}\, z \geq 0\}$). In [39], the use of hypocoercive techniques enabled to establish a first description of the spectrum of P_h near 0 which, in the spirit of the case of other non self-adjoint operators studied in [22], appears in particular to be discrete. More precisely, the following result is shown in [39] :

Theorem 7.1.2. *Assume that Hypothesis 7.1.1 is satisfied and recall the notation (7.1.9). Then the operator (P_h, D) admits 0 as a simple eigenvalue. Moreover, there exists $c > 0$ and $h_0 > 0$ such that for all $0 < h \leq h_0$, we have that $\mathrm{Spec}(P_h) \cap \{\mathrm{Re}\, z \leq ch\}$ consists of exactly n_0 eigenvalues (counted with algebraic multiplicity) which are real and exponentially small with respect to $1/h$. Finally, for all $0 < \tilde{c} \leq c$, the resolvent estimate*

$$(P_h - z)^{-1} = O(h^{-1})$$

holds uniformly in $\{\mathrm{Re}\, z \leq ch\} \backslash B(0, \tilde{c}h)$.

In order to study the long time behavior of the solutions of (7.1.5), we need a precise description of the small spectrum of P_h. To this aim, we construct in Section 7.3 a family of accurate quasimodes localized around the minima of V that enables us to establish sharp asymptotics of the small eigenvalues of P_h. This will lead to the following Theorem which is the main result of this paper. Before we can state it, let us introduce a few notations that we will use throughout the paper. We denote

$$W(x,v) = \frac{V(x)}{2} + \frac{v^2}{4} \tag{7.1.10}$$

the global potential on $\mathbb{R}^{2d}$ and for $x \in \mathbb{R}^d$,

$$\mathcal{V}_x \text{ (resp. } \mathcal{W}_x\text{) the Hessian of } V \text{ at } x \text{ (resp. the Hessian of } W \text{ at } (x,0)\text{)}. \tag{7.1.11}$$

When $\mathbf{s} \in \mathbb{R}^d$ is a saddle point of V (i.e $\mathbf{s} \in \mathtt{U}^{(1)}$), we also denote

$$\tau_{\mathbf{s}} \text{ the only negative eigenvalue of } \mathcal{V}_{\mathbf{s}}. \tag{7.1.12}$$

For the sake of simplicity, we will make in the statement of the Theorem an additionnal assumption (Hypothesis 7.2.5) on the topology of the potential V that could actually be omitted (see [31] or [4]). It implies in particular that V has a unique global minimum that we denote $\underline{\mathbf{m}}$.
According to Theorem 7.1.2, we can associate to each $\mathbf{m} \in \mathtt{U}^{(0)} \backslash \{\underline{\mathbf{m}}\}$ a non zero exponentially small eigenvalue of P_h that we denote $\lambda(\mathbf{m}, h)$.

Theorem 7.1.3. *Suppose that Hypotheses 7.1.1 and 7.2.5 are satisfied and recall the notations (7.1.11)-(7.1.12). The exponentially small eigenvalues of P_h satisfy the following equivalent in the limit $h \to 0$:*

$$\lambda(\mathbf{m}, h) \sim h\varrho(\mathbf{m})\, e^{\frac{-2S(\mathbf{m})}{h}}$$

with

$$\varrho(\mathbf{m}) = \frac{1}{\pi} \sum_{\mathbf{s} \in \mathbf{j}(\mathbf{m})} \left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right)^{\frac{1}{\sqrt{|\tau_{\mathbf{s}}|}}} \left(\frac{\det \mathcal{V}_{\mathbf{m}}}{|\det \mathcal{V}_{\mathbf{s}}|}\right)^{1/2} \int_{\gamma_1 \leq z \leq \gamma < 1} k_0^{\mathbf{s}}(\gamma) k_0^{\mathbf{s}}(z) \ln\left(2\,\frac{(1+z)(1+\gamma)}{1+3z+3\gamma+z\gamma}\right) \mathrm{d}z\mathrm{d}\gamma$$

where

$$k_0^{\mathbf{s}}(z) = \frac{2\sqrt{2}}{\sqrt{|\tau_{\mathbf{s}}|}(z-\gamma_2)^2} \left(\frac{z-\gamma_1}{z-\gamma_2}\right)^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}-1} \quad ; \quad \gamma_1 = -3+2\sqrt{2} \quad ; \quad \gamma_2 = -3-2\sqrt{2}$$

and the maps S and $\mathbf{j}$ are defined in Definition C.0.7.

Finally, following [39], we use the sharp localization obtained in Theorem 7.1.3 in order to discuss the phenomena of return to equilibrium and metastability for the solutions of (7.1.5). More precisely, we are able to give a sharp rate of convergence of the semigroup $e^{-tP_h/h}$ towards $\mathbb{P}_1$, the orthogonal projector on Ker P_h : denoting λ^* the smallest non zero eigenvalue of P_h, we establish that the rate of return to equilibrium is essentially given by λ^*/h :

Corollary 7.1.4. *Under the assumptions of Theorem 7.1.3, there exists $h_0 > 0$ such that for all $0 < h \leq h_0$ and $t \geq 0$,*

$$\|e^{-tP_h/h} - \mathbb{P}_1\| \leq Ce^{-t\lambda^*/h}.$$

Besides, in the spirit of [4, 36], we also show the metastable behavior of the solutions of (7.1.5) :

Corollary 7.1.5. *Suppose that the assumptions of Theorem 7.1.3 hold true. Let us consider some local minima $\mathbf{m}_1 = \underline{\mathbf{m}}$, $\mathbf{m}_2$, ..., $\mathbf{m}_p$ such that*

$$S(\mathtt{U}^{(0)}) = \{+\infty = S(\mathbf{m}_1) > S(\mathbf{m}_2) > \cdots > S(\mathbf{m}_p)\}$$

for the map S from Definition C.0.7. For $2 \leq k \leq p$, denote $\mathbb{P}_k$ the spectral projection (which is not necessarily orthogonal) associated to the eigenvalues of P_h that are $O(e^{-2\frac{S(\mathbf{m}_k)}{h}})$. Then for any times $(t_k^{\pm})_{1\leq k\leq p}$ satisfying

$$t_p^- \geq |\ln(h^\infty)| \quad \text{and} \quad t_k^- \geq |\ln(h^\infty)|e^{2\frac{S(\mathbf{m}_{k+1})}{h}} \quad \text{for} \quad k = 1, \ldots, p-1$$

as well as

$$t_1^+ = +\infty \qquad \text{and} \qquad t_k^+ = O\Big(h^\infty e^{2\frac{S(\mathbf{m}_k)}{h}}\Big) \qquad \text{for} \quad k = 2, \ldots, p$$

one has

$$e^{-tP_h/h} = \mathbb{P}_k + O(h^\infty) \qquad \text{on } [t_k^-, t_k^+].$$

In other words, we have shown the existence of timescales on which, during its convergence towards the global equilibrium, the solution of (7.1.5) will essentially visit the metastable spaces associated to the small eigenvalues of P_h.

Another perspective would then be to study the case of collision operators satisfying the local conservation laws of physics, such as the *full linear Boltzmann operator*

$$Q_h^{FL} = h(\mathrm{Id} - \Pi_h^{FL})$$

with Π_h^{FL} the orthogonal projector on the *collision invariants* subspace

$$\mathrm{Vect}_{\mathbb{R}_v^d}\left\{e^{-\frac{v^2}{4h}},\, v_1 e^{-\frac{v^2}{4h}},\, \ldots,\, v_d e^{-\frac{v^2}{4h}},\, v^2 e^{-\frac{v^2}{4h}}\right\} L^2(\mathbb{R}_x^d)$$

which was recently studied in [7] at fixed temperature.

7.2 Preliminaries

From now on, the letter r will denote a small universal positive constant whose value may decrease as we progress in this paper (one can think of r as $1/C$).

7.2.1 Naive approach

In order to investigate a first natural approach to our problem consisting in trying to reproduce the method from [36] which was itself inspired by [4, 26], let us make for simplicity and for this subsection only an additional assumption.

Hypothesis 7.2.1. *The potential V has exactly one saddle point* $\mathbf{s}$.

Roughly speaking, this approach consists in introducing a linear form $\ell(x,v) = \ell_x \cdot (x - \mathbf{s}) + \ell_v \cdot v$ in the variables $(x - \mathbf{s}, v)$ as well as a gaussian cut-off θ which is essentially given by

$$\theta(x,v) = \int_0^{\ell(x,v)} \mathrm{e}^{-\frac{s^2}{2h}} \,\mathrm{d}s.$$

With the notation (7.1.10), the idea is then to introduce the so-called gaussian quasimode

$$\varphi(x,v) = \theta(x,v)\, \mathrm{e}^{-\frac{W(x,v)}{h}}$$

and compute $P_h\varphi$ in order to then choose the linear form ℓ minimizing the norm of $P_h\varphi$. We already know from [36] (proof of Proposition 3.13) that

$$(7.2.1)\quad X_0^h\varphi(x,v) = h\, p_\ell(x,v)\mathrm{e}^{-\frac{1}{h}\left(W(x,v)+\frac{1}{2}\ell^2(x,v)\right)}\big(1 + O(h)\big) \qquad \text{with } p_\ell = O_{L^\infty}(1), \quad |x-\mathbf{s}|, |v| < r.$$

It is also shown that the collision operator studied in this reference, that we denote $Q_h^{S^0}$, satisfies a similar result :

$$(7.2.2)\quad Q_h^{S^0}\varphi(x,v) = h\, q_\ell(x,v)\mathrm{e}^{-\frac{1}{h}\left(W(x,v)+\frac{1}{2}\ell^2(x,v)\right)}\big(1 + O(h)\big) \qquad \text{with } q_\ell = O_{L^\infty}(1), \quad |x-\mathbf{s}|, |v| < r$$

and it is then sufficient in that case to choose ℓ so that $p_\ell = -q_\ell$.
In our case, although Q_h may appear as a quite simple operator as it is just an orthogonal projection, in order to perform a computation similar to (7.2.2), it will be more convenient to adopt a microlocal point of view. This is the point of the two following results which are proven in Appendix 7.6.1.

Proposition 7.2.2. *Let us denote*

$$b_h = h\partial_v + v/2.$$

There exists a symbol $m_h \in S^{1/2}\big(\langle v,\eta\rangle^{-2}\big)$ *given by*

$$m_h(v,\eta) = 2\int_0^1 (y+1)^{d-2}\mathrm{e}^{-\frac{y}{h}\left(\frac{v^2}{2}+2\eta^2\right)}\mathrm{d}y$$

such that

$$Q_h = b_h^* \circ \mathrm{Op}_h(m_h\,\mathrm{Id}) \circ b_h.$$

Corollary 7.2.3. *One has*

$$Q_h = \mathrm{Op}_h(g_h) \circ b_h$$

with

$$g_h(v,\eta) = \int_0^1 (y+1)^{d-1}\mathrm{e}^{-\frac{y}{h}\left(\frac{v^2}{2}+2\eta^2\right)}\mathrm{d}y\,(-2i\eta^t + v^t) \in S^{1/2}\big(\langle v,\eta\rangle^{-1}\big).$$

We are now in position to establish the following fundamental computation which shows that the balancing obtained between $X_0^h\varphi$ and $Q_h^{S^0}\varphi$ cannot happen between $X_0^h\varphi$ and $Q_h\varphi$. This will motivate the introduction of some new quasimodes later on.

Lemma 7.2.4. *Assume for simplicity that Hypothesis 7.2.1 holds true and let* ℓ *a linear form in the variables* $(x - \mathbf{s}, v)$*. We have*

$$Q_h\varphi(x,v) = -h\int_0^1 \partial_y(L_y)\,\mathrm{e}^{-\frac{W(x,v)+\frac{1}{2}L_y^2(x,v)}{h}}\,\mathrm{d}y \cdot \begin{pmatrix} x-\mathbf{s} \\ v \end{pmatrix}$$

where with a slight abuse of notations, L_y *denotes both the linear form*

$$L_y(x,v) = \frac{(1+y)\ell_x\cdot(x-\mathbf{s}) + (1-y)\ell_v\cdot v}{\Big(4y\ell_v^2 + (y+1)^2\Big)^{1/2}}$$

and the vector representing it. Moreover, denoting

$$m_{y,h}(v,\eta) = 2(y+1)^{d-2}\mathrm{e}^{-\frac{y}{h}\left(\frac{v^2}{2}+2\eta^2\right)}, \tag{7.2.3}$$

we have

$$\mathrm{Op}_h(m_{y,h})\circ b_h\varphi(x,v) = 2h(2\pi h)^{-d/2}\mathrm{e}^{-\frac{V(x)}{2h}}\frac{(y+1)^{d-2}}{(4y)^{\frac{d}{2}}} \times \int_{v'\in\mathbb{R}^d}\mathrm{e}^{-\frac{1}{h}\left(\frac{v'^2}{4}+\frac{y}{8}(v+v')^2+\frac{(v-v')^2}{8y}+\frac{1}{2}\ell^2(x,v')\right)}\,\mathrm{d}v'\,\ell_v. \tag{7.2.4}$$

Proof. According to Corollary 7.2.3, we have

$$\begin{aligned}Q_h\varphi(x,v) &= \mathrm{Op}_h(g_h)\Big[h\partial_v\theta\mathrm{e}^{-W/h}\Big](x,v)\\ &= h\mathrm{Op}_h(g_h)\Big[\mathrm{e}^{-\frac{1}{h}\left(W+\frac{1}{2}\ell^2\right)}\ell_v\Big](x,v)\\ &= h(2\pi h)^{-d}\int_{v'\in\mathbb{R}^d}\int_{\eta\in\mathbb{R}^d}\mathrm{e}^{\frac{i}{h}(v-v')\cdot\eta}g_h\Big(\frac{v+v'}{2},\eta\Big)\mathrm{e}^{-\frac{1}{h}\left(W(x,v')+\frac{1}{2}\ell^2(x,v')\right)}\,\mathrm{d}v'\mathrm{d}\eta\,\ell_v.\end{aligned}$$

Let us now compute the integral in η with the expression of g_h from Corollary 7.2.3 :

$$\begin{aligned}\int_{\eta\in\mathbb{R}^d}\mathrm{e}^{\frac{i}{h}(v-v')\cdot\eta}g_h\Big(\frac{v+v'}{2},\eta\Big)\mathrm{d}\eta &= \int_0^1(y+1)^{d-1}\mathrm{e}^{-\frac{y(v+v')^2}{8h}}\Bigg[\frac{(v+v')^t}{2}\int_{\eta\in\mathbb{R}^d}\mathrm{e}^{\frac{i}{h}(v-v')\cdot\eta}\mathrm{e}^{-\frac{2y\eta^2}{h}}\mathrm{d}\eta\\ &\qquad -2i\int_{\eta\in\mathbb{R}^d}\eta^t\mathrm{e}^{\frac{i}{h}(v-v')\cdot\eta}\mathrm{e}^{-\frac{2y\eta^2}{h}}\mathrm{d}\eta\Bigg]\mathrm{d}y\\ &= \int_0^1(y+1)^{d-1}\mathrm{e}^{-\frac{y(v+v')^2}{8h}}\left[\frac{(v+v')^t}{2}+\frac{(v-v')^t}{2y}\right]\int_{\eta\in\mathbb{R}^d}\mathrm{e}^{\frac{i}{h}(v-v')\cdot\eta}\mathrm{e}^{-\frac{2y\eta^2}{h}}\mathrm{d}\eta\mathrm{d}y\\ &= 2(2\pi h)^{d/2}\int_0^1\frac{(y+1)^{d-1}}{(4y)^{\frac{d}{2}+1}}\Big((v+v')y+v-v'\Big)^t\mathrm{e}^{-\frac{1}{8h}\left(y(v+v')^2+\frac{(v-v')^2}{y}\right)}\mathrm{d}y.\end{aligned}$$

Hence, we get

$$Q_h\varphi(x,v) = 2h(2\pi h)^{-d/2}\mathrm{e}^{-\frac{V(x)}{2h}}\int_0^1\frac{(y+1)^{d-1}}{(4y)^{\frac{d}{2}+1}}\int_{v'\in\mathbb{R}^d}\Big((v+v')y+v-v'\Big) \times \mathrm{e}^{-\frac{1}{h}\left(\frac{v'^2}{4}+\frac{y}{8}(v+v')^2+\frac{(v-v')^2}{8y}+\frac{1}{2}\ell^2(x,v')\right)}\,\mathrm{d}v'\mathrm{d}y\cdot\ell_v \tag{7.2.5}$$

and (7.2.4) is now a straightforward adaptation of (7.2.5) with m_h instead of g_h. Denoting $x_{\mathbf{s}} = x-\mathbf{s}$,

$$M_y = \frac{1}{2}\mathrm{Id}+\ell_v\ell_v^t+\frac{y^2+1}{4y}\mathrm{Id}\qquad\text{and}\qquad u_y(x_{\mathbf{s}},v) = \ell_x\cdot x_{\mathbf{s}}\,\ell_v+\frac{y^2-1}{4y}v,$$

(7.2.5) becomes by the change of variables $w = v'+M_y^{-1}u_y(x_{\mathbf{s}},v)$

$$\begin{aligned}Q_h\varphi(x,v) &= 2h(2\pi h)^{-d/2}\mathrm{e}^{-\frac{V(x)}{2h}}\int_0^1\frac{(y+1)^{d-1}}{(4y)^{\frac{d}{2}+1}}\\ &\quad\times\exp\left[\frac{-1}{2h}\Big(\ell_x\ell_x^t x_{\mathbf{s}}\cdot x_{\mathbf{s}}+\frac{y^2+1}{4y}v^2-M_y^{-1}u_y(x_{\mathbf{s}},v)\cdot u_y(x_{\mathbf{s}},v)\Big)\right]\\ &\quad\times\int_{w\in\mathbb{R}^d}\Big[\Big(v-M_y^{-1}u_y(x_{\mathbf{s}},v)\Big)y+v+M_y^{-1}u_y(x_{\mathbf{s}},v)\Big]\mathrm{e}^{-\frac{M_yw\cdot w}{2h}}\mathrm{d}w\mathrm{d}y\cdot\ell_v\\ &= 2h\mathrm{e}^{-\frac{V(x)}{2h}}\int_0^1\frac{(y+1)^{d-1}}{(4y)^{\frac{d}{2}+1}}\det(M_y)^{-1/2}\Big((1+y)v+(1-y)M_y^{-1}u_y(x_{\mathbf{s}},v)\Big)\cdot\ell_v \qquad (7.2.6)\\ &\quad\times\exp\left[\frac{-1}{2h}\Big(\ell_x\ell_x^t x_{\mathbf{s}}\cdot x_{\mathbf{s}}+\frac{y^2+1}{4y}v^2-M_y^{-1}u_y(x_{\mathbf{s}},v)\cdot u_y(x_{\mathbf{s}},v)\Big)\right]\mathrm{d}y\end{aligned}$$

Now
$$\frac{(y+1)^{d-1}}{(4y)^{\frac{d}{2}+1}}\det(M_y)^{-1/2} = \frac{1}{4y\Big(4y\ell_v^2+(y+1)^2\Big)^{1/2}}$$
while

(7.2.7)
$$M_y^{-1}\ell_v = \frac{4y}{4y\ell_v^2+(y+1)^2}\,\ell_v$$

so the prefactor in the integral from (7.2.6) becomes
$$\frac{1}{4y\Big(4y\ell_v^2+(y+1)^2\Big)^{1/2}}\left[\frac{4y(1-y)\ell_v^2}{4y\ell_v^2+(y+1)^2}\ell_x\cdot x_{\mathbf{s}} + \Big((1+y)+\frac{(1-y)(y^2-1)}{4y\ell_v^2+(y+1)^2}\Big)\ell_v\cdot v\right]$$
which is further equal to

(7.2.8)
$$\frac{(1-y)\ell_v^2\ell_x\cdot x_{\mathbf{s}}+(1+y)(1+\ell_v^2)\ell_v\cdot v}{\Big(4y\ell_v^2+(y+1)^2\Big)^{3/2}} = -\frac{1}{2}\partial_y(L_y)\cdot\begin{pmatrix}x_{\mathbf{s}}\\ v\end{pmatrix}.$$

Thus, it only remains to show that the exponentials coincide, i.e
$$\ell_x\ell_x^t x_{\mathbf{s}}\cdot x_{\mathbf{s}}+\frac{y^2+1}{4y}v^2 - M_y^{-1}u_y(x_{\mathbf{s}},v)\cdot u_y(x_{\mathbf{s}},v) = \frac{v^2}{2}+L_y^2(x,v)$$
or equivalently

(7.2.9)
$$\ell_x\ell_x^t x_{\mathbf{s}}\cdot x_{\mathbf{s}}+\frac{(y-1)^2}{4y}v^2 - M_y^{-1}u_y(x_{\mathbf{s}},v)\cdot u_y(x_{\mathbf{s}},v) = \frac{\Big((1+y)\ell_x\cdot x_{\mathbf{s}}+(1-y)\ell_v\cdot v\Big)^2}{4y\ell_v^2+(y+1)^2}.$$

Using (7.2.7), we already obtain
$$M_y^{-1}u_y(x_{\mathbf{s}},v)\cdot u_y(x_{\mathbf{s}},v) = \frac{4y\ell_v^2}{4y\ell_v^2+(y+1)^2}\ell_x\ell_x^t x_{\mathbf{s}}\cdot x_{\mathbf{s}} + 2\frac{y^2-1}{4y\ell_v^2+(y+1)^2}\ell_x\cdot x_{\mathbf{s}}\ell_v\cdot v+\frac{(y^2-1)^2}{16y^2}M_y^{-1}v\cdot v$$
so the LHS of (7.2.9) becomes

(7.2.10)
$$\frac{(1+y)^2}{4y\ell_v^2+(y+1)^2}(\ell_x\cdot x_{\mathbf{s}})^2+2\frac{1-y^2}{4y\ell_v^2+(y+1)^2}\ell_x\cdot x_{\mathbf{s}}\ell_v\cdot v+\Big(\frac{(y-1)^2}{4y}-\frac{(y^2-1)^2}{16y^2}M_y^{-1}\Big)v\cdot v.$$

Finally, still using (7.2.7), one can easily check that
$$\frac{(y-1)^2}{4y}-\frac{(y^2-1)^2}{16y^2}M_y^{-1} = \frac{(1-y)^2}{4y\ell_v^2+(y+1)^2}\ell_v\ell_v^t$$
so (7.2.10) equals the RHS of (7.2.9) and the proof is complete. □

This result shows that unlike in the case of some S^0 collisions operators as studied in [36] (or even in the case of differential operators [4, 26]), here the action of Q_h on the quasimode φ does not yield a precise exponential, but rather a superposition of exponentials with the linear form in the phase varying. This suggests the introduction of some new quasimodes given by a superposition of functions similar to φ with the linear form varying.

7.2.2 Labeling of the potential minima

We now drop Hypothesis 7.2.1. The constructions of our quasimodes will rely on the labeling procedure described in Appendix C as well as the topological constructions that go with it. Following [6, 18, 23, 26], we can state our last assumption that allows us to treat the generic case.

Hypothesis 7.2.5. *We assume the following :*

a) Each $\mathbf{m}_{k,j}$ is the only global minimum of V on the CC of $\{\frac{V}{2}<\sigma_k\}$ to which it belongs.

b) For all $\mathbf{m}\neq\mathbf{m}'\in\mathtt{U}^{(0)}$, the sets $\mathbf{j}(\mathbf{m})$ and $\mathbf{j}(\mathbf{m}')$ do not intersect.

According to Proposition 3.9 from [36], this hypothesis is equivalent to the facts that $(\mathbf{m},0)$ is the only global minimum of $W|_{E(\mathbf{m})}$ and $\mathbf{j}^W(\mathbf{m})\cap\mathbf{j}^W(\mathbf{m}')=\emptyset$ which is what we use in practice.

7.3 Accurate quasimodes

7.3.1 Gaussian quasimodes superposition

By Hypothesis 7.2.5, the potential V has a unique global minimum that we denote $\underline{\mathbf{m}}$. For $r > 0$, denote $\tilde{r}$ a positive number such that for all $\mathbf{m} \in \mathtt{U}^{(0)} \backslash \{\underline{\mathbf{m}}\}$ and $\mathbf{s} \in \mathbf{j}(\mathbf{m})$,

$$W(x,v) \geq \boldsymbol{\sigma}(\mathbf{m}) + \frac{r^2}{8} \quad \text{as soon as } |x-\mathbf{s}| < \tilde{r} \text{ and } |v| \geq r. \tag{7.3.1}$$

We also denote for $x \in \mathbb{R}^d$

$$B_0(x,r) = B(x,\tilde{r}) \times B(0,r) \subseteq \mathbb{R}^{2d}.$$

Let $\mathbf{m} \in \mathtt{U}^{(0)} \backslash \{\underline{\mathbf{m}}\}$; for each $\mathbf{s} \in \mathbf{j}(\mathbf{m})$ we introduce a vector $\ell^{\mathbf{s}} = (\ell^{\mathbf{s}}_x, \ell^{\mathbf{s}}_v) \in \mathbb{R}^{2d}$ which will represent a linear form involved in the construction of our quasimodes. Note that thanks to item b) from Hypothesis 7.2.5, each $\ell^{\mathbf{s}}$ corresponds to a unique $\mathbf{m} \in \mathtt{U}^{(0)} \backslash \{\underline{\mathbf{m}}\}$. In the spirit of [4, 26, 36] and more precisely in view of (7.2.1)-(7.2.2), we want $\mathbf{s}$ to be a local minimum of the function $W(x,v) + (\ell^{\mathbf{s}}_x \cdot (x-\mathbf{s}) + \ell^{\mathbf{s}}_v \cdot v)^2/2$ so according to Lemma 7.6.2 and using the notation (7.1.11), we take $\ell^{\mathbf{s}}$ satisfying

$$-\mathcal{V}_{\mathbf{s}}^{-1} \ell^{\mathbf{s}}_x \cdot \ell^{\mathbf{s}}_x - |\ell^{\mathbf{s}}_v|^2 > \frac{1}{2}.$$

This condition would be sufficient to develop a framework for the construction of our quasimodes. However, it would appear later on when establishing a result analogous to the one of Lemma 7.3.6 that the optimal choice of $\ell^{\mathbf{s}}$ would actually satisfy

$$-\mathcal{V}_{\mathbf{s}}^{-1} \ell^{\mathbf{s}}_x \cdot \ell^{\mathbf{s}}_x - |\ell^{\mathbf{s}}_v|^2 = 1.$$

Similarly, one could show in this framework from the analogous of (7.3.9) that our quasimodes would not depend on the norm of $\ell^{\mathbf{s}}$. Thus, we set

$$|\ell^{\mathbf{s}}_v|^2 = 1 \tag{7.3.2}$$

as well as

$$-\mathcal{V}_{\mathbf{s}}^{-1} \ell^{\mathbf{s}}_x \cdot \ell^{\mathbf{s}}_x = 2 \tag{7.3.3}$$

straight away as it leads to significant simplifications in the study.
We now introduce the polynomial

$$P(\gamma) = 4\gamma + (\gamma+1)^2 \tag{7.3.4}$$

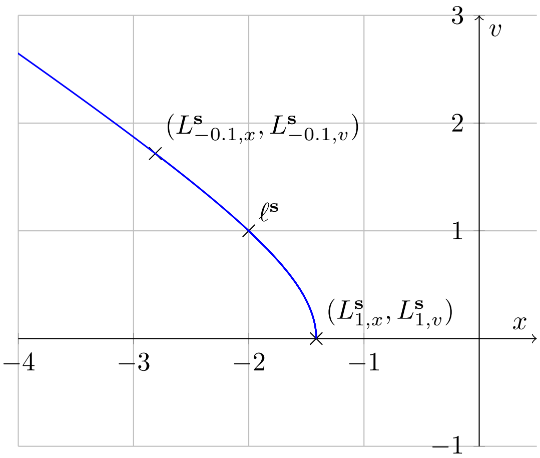
and its two roots

$$\gamma_1 = -3 + 2\sqrt{2} \in (-1,0) \qquad \text{and} \qquad \gamma_2 = -3 - 2\sqrt{2} < -1.$$

In the spirit of Lemma 7.2.4, we also introduce for $\gamma \in (\gamma_1, 1]$ the vector $(L^{\mathbf{s}}_{\gamma;x}\,, L^{\mathbf{s}}_{\gamma;v}) \in \mathbb{R}^{2d}$ where

$$L^{\mathbf{s}}_{\gamma;x} = \frac{1+\gamma}{P(\gamma)^{1/2}}\, \ell^{\mathbf{s}}_x \qquad \text{and} \qquad L^{\mathbf{s}}_{\gamma;v} = \frac{1-\gamma}{P(\gamma)^{1/2}}\, \ell^{\mathbf{s}}_v. \tag{7.3.5}$$

Note that $(L^{\mathbf{s}}_{0;x}\,, L^{\mathbf{s}}_{0;v}) = \ell^{\mathbf{s}}$. Lemma 7.2.4 would actually suggest to consider only $\gamma \in [0,1]$, but doing so it would appear with the notation (7.3.45) that (7.3.47) has no non-trivial solution, which is not true anymore when working on $(\gamma_1, 1]$. We do not consider γ outside $(\gamma_1, 1]$ as it would add a condition similar to (7.3.46) which would be incompatible with (7.3.46). Here is the picture of an example in the case $d = 1$:



Since by Hypothesis 7.1.1 we have that V is a Morse function, there exists according to the Morse Lemma a smooth diffeomorphism $\phi_{\mathbf{s}}$ defined on $B(\mathbf{s}, \tilde{r})$, sending $\mathbf{s}$ on 0, whose differential at $\mathbf{s}$ is the identity and such that

$$V \circ \phi_{\mathbf{s}}^{-1} = V(\mathbf{s}) + \frac{1}{2}\langle \mathcal{V}_{\mathbf{s}} \cdot, \cdot \rangle. \tag{7.3.6}$$

For shortness, we will use for $x \in B(\mathbf{s}, \tilde{r})$ the notation

$$\tilde{x}_{\mathbf{s}} = \phi_{\mathbf{s}}(x) \tag{7.3.7}$$

and we introduce the smooth function $L^{\mathbf{s}}_{\gamma}$ supported in $B(\mathbf{s}, 2\tilde{r}) \times \mathbb{R}^d_v$ and given when x is close to $\mathbf{s}$ by the twisted linear form :

$$L^{\mathbf{s}}_{\gamma}(x, v) = L^{\mathbf{s}}_{\gamma;x} \cdot \tilde{x}_{\mathbf{s}} + L^{\mathbf{s}}_{\gamma;v} \cdot v \qquad \text{for } (x, v) \in B(\mathbf{s}, \tilde{r}) \times \mathbb{R}^d_v.$$

Now, let us denote $\zeta \in \mathcal{C}^{\infty}_c(\mathbb{R}, [0,1])$ an even cut-off function supported in $[-\delta, \delta]$ that is equal to 1 on $[-\delta/2, \delta/2]$ where $\delta > 0$ is a parameter to be fixed later. As we will not be able to produce some remainder terms that are uniform with respect to $\gamma \in (\gamma_1, 1]$, we will work on $[\gamma_1 + \nu, 1]$ with

$$\nu > 0 \text{ that will be fixed small enough before letting } h \to 0.$$

Consider also a probability density $k^{\mathbf{s}}_{\nu}$ on $[\gamma_1 + \nu, 1]$ as well as the quantity

$$A^{\mathbf{s}}_{\nu,h} = \int_{\gamma_1+\nu}^{1} k^{\mathbf{s}}_{\nu}(\gamma) \int_0^{\infty} \zeta\Big(\frac{s}{N^{\mathbf{s}}(\gamma)}\Big) \mathrm{e}^{-\frac{s^2}{2h}} \,\mathrm{d}s \,\mathrm{d}\gamma = \frac{\sqrt{\pi h}}{\sqrt{2}}\big(1 + O(\mathrm{e}^{-\alpha/h})\big) \qquad \text{for some } \alpha > 0, \tag{7.3.8}$$

where

$$N^{\mathbf{s}}(\gamma) = \Big(|L^{\mathbf{s}}_{\gamma;x}|^2 + |L^{\mathbf{s}}_{\gamma;v}|^2\Big)^{1/2} \geq \frac{1}{C}.$$

We will also use the notation

$$U^{\mathbf{s}}_{\gamma} = \frac{L^{\mathbf{s}}_{\gamma}}{N^{\mathbf{s}}(\gamma)}.$$

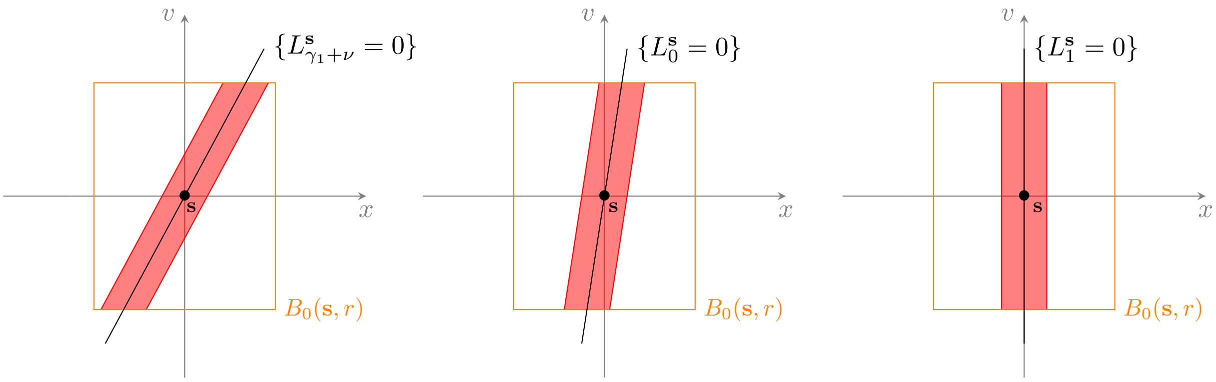
We now define for each $\mathbf{m} \in \mathtt{U}^{(0)} \backslash \{\underline{\mathbf{m}}\}$ the *gaussian cut-off superposition* $\theta^{\mathbf{m}}_{\nu,h}$ as follows : if (x, v) belongs to

$$\bigcup_{\gamma \in [\gamma_1+\nu, 1]} \{|U^{\mathbf{s}}_{\gamma}| \leq 2\delta\} \cap B_0(\mathbf{s}, r)$$

for some $\mathbf{s} \in \mathbf{j}(\mathbf{m})$, then

$$\theta^{\mathbf{m}}_{\nu,h}(x, v) = \frac{1}{2}\bigg(1 + (A^{\mathbf{s}}_{\nu,h})^{-1} \int_{\gamma_1+\nu}^{1} k^{\mathbf{s}}_{\nu}(\gamma) \int_0^{L^{\mathbf{s}}_{\gamma}(x,v)} \zeta\Big(\frac{s}{N^{\mathbf{s}}(\gamma)}\Big) \mathrm{e}^{-s^2/2h} \mathrm{d}s \,\mathrm{d}\gamma\bigg). \tag{7.3.9}$$

Here are some pictures of the set $\{|U^{\mathbf{s}}_{\gamma}| \leq 2\delta\} \cap B_0(\mathbf{s}, r)$ for $\gamma = \gamma_1 + \nu$; $\gamma = 0$ and $\gamma = 1$:



Furthermore, we set

$$\theta^{\mathbf{m}}_{\nu,h}=1 \quad \text{on } \Big(E(\mathbf{m})+B(0,\varepsilon)\Big)\Big\backslash\Big(\bigsqcup_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\Big(\bigcup_{\gamma\in[\gamma_1+\nu,1]}\{|U^{\mathbf{s}}_\gamma|\le 2\delta\}\cap B_0(\mathbf{s},r)\Big)\Big) \tag{7.3.10}$$

with $\varepsilon=\varepsilon(r)>0$ to be fixed later and

$$\theta^{\mathbf{m}}_{\nu,h}=0 \quad \text{everywhere else.} \tag{7.3.11}$$

Note that $\theta^{\mathbf{m}}_{\nu,h}$ takes values in $[0,1]$ and that, thanks to (7.3.9), we also have

$$\theta^{\mathbf{m}}_{\nu,h}=1 \quad \text{on } \Big(\bigcup_{\gamma\in[\gamma_1+\nu,1]}\{|U^{\mathbf{s}}_\gamma|\le 2\delta\}\cap B_0(\mathbf{s},r)\Big)\bigcap\Big(\bigcap_{\gamma\in[\gamma_1+\nu,1]}\{U^{\mathbf{s}}_\gamma\ge\delta\}\Big)$$

and

$$\theta^{\mathbf{m}}_{\nu,h}=0 \quad \text{on } \Big(\bigcup_{\gamma\in[\gamma_1+\nu,1]}\{|U^{\mathbf{s}}_\gamma|\le 2\delta\}\cap B_0(\mathbf{s},r)\Big)\bigcap\Big(\bigcap_{\gamma\in[\gamma_1+\nu,1]}\{U^{\mathbf{s}}_\gamma\le-\delta\}\Big).$$

Denote Ω the CC of $\{W\le\boldsymbol{\sigma}(\mathbf{m})\}$ containing $\mathbf{m}$. The CCs of $\{W\le\boldsymbol{\sigma}(\mathbf{m})\}$ are separated so for $\varepsilon>0$ small enough, there exists $\tilde\varepsilon>0$ such that

$$\min\big\{W(x,v)\,;\,\operatorname{dist}\big((x,v),\Omega\big)=\varepsilon\big\}=\boldsymbol{\sigma}(\mathbf{m})+2\tilde\varepsilon.$$

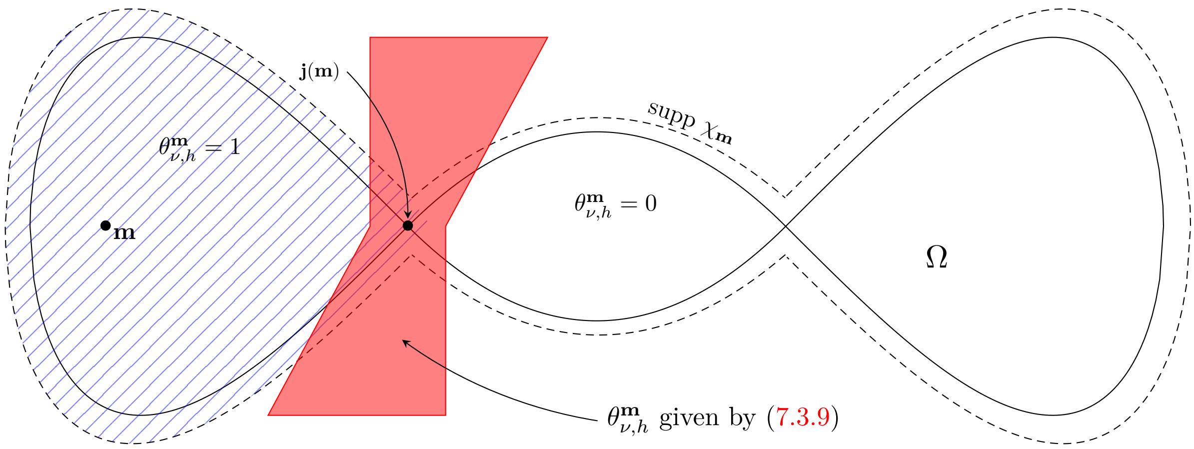
Thus the distance between $\{W\le\boldsymbol{\sigma}(\mathbf{m})+\tilde\varepsilon\}\cap\big(\Omega+B(0,\varepsilon)\big)$ and $\partial\big(\Omega+B(0,\varepsilon)\big)$ is positive and we can consider a cut-off function

$$\chi_{\mathbf{m}}\in\mathcal{C}^\infty_c(\mathbb{R}^{2d},[0,1])$$

such that

$$\chi_{\mathbf{m}}=1 \text{ on } \{W\le\boldsymbol{\sigma}(\mathbf{m})+\tilde\varepsilon\}\cap\big(\Omega+B(0,\varepsilon)\big) \qquad \text{and} \qquad \operatorname{supp}\chi_{\mathbf{m}}\subset\big(\Omega+B(0,\varepsilon)\big). \tag{7.3.12}$$

To sum up, we have the following picture :



The following Lemma will among other things help us discuss the regularity of $\theta^{\mathbf{m}}_{\nu,h}$.

Lemma 7.3.1. *Recall the notaion (7.1.11). For all $\gamma \in (\gamma_1, 1]$, we have*

$$-\mathcal{V}_{\mathbf{s}}^{-1} L^{\mathbf{s}}_{\gamma,x} \cdot L^{\mathbf{s}}_{\gamma,x} - (L^{\mathbf{s}}_{\gamma,v})^2 = 1.$$

In particular, according to Lemma 7.6.2, $(\mathbf{s}, 0)$ is a non degenerate minimum of $W + \frac{1}{2}(L^{\mathbf{s}}_\gamma)^2$ and the associated hessian has determinant

$$2^{-2d}\big|\det \mathcal{V}_{\mathbf{s}}\big|.$$

Proof. It suffices to use (7.3.2) and (7.3.3) :

$$-\mathcal{V}_{\mathbf{s}}^{-1} L^{\mathbf{s}}_{\gamma,x} \cdot L^{\mathbf{s}}_{\gamma,x} - (L^{\mathbf{s}}_{\gamma,v})^2 = 2\,\frac{(1+\gamma)^2}{P(\gamma)} - \frac{(1-\gamma)^2}{P(\gamma)} = \frac{P(\gamma)}{P(\gamma)} = 1.$$

For the computation of the determinant, it is sufficient to notice that, with the notation (7.1.11), the hessian of $W + \frac{1}{2}(L^{\mathbf{s}}_\gamma)^2$ at $(\mathbf{s}, 0)$ is

$$\mathcal{W}_{\mathbf{s}} + \begin{pmatrix} L^{\mathbf{s}}_{\gamma;x} \\ L^{\mathbf{s}}_{\gamma;v} \end{pmatrix} \begin{pmatrix} L^{\mathbf{s}}_{\gamma;x} \\ L^{\mathbf{s}}_{\gamma;v} \end{pmatrix}^t$$

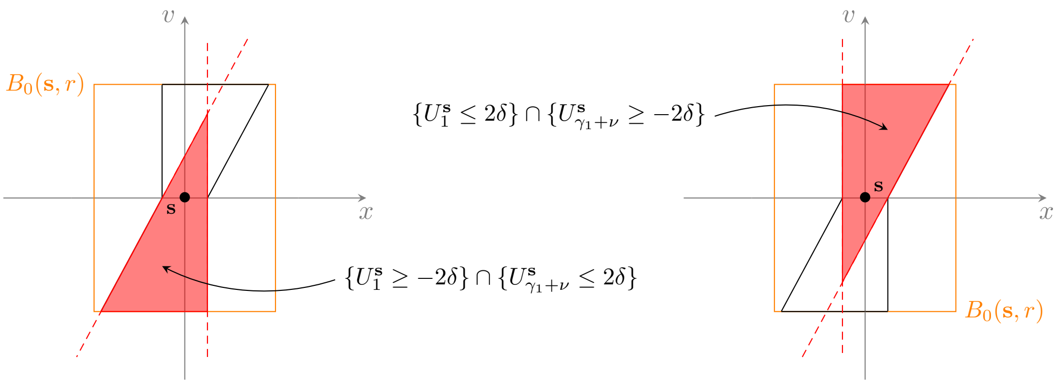
and apply Lemma 7.6.2. □

Proposition 7.3.2. *Up to changing the sign of $\ell^{\mathbf{s}}$, for all $\nu \in (0, |\gamma_1|)$, we can choose $\varepsilon > 0$ and $\delta > 0$ small enough so that the function $\theta^{\mathbf{m}}_{\nu,h}$ is smooth on the neighborhood of the support of $\chi_{\mathbf{m}}$ given by $\Omega + B(0, \varepsilon)$.*

Proof. Recall that by item b) from Hypothesis 7.2.5, each $\ell^{\mathbf{s}}$ corresponds to a unique $\mathbf{m} \in \mathtt{U}^{(0)} \backslash \{\underline{\mathbf{m}}\}$. Let us first show that in $B_0(\mathbf{s}, r)$, we have

$$\bigcup_{\gamma \in [\gamma_1+\nu, 1]} \{|U^{\mathbf{s}}_\gamma| \leq 2\delta\} = \big(\{U^{\mathbf{s}}_1 \geq -2\delta\} \cap \{U^{\mathbf{s}}_{\gamma_1+\nu} \leq 2\delta\}\big) \cup \big(\{U^{\mathbf{s}}_1 \leq 2\delta\} \cap \{U^{\mathbf{s}}_{\gamma_1+\nu} \geq -2\delta\}\big) \tag{7.3.13}$$

(so in particular, this set is closed).



Let $(x, v) \in \{U^{\mathbf{s}}_1 \geq -2\delta\} \cap \{U^{\mathbf{s}}_{\gamma_1+\nu} \leq 2\delta\}$. If $U^{\mathbf{s}}_1(x, v) \leq 2\delta$, then $(x, v) \in \{|U^{\mathbf{s}}_1| \leq 2\delta\}$ and similarly, if $U^{\mathbf{s}}_{\gamma_1+\nu}(x, v) \geq -2\delta$, then $(x, v) \in \{|U^{\mathbf{s}}_{\gamma_1+\nu}| \leq 2\delta\}$. Now if $U^{\mathbf{s}}_1(x, v) > 2\delta$ and $U^{\mathbf{s}}_{\gamma_1+\nu}(x, v) < -2\delta$, by the intermediate value theorem, there exists $\gamma \in [\gamma_1 + \nu, 1]$ such that $U^{\mathbf{s}}_\gamma(x, v) = 0$ so in particular $(x, v) \in \{|U^{\mathbf{s}}_\gamma| \leq 2\delta\}$. Thus, we have shown that

$$\{U^{\mathbf{s}}_1 \geq -2\delta\} \cap \{U^{\mathbf{s}}_{\gamma_1+\nu} \leq 2\delta\} \subseteq \bigcup_{\gamma \in [\gamma_1+\nu, 1]} \{|U^{\mathbf{s}}_\gamma| \leq 2\delta\} \tag{7.3.14}$$

and clearly the same strategy of proof enables to show that

$$\{U^{\mathbf{s}}_1 \leq 2\delta\} \cap \{U^{\mathbf{s}}_{\gamma_1+\nu} \geq -2\delta\} \subseteq \bigcup_{\gamma \in [\gamma_1+\nu, 1]} \{|U^{\mathbf{s}}_\gamma| \leq 2\delta\}. \tag{7.3.15}$$

Conversely, let

$$(x,v) \notin \left(\{U_1^{\mathbf{s}} \geq -2\delta\} \cap \{U_{\gamma_1+\nu}^{\mathbf{s}} \leq 2\delta\}\right) \cup \left(\{U_1^{\mathbf{s}} \leq 2\delta\} \cap \{U_{\gamma_1+\nu}^{\mathbf{s}} \geq -2\delta\}\right).$$

Since $\{U_1^{\mathbf{s}} < -2\delta\} \cap \{U_1^{\mathbf{s}} > 2\delta\}$ and $\{U_{\gamma_1+\nu}^{\mathbf{s}} < -2\delta\} \cap \{U_{\gamma_1+\nu}^{\mathbf{s}} > 2\delta\}$ are empty, we have

$$(x,v) \in \{U_1^{\mathbf{s}} < -2\delta\} \cap \{U_{\gamma_1+\nu}^{\mathbf{s}} < -2\delta\} \quad \text{or} \quad (x,v) \in \{U_{\gamma_1+\nu}^{\mathbf{s}} > 2\delta\} \cap \{U_1^{\mathbf{s}} > 2\delta\}. \tag{7.3.16}$$

Besides, using (7.3.2), one can check that the sign of $\partial_\gamma U_\gamma^{\mathbf{s}}(x,v)$ is given by

$$\ell_x^{\mathbf{s}} \cdot \tilde{x}_{\mathbf{s}} - |\ell_x^{\mathbf{s}}|^2 \ell_v^{\mathbf{s}} \cdot v - \left(\ell_x^{\mathbf{s}} \cdot \tilde{x}_{\mathbf{s}} + |\ell_x^{\mathbf{s}}|^2 \ell_v^{\mathbf{s}} \cdot v\right)\gamma \tag{7.3.17}$$

which vanishes at most once in $(\gamma_1+\nu, 1)$. If it does not vanish in $(\gamma_1+\nu, 1)$, then by monotonicity (7.3.16) implies that for any $\gamma \in [\gamma_1+\nu, 1]$, we have $(x,v) \notin \{|U_\gamma^{\mathbf{s}}| \leq 2\delta\}$. Now in the case where the expression from (7.3.17) vanishes at some point in $(\gamma_1+\nu, 1)$, its values at $\gamma_1+\nu$ and 1 have opposite signs, i.e

$$|\ell_x^{\mathbf{s}}|^2 \ell_v^{\mathbf{s}} \cdot v \left((1-\gamma_1-\nu)\ell_x^{\mathbf{s}} \cdot \tilde{x}_{\mathbf{s}} - |\ell_x^{\mathbf{s}}|^2 (1+\gamma_1+\nu)\ell_v^{\mathbf{s}} \cdot v\right) > 0. \tag{7.3.18}$$

When both factors from (7.3.18) are positive, we have $\ell_x^{\mathbf{s}} \cdot \tilde{x}_{\mathbf{s}} > 0$ so $U_1^{\mathbf{s}}(x,v) > 0$ and it follows that $(x,v) \in \{U_{\gamma_1+\nu}^{\mathbf{s}} > 2\delta\} \cap \{U_1^{\mathbf{s}} > 2\delta\}$. Moreover, we also have in that case that the minimum of $\gamma \mapsto U_\gamma^{\mathbf{s}}(x,v)$ on $[\gamma_1+\nu, 1]$ is attained on the boundary of the interval since $\partial_\gamma U_\gamma^{\mathbf{s}}(x,v)|_{\gamma=1} < 0$, so for any $\gamma \in [\gamma_1+\nu, 1]$ it holds $(x,v) \in \{U_\gamma^{\mathbf{s}} > 2\delta\}$. Here again, the same strategy of proof enables to show that if both factors from (7.3.18) are negative, then for any $\gamma \in [\gamma_1+\nu, 1]$, it holds $(x,v) \in \{U_\gamma^{\mathbf{s}} < -2\delta\}$. Combined with (7.3.14) and (7.3.15), this proves (7.3.13).

From (7.3.9), (7.3.10), (7.3.11) and (7.3.13), we see that the only parts on which it is not clear that $\theta_{\nu,h}^{\mathbf{m}}$ is smooth are

$$F_1 = \bigsqcup_{\mathbf{s} \in \mathbf{j}(\mathbf{m})} \left(\{U_1^{\mathbf{s}} = 2\delta\} \cap \{U_{\gamma_1+\nu}^{\mathbf{s}} \geq 2\delta\} \cap B_0(\mathbf{s}, r)\right),$$

$$F_2 = \bigsqcup_{\mathbf{s} \in \mathbf{j}(\mathbf{m})} \left(\{U_1^{\mathbf{s}} \geq 2\delta\} \cap \{U_{\gamma_1+\nu}^{\mathbf{s}} = 2\delta\} \cap B_0(\mathbf{s}, r)\right),$$

$$F_3 = \bigsqcup_{\mathbf{s} \in \mathbf{j}(\mathbf{m})} \left(\{U_1^{\mathbf{s}} = -2\delta\} \cap \{U_{\gamma_1+\nu}^{\mathbf{s}} \leq -2\delta\} \cap B_0(\mathbf{s}, r)\right),$$

$$F_4 = \bigsqcup_{\mathbf{s} \in \mathbf{j}(\mathbf{m})} \left(\{U_1^{\mathbf{s}} \leq -2\delta\} \cap \{U_{\gamma_1+\nu}^{\mathbf{s}} = -2\delta\} \cap B_0(\mathbf{s}, r)\right),$$

$$F_5 = \bigsqcup_{\mathbf{s} \in \mathbf{j}(\mathbf{m})} \left(\bigcup_{\gamma \in [\gamma_1+\nu, 1]} \{|U_\gamma^{\mathbf{s}}| \leq 2\delta\} \cap \partial B_0(\mathbf{s}, r)\right)$$

$$\text{and} \quad F_6 = \partial\Big(E(\mathbf{m}) + B(0,\varepsilon)\Big) \Big\backslash \left(\bigsqcup_{\mathbf{s} \in \mathbf{j}(\mathbf{m})} \left(\bigcup_{\gamma \in [\gamma_1+\nu, 1]} \{|U_\gamma^{\mathbf{s}}| \leq 2\delta\} \cap B_0(\mathbf{s}, r)\right)\right).$$

Note that (7.3.13) suggested to put $\{U_1^{\mathbf{s}} = 2\delta\} \cap \{U_{\gamma_1+\nu}^{\mathbf{s}} \geq -2\delta\} \cap B_0(\mathbf{s}, r)$ in the definition of F_1, but we allowed ourselves to discard the part $\{U_1^{\mathbf{s}} = 2\delta\} \cap \{U_{\gamma_1+\nu}^{\mathbf{s}} \in [-2\delta, 2\delta)\} \cap B_0(\mathbf{s}, r)$ since it is included in the interior of $\{U_1^{\mathbf{s}} \geq -2\delta\} \cap \{U_{\gamma_1+\nu}^{\mathbf{s}} \leq 2\delta\} \cap B_0(\mathbf{s}, r)$ (and we did similarly for F_2, F_3 and F_4).

Now, let $\mathbf{s} \in \mathbf{j}(\mathbf{m})$ and $(\gamma, x, v) \in [\gamma_1+\nu, 1] \times \overline{B_0(\mathbf{s}, r)} \backslash \{(\mathbf{s}, 0)\}$ such that $U_\gamma^{\mathbf{s}}(x,v) = L_\gamma^{\mathbf{s}}(x,v) = 0$. Using Lemma 7.3.1, we see that if $r > 0$ is small enough,

$$W(x,v) = W(x,v) + \frac{1}{2} L_\gamma^{\mathbf{s}}(x,v)^2 > W(\mathbf{s}, 0). \tag{7.3.19}$$

Hence, for all $\gamma \in [\gamma_1+\nu, 1]$, the set $\{U_\gamma^{\mathbf{s}} = 0\} \cap B_0(\mathbf{s}, r)$ is contained in $\{W \geq \boldsymbol{\sigma}(\mathbf{m})\}$. Assume by contradiction that for any $r > 0$, the function $U_\gamma^{\mathbf{s}}$ takes both positive and negative values on $E(\mathbf{m}) \cap B_0(\mathbf{s}, r)$. Then according to Lemma C.0.1, the two CCs of $\mathcal{O}_r \cap \{W < \boldsymbol{\sigma}(\mathbf{m})\}$ are both included in $E(\mathbf{m})$ (the one on which $U_\gamma^{\mathbf{s}} > 0$ and the one where $U_\gamma^{\mathbf{s}} < 0$). This is a contradiction with the fact that $\mathbf{s} \in \mathtt{V}^{(1)}$. Therefore $U_\gamma^{\mathbf{s}}$ has a sign on $E(\mathbf{m}) \cap B_0(\mathbf{s}, r)$ and since it depends smoothly on γ and cannot vanish on $E(\mathbf{m}) \cap B_0(\mathbf{s}, r)$, this sign does not depend on γ. In particular, it is given by the sign of $U_0^{\mathbf{s}}$ on $E(\mathbf{m})\cap B_0(\mathbf{s},r)$ so taking $\ell^{\mathbf{s}}$ such that

$$\ell^{\mathbf{s}}\cdot(\phi_{\mathbf{s}}(x_0),v_0)>0 \tag{7.3.20}$$

for some $(x_0,v_0)\in E(\mathbf{m})\cap B_0(\mathbf{s},r)$, we get that for each $\gamma\in[\gamma_1+\nu,1]$, the function $U_\gamma^{\mathbf{s}}$ is positive on $E(\mathbf{m})\cap B_0(\mathbf{s},r)$. We can then choose $\varepsilon(\delta)>0$ small enough so that

$$\Big(\big(E(\mathbf{m})+B(0,\varepsilon)\big)\cap B_0(\mathbf{s},r)\Big)\subseteq\big\{U_1^{\mathbf{s}}\geq-\delta\big\}\cap\big\{U_{\gamma_1+\nu}^{\mathbf{s}}\geq-\delta\big\}. \tag{7.3.21}$$

Similarly, if we denote $\Omega_{\mathbf{s}}$ the other CC of $\{W<\boldsymbol{\sigma}(\mathbf{m})\}$ which contains $(\mathbf{s},0)$ on its boundary, one can check that $\big(\phi_{\mathbf{s}}^{-1}(-\phi_{\mathbf{s}}(x_0)),-v_0\big)\in\Omega_{\mathbf{s}}\cap B_0(\mathbf{s},r)\cap\{U_0^{\mathbf{s}}<0\}$ where (x_0,v_0) was introduced in (7.3.20) so $U_\gamma^{\mathbf{s}}$ is negative on $\Omega_{\mathbf{s}}\cap B_0(\mathbf{s},r)$ and

$$\Big(\big(\Omega_{\mathbf{s}}+B(0,\varepsilon)\big)\cap B_0(\mathbf{s},r)\Big)\subseteq\big\{U_1^{\mathbf{s}}\leq\delta\big\}\cap\big\{U_{\gamma_1+\nu}^{\mathbf{s}}\leq\delta\big\}. \tag{7.3.22}$$

Choosing once again $\varepsilon(r)$ small enough, we can even assume that

$$\Big(\overline{E(\mathbf{m})+B(0,\varepsilon)}\,\cap\,\overline{\Omega_{\mathbf{s}}+B(0,\varepsilon)}\Big)\subseteq\mathbf{j}^W(\mathbf{m})+B_0(0,r) \tag{7.3.23}$$

(see [36], Lemma 3.2 for more details). We first prove that $\theta_{\nu,h}^{\mathbf{m}}$ is smooth on $F_1\cap(\Omega+B(0,\varepsilon))$: let $\mathbf{s}\in\mathbf{j}(\mathbf{m})$ and $(x,v)\in B_0(\mathbf{s},r)\cap\{U_1^{\mathbf{s}}=2\delta\}\cap\{U_{\gamma_1+\nu}^{\mathbf{s}}\geq 2\delta\}\cap(\Omega+B(0,\varepsilon))$. According to (7.3.22), there exists a small ball B centered in (x,v) such that

$$B\subset\Big(B_0(\mathbf{s},r)\cap\{U_1^{\mathbf{s}}>\delta\}\cap\{U_{\gamma_1+\nu}^{\mathbf{s}}>\delta\}\cap\big(E(\mathbf{m})+B(0,\varepsilon)\big)\Big).$$

Thus, according to (7.3.9), (7.3.10) and (7.3.13) with δ instead of 2δ, we have $\theta_{\nu,h}^{\mathbf{m}}=1$ on B so $\theta_{\nu,h}^{\mathbf{m}}$ is smooth at (x,v). Obviously, the same goes for $F_2\cap(\Omega+B(0,\varepsilon))$ and similarly, for $(x,v)\in(F_3\cup F_4)\cap(\Omega+B(0,\varepsilon))$, we can show that $\theta_{\nu,h}^{\mathbf{m}}=0$ in a neighborhood of (x,v).
Now we show that F_5 does not meet $\Omega+B(0,\varepsilon)$. Recall that Ω denotes the CC of $\{W\leq\boldsymbol{\sigma}(\mathbf{m})\}$ containing $\mathbf{m}$. For $\mathbf{s}\in\mathbf{j}(\mathbf{m})$, we can deduce from (7.3.19) that if $(\gamma,x,v)\in[\gamma_1+\nu,1]\times\partial B_0(\mathbf{s},r)$ is such that $U_\gamma^{\mathbf{s}}(x,v)=0$, then $(x,v)\notin\Omega$. Hence $(\gamma,x,v)\mapsto|U_\gamma^{\mathbf{s}}(x,v)|$ must attain a positive minimum on $[\gamma_1+\nu,1]\times(\partial B_0(\mathbf{s},r)\cap\Omega)$, so we can choose $\delta(r,\nu)>0$ independent of γ such that for all $\gamma\in[\gamma_1+\nu,1]$, the set $\partial B_0(\mathbf{s},r)\cap\{|U_\gamma^{\mathbf{s}}|\leq 2\delta\}$ does not intersect Ω. It follows that we can choose $\varepsilon(\delta)>0$ such that

$$F_5\subseteq\big(\mathbb{R}^{2d}\backslash\overline{\Omega+B(0,\varepsilon)}\big).$$

It only remains to prove that, as for F_5, the set F_6 does not meet $\Omega+B(0,\varepsilon)$. If $(x,v)\in F_6\cap B_0(\mathbf{s},r)$, (7.3.21) and (7.3.13) imply that $(x,v)\in\{U_1^{\mathbf{s}}\geq 2\delta\}\cap\{U_{\gamma_1+\nu}^{\mathbf{s}}\geq 2\delta\}$ so using (7.3.22), we see that (x,v) is outside $\Omega_{\mathbf{s}}+B(0,\varepsilon)$. Since it is not in $(E(\mathbf{m})+B(0,\varepsilon))$ either, it is outside $\Omega+B(0,\varepsilon)$. Now if $(x,v)\in F_6\backslash\big(\mathbf{j}^W(\mathbf{m})+B_0(0,r)\big)$, (7.3.23) implies that (x,v) is outside $\cup_{\mathbf{j}(\mathbf{m})}(\Omega_{\mathbf{s}}+B(0,\varepsilon))$ so it is also outside $\Omega+B(0,\varepsilon)$ for ε small enough and the proof is complete. □

From now on, we fix the sign of $\ell^{\mathbf{s}}$ as well as $\varepsilon>0$ and $\delta>0$ such that the conclusion of Proposition 7.3.2 holds true. In particular, even though we do not make it appear in the notations, the functions $\chi_{\mathbf{m}}$ and ζ now depend on ν. Finally, we denote

$$W^{\mathbf{m}}(x,v)=W(x,v)-V(\mathbf{m})/2 \tag{7.3.24}$$

and it is clear from (7.3.12) that

$$W^{\mathbf{m}}\geq S(\mathbf{m})+\tilde{\varepsilon}\qquad\text{on supp }\nabla\chi_{\mathbf{m}}. \tag{7.3.25}$$

Our quasimodes will be the L^2-renormalizations of the functions

$$f_{\nu,h}^{\mathbf{m}}(x,v)=\chi_{\mathbf{m}}(x,v)\theta_{\nu,h}^{\mathbf{m}}(x,v)\mathrm{e}^{-W^{\mathbf{m}}(x,v)/h}\quad;\quad\mathbf{m}\in\mathtt{U}^{(0)}\backslash\{\underline{\mathbf{m}}\} \tag{7.3.26}$$

and for $\mathbf{m}=\underline{\mathbf{m}}$,

$$f_{\underline{\mathbf{m}},h}(x,v)=\mathrm{e}^{-W^{\underline{\mathbf{m}}}(x,v)/h}\in\operatorname{Ker}P_h.$$

Note that for $\mathbf{m}\neq\underline{\mathbf{m}}$, we have $f_{\nu,h}^{\mathbf{m}}\in\mathcal{C}_c^\infty(\mathbb{R}^{2d})$ thanks to Proposition 7.3.2 and

$$\operatorname{supp}f_{\nu,h}^{\mathbf{m}}\subseteq E(\mathbf{m})+B(0,\varepsilon') \tag{7.3.27}$$

where $\varepsilon'=\max(\varepsilon,r)$.

7.3.2 Action of the operator P_h

Let us fix $\mathbf{m} \in \mathtt{U}^{(0)} \backslash \{\underline{\mathbf{m}}\}$. For $\gamma \in (\gamma_1, 1]$, we will denote

$$\widetilde{W}^{\mathbf{m}}_\gamma(x,v) = W^{\mathbf{m}}(x,v) + \frac{1}{2} \sum_{\mathbf{s} \in \mathbf{j}(\mathbf{m})} L^{\mathbf{s}}_\gamma(x,v)^2. \tag{7.3.28}$$

For $\mathbf{s} \in \mathbf{j}(\mathbf{m})$ and $x \in B(\mathbf{s}, \tilde{r})$ we also denote

$$\tilde{\theta}^{\mathbf{s}}_{\gamma,h}(x,v) = \int_0^{L^{\mathbf{s}}_\gamma(x,v)} \mathrm{e}^{-\frac{s^2}{2h}} \mathrm{d}s. \tag{7.3.29}$$

We now have to compute $P_h f^{\mathbf{m}}_{\nu,h}$. We will see fairly easily thanks to (7.3.35) that X^h_0 applied to $f^{\mathbf{m}}_{\nu,h}$ will yield a superposition of the exponentials

$$\Big(\mathrm{e}^{-\widetilde{W}^{\mathbf{m}}_\gamma/h}\Big)_{\gamma \in [\gamma_1+\nu,1]}. \tag{7.3.30}$$

In view of (7.3.9), we see that the computation of $Q_h f^{\mathbf{m}}_{\nu,h}$ will essentially boil down to the one of $Q_h(\tilde{\theta}^{\mathbf{s}}_{\gamma,h} \mathrm{e}^{-W^{\mathbf{m}}/h})$ which we are already able to do thanks to Lemma 7.2.4 :

$$Q_h(\tilde{\theta}^{\mathbf{s}}_{\gamma,h} \mathrm{e}^{-W^{\mathbf{m}}/h}) = -h \int_0^1 \partial_y \mathscr{L}^{\mathbf{s}}(\gamma,y) \exp\Big[-\frac{1}{h}\Big(W^{\mathbf{m}}(x,v) + \frac{1}{2}\big[\mathscr{L}^{\mathbf{s}}(\gamma,y)\cdot(\tilde{x}_{\mathbf{s}},v)\big]^2\Big)\Big] \mathrm{d}y \cdot \begin{pmatrix} \tilde{x}_{\mathbf{s}} \\ v \end{pmatrix}$$

where $\mathscr{L}^{\mathbf{s}}(\gamma,y)$ stands for the vector

$$\left(\frac{1+y}{\big(4|L^{\mathbf{s}}_{\gamma,v}|^2 y + (y+1)^2\big)^{1/2}} L^{\mathbf{s}}_{\gamma,x} \quad ; \quad \frac{1-y}{\big(4|L^{\mathbf{s}}_{\gamma,v}|^2 y + (y+1)^2\big)^{1/2}} L^{\mathbf{s}}_{\gamma,v} \right). \tag{7.3.31}$$

Here we disregarded the fact that the linear form L_γ is twisted in x as Q_h only acts in v. Our concern is now to see whether the functions

$$\left(\exp\Big[-\frac{1}{h}\Big(W^{\mathbf{m}}(x,v) + \frac{1}{2}\big[\mathscr{L}^{\mathbf{s}}(\gamma,y)\cdot(\tilde{x}_{\mathbf{s}},v)\big]^2\Big)\Big] \right)_{\gamma \in [\gamma_1+\nu,1]\,;\, y \in [0,1]}$$

belong to the family (7.3.30) as we hoped for some compensations between $X^h_0 f^{\mathbf{m}}_{\nu,h}$ and $Q_h f^{\mathbf{m}}_{\nu,h}$. It appears to be the case as, denoting for $\gamma \in (\gamma_1, 1]$ and $y \in [0,1]$

$$\Gamma_\gamma(y) = \frac{y+\gamma}{1+y\gamma}, \tag{7.3.32}$$

an easy computation shows that

$$\mathscr{L}^{\mathbf{s}}(\gamma,y) = \big(L^{\mathbf{s}}_{\Gamma_\gamma(y),x}\,;\, L^{\mathbf{s}}_{\Gamma_\gamma(y),v}\big). \tag{7.3.33}$$

We sum up the above discussion in the following updated version of Lemma 7.2.4.

Lemma 7.3.3. *With the notations* (7.3.29), (7.3.32) *and* (7.3.28), *we have*

$$\mathrm{Op}_h(g)\Big(\mathrm{e}^{-\frac{\widetilde{W}^{\mathbf{m}}_\gamma}{h}} L^{\mathbf{s}}_{\gamma,v}\Big) = Q_h(\tilde{\theta}^{\mathbf{s}}_{\gamma,h} \mathrm{e}^{-W^{\mathbf{m}}/h})(x,v) = -h \int_0^1 \partial_y (L_{\Gamma_\gamma(y)})\, \mathrm{e}^{-\frac{\widetilde{W}^{\mathbf{m}}_{\Gamma_\gamma(y)}}{h}} \mathrm{d}y \cdot \begin{pmatrix} \tilde{x}_{\mathbf{s}} \\ v \end{pmatrix}.$$

Moreover,

$$\mathrm{Op}_h(m_{y,h}\,\mathrm{Id}) \circ b_h\Big(\tilde{\theta}^{\mathbf{s}}_{\gamma,h} \mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}}\Big) = 2h(2\pi h)^{-d/2} \mathrm{e}^{-\frac{V(x)-V(\mathbf{m})}{2h}} \frac{(y+1)^{d-2}}{(4y)^{\frac{d}{2}}} \times \int_{v' \in \mathbb{R}^d} \mathrm{e}^{-\frac{1}{h}\left(\frac{v'^2}{4} + \frac{y}{8}(v+v')^2 + \frac{(v-v')^2}{8y} + \frac{1}{2} L^{\mathbf{s}}_\gamma(x,v')^2\right)} \mathrm{d}v' L^{\mathbf{s}}_{\gamma,v}. \tag{7.3.34}$$

We are now in position to give a precise computation of $P_h f^{\mathbf{m}}_{\nu,h}$.

Proposition 7.3.4. *Let $f^{\mathbf{m}}_{\nu,h}$ be the quasimode defined in (7.3.26) and recall the notations (7.3.7) and (7.3.28). There exist some functions $R^{\mathbf{m}}_{\nu,h}$ and $(\omega^{\mathbf{m}}_{\nu,z})_{z\in[\gamma_1+\nu,1]}$ in $L^2(\mathbb{R}^{2d})$ such that*

a) The function $P_h f^{\mathbf{m}}_{\nu,h} - R^{\mathbf{m}}_{\nu,h}$ is supported in $\mathbf{j}^W(\mathbf{m}) + B_0(0,r)$.

b) The function $R^{\mathbf{m}}_{\nu,h}$ is $O_{\nu,L^2}\Big(h^{\frac{3+d}{2}} \mathrm{e}^{-\frac{S(\mathbf{m})}{h}}\Big)$.

c) For $(x,v)\in \mathbf{j}^W(\mathbf{m}) + B_0(0,r)$, one has

$$(P_h f^{\mathbf{m}}_{\nu,h} - R^{\mathbf{m}}_{\nu,h})(x,v) = \Big(\frac{h}{2\pi}\Big)^{1/2}\int_{\gamma_1+\nu}^{1} \omega^{\mathbf{m}}_{\nu,z}(x,v)\,\exp\Big[-\frac{1}{h}\widetilde{W}^{\mathbf{m}}_z(x,v)\Big]\mathrm{d}z$$

where, using the notation (7.1.11), we have the expression

$$\omega^{\mathbf{m}}_{\nu,z}(x,v) = \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\bigg[k^{\mathbf{s}}_\nu(z)\begin{pmatrix}0 & -\mathcal{V}_{\mathbf{s}}\\ \mathrm{Id} & 0\end{pmatrix}\begin{pmatrix}L^{\mathbf{s}}_{z;x}\\ L^{\mathbf{s}}_{z;v}\end{pmatrix} - \int_{\gamma_1+\nu}^{z} k^{\mathbf{s}}_\nu(\gamma)\,\mathrm{d}\gamma\begin{pmatrix}\partial_z L^{\mathbf{s}}_{z;x}\\ \partial_z L^{\mathbf{s}}_{z;v}\end{pmatrix}\bigg]\cdot\begin{pmatrix}\tilde{x}_{\mathbf{s}}\\ v\end{pmatrix}.$$

Proof. In order to lighten the notations, we will drop some of the exponents and indexes $\mathbf{m}$, $\mathbf{s}$, ν and h in the proof. We know that θ is smooth on the support of χ and since θ is constant outside of $B_0(\mathbf{s},r)$, we have

$$\nabla\theta = \frac{1}{2}\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}(A^{\mathbf{s}}_h)^{-1}\int_{\gamma_1+\nu}^{1} k^{\mathbf{s}}(\gamma)\zeta(U^{\mathbf{s}}_\gamma)\mathrm{e}^{-(L^{\mathbf{s}}_\gamma)^2/2h}\,\nabla L^{\mathbf{s}}_\gamma\,\mathbf{1}_{B_0(\mathbf{s},r)}\,\mathrm{d}\gamma. \tag{7.3.35}$$

Using Corollary 7.2.3, we can then begin by computing

$$\begin{aligned} Q_h(f) &= h\mathrm{Op}_h(g)\big((\partial_v\theta)\chi\mathrm{e}^{-W^{\mathbf{m}}/h} + (\partial_v\chi)\theta\mathrm{e}^{-W^{\mathbf{m}}/h}\big)\\ &= \frac{h}{2}\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}(A^{\mathbf{s}}_h)^{-1}\int_{\gamma_1+\nu}^{1} k^{\mathbf{s}}(\gamma)\mathrm{Op}_h(g)\Big(\chi\zeta(U^{\mathbf{s}}_\gamma)\,\mathrm{e}^{-\frac{\widetilde{W}^{\mathbf{m}}_\gamma}{h}}\mathbf{1}_{B_0(\mathbf{s},r)}L^{\mathbf{s}}_{\gamma,v}\Big)\,\mathrm{d}\gamma + O_\nu\Big(h\mathrm{e}^{-\frac{S(\mathbf{m})+\tilde{\varepsilon}}{h}}\Big) \end{aligned} \tag{7.3.36}$$

as χ now depends on ν, where we used (7.3.25) as well as the fact that $\mathrm{Op}_h(g)$ is bounded uniformly in h since $g\in S^{1/2}(\langle(v,\eta)\rangle^{-1})$. Now, since $\chi\zeta(U^{\mathbf{s}}_\gamma)-1 = O_\nu\big((x-\mathbf{s},v)^2\big)$, we have thanks to Lemma 7.3.1 and by a standard Laplace method that

$$\big(\chi\zeta(U^{\mathbf{s}}_\gamma)-1\big)\mathrm{e}^{-\frac{\widetilde{W}^{\mathbf{m}}_\gamma}{h}}\mathbf{1}_{B_0(\mathbf{s},r)}\nabla L^{\mathbf{s}}_\gamma = O_\nu\Big(h^{1+\frac{d}{2}}\mathrm{e}^{-\frac{S(\mathbf{m})}{h}}\Big). \tag{7.3.37}$$

Hence, still by the boundedness of $\mathrm{Op}_h(g)$, we get that

$$\mathrm{Op}_h(g)\Big(\chi\zeta(U^{\mathbf{s}}_\gamma)\,\mathrm{e}^{-\frac{\widetilde{W}^{\mathbf{m}}_\gamma}{h}}\mathbf{1}_{B_0(\mathbf{s},r)}L^{\mathbf{s}}_{\gamma,v}\Big) = \mathrm{Op}_h(g)\Big(\mathrm{e}^{-\frac{\widetilde{W}^{\mathbf{m}}_\gamma}{h}}\mathbf{1}_{B_0(\mathbf{s},r)}L^{\mathbf{s}}_{\gamma,v}\Big) + O_\nu\Big(h^{1+\frac{d}{2}}\mathrm{e}^{-\frac{S(\mathbf{m})}{h}}\Big). \tag{7.3.38}$$

In the same spirit, we can write

$$\mathbf{1}_{B_0(\mathbf{s},r)}L^{\mathbf{s}}_{\gamma,v} = \mathbf{1}_{|x-\mathbf{s}|<\tilde{r}}(\mathbf{1}_{|v|<r}-1+1)L^{\mathbf{s}}_{\gamma,v} = \mathbf{1}_{|x-\mathbf{s}|<\tilde{r}}\,L^{\mathbf{s}}_{\gamma,v} + \rho_\gamma$$

with ρ_γ supported in $\{(x,v);\, |x-\mathbf{s}|<\tilde{r}$ and $|v|\geq r\}$ and such that $\|\rho_\gamma\|_\infty\leq C_\nu$, so using the boudedness of $\mathrm{Op}_h(g)$ again and the fact that it is local in the variable x, as well as (7.3.1), we get

$$\mathrm{Op}_h(g)\Big(\mathrm{e}^{-\frac{\widetilde{W}^{\mathbf{m}}_\gamma}{h}}\mathbf{1}_{B_0(\mathbf{s},r)}L^{\mathbf{s}}_{\gamma,v}\Big) = \mathrm{Op}_h(g)\Big(\mathrm{e}^{-\frac{\widetilde{W}^{\mathbf{m}}_\gamma}{h}}L^{\mathbf{s}}_{\gamma,v}\Big)\mathbf{1}_{|x-\mathbf{s}|<\tilde{r}} + O_\nu\Big(h^{1+\frac{d}{2}}\mathrm{e}^{-\frac{S(\mathbf{m})}{h}}\Big). \tag{7.3.39}$$

Hence, putting (7.3.38) and (7.3.39) together and using (7.3.8), we get that (7.3.36) becomes

$$Q_h(f) = \Big(\frac{h}{2\pi}\Big)^{1/2}\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\int_{\gamma_1+\nu}^{1} k^{\mathbf{s}}(\gamma)\mathrm{Op}_h(g)\Big(\mathrm{e}^{-\frac{\widetilde{W}^{\mathbf{m}}_\gamma}{h}}L^{\mathbf{s}}_{\gamma,v}\Big)\mathrm{d}\gamma\,\mathbf{1}_{|x-\mathbf{s}|<\tilde{r}} + O_\nu\Big(h^{\frac{3+d}{2}}\mathrm{e}^{-\frac{S(\mathbf{m})}{h}}\Big) \tag{7.3.40}$$

which further gives

$$(7.3.41)\quad Q_h(f)+O_\nu\Big(h^{\frac{3+d}{2}}\mathrm{e}^{-\frac{S(\mathbf{m})}{h}}\Big)=$$
$$-\Big(\frac{h}{2\pi}\Big)^{1/2}\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\int_{\gamma_1+\nu}^{1}k^{\mathbf{s}}(\gamma)\int_0^1\partial_y(L_{\Gamma_\gamma(y)})\,\exp\Big[-\frac{1}{h}\widetilde{W}^{\mathbf{m}}_{\Gamma_\gamma(y)}\Big]\,\mathrm{d}y\cdot\begin{pmatrix}\tilde{x}_{\mathbf{s}}\\ v\end{pmatrix}\mathrm{d}\gamma\,\mathbf{1}_{|x-\mathbf{s}|<\tilde{r}}$$

thanks to Lemma 7.3.3. By the change of variable $z=\Gamma_\gamma(y)$, the integral in y from (7.3.41) becomes

$$\int_\gamma^1\partial_z(L^{\mathbf{s}}_z)\,\exp\Big[-\widetilde{W}^{\mathbf{m}}_z(x,v)/h\Big]\,\mathrm{d}z.$$

Therefore, switching the order of integration and using (7.3.1) again, (7.3.41) yields that up to a $O_{\nu,L^2}(h^{\frac{3+d}{2}}\mathrm{e}^{-S(\mathbf{m})/h})$, the function $Q_h(f)$ satisfies

$$(7.3.42)\quad Q_h(f)(x,v)=-\Big(\frac{h}{2\pi}\Big)^{1/2}\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\int_{\gamma_1+\nu}^{1}\int_{\gamma_1+\nu}^{z}k^{\mathbf{s}}(\gamma)\,\mathrm{d}\gamma\,\partial_z(L^{\mathbf{s}}_z)\cdot\begin{pmatrix}\tilde{x}_{\mathbf{s}}\\ v\end{pmatrix}\mathrm{e}^{-\frac{\widetilde{W}^{\mathbf{m}}_z(x,v)}{h}}\mathrm{d}z\,\mathbf{1}_{B_0(\mathbf{s},r)}(x,v)$$

Now the computation for the transport term is easier : according to (7.3.35), we have

$$\begin{aligned}X^h_0f&=h\begin{pmatrix}v\\-\partial_xV\end{pmatrix}\cdot\nabla f\\&=h\begin{pmatrix}v\\-\partial_xV\end{pmatrix}\cdot\nabla\theta\,\chi\mathrm{e}^{-W^{\mathbf{m}}/h}+h\begin{pmatrix}v\\-\partial_xV\end{pmatrix}\cdot\nabla\chi\,\theta\mathrm{e}^{-W^{\mathbf{m}}/h}\\&=\frac{h}{2}\chi\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}(A^{\mathbf{s}}_h)^{-1}\int_{\gamma_1+\nu}^{1}k^{\mathbf{s}}(z)\zeta(U^{\mathbf{s}}_z)\begin{pmatrix}v\\-\partial_xV\end{pmatrix}\cdot\nabla L^{\mathbf{s}}_z\,\mathrm{e}^{-\frac{\widetilde{W}^{\mathbf{m}}_z}{h}}\mathbf{1}_{B_0(\mathbf{s},r)}\,\mathrm{d}z+O_\nu\Big(h\mathrm{e}^{-\frac{S(\mathbf{m})+\tilde{\varepsilon}}{h}}\Big)\end{aligned}$$

thanks to (7.3.25). Here again, we can use (7.3.8) and (7.3.37) to get

$$X^h_0f=\Big(\frac{h}{2\pi}\Big)^{1/2}\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\int_{\gamma_1+\nu}^{1}k^{\mathbf{s}}(z)\begin{pmatrix}v\\-\partial_xV\end{pmatrix}\cdot\nabla L^{\mathbf{s}}_z\,\mathrm{e}^{-\frac{\widetilde{W}^{\mathbf{m}}_z}{h}}\mathbf{1}_{B_0(\mathbf{s},r)}\,\mathrm{d}z+O_\nu\Big(h^{\frac{3+d}{2}}\mathrm{e}^{-\frac{S(\mathbf{m})}{h}}\Big).$$

Recalling that the differential of $\phi_{\mathbf{s}}$ at $\mathbf{s}$ is the identity, the last step consists in using (7.3.6) to write

$$\begin{aligned}\begin{pmatrix}v\\-\partial_xV\end{pmatrix}\cdot\nabla L^{\mathbf{s}}_z&=\begin{pmatrix}0&\mathrm{Id}\\-\mathcal{V}_{\mathbf{s}}&0\end{pmatrix}\begin{pmatrix}\tilde{x}_{\mathbf{s}}\\ v\end{pmatrix}\cdot\begin{pmatrix}L^{\mathbf{s}}_{z,x}\\ L^{\mathbf{s}}_{z,v}\end{pmatrix}+O_\nu\Big((\tilde{x}_{\mathbf{s}},v)^2\Big)\\&=\begin{pmatrix}0&-\mathcal{V}_{\mathbf{s}}\\ \mathrm{Id}&0\end{pmatrix}\begin{pmatrix}L^{\mathbf{s}}_{z,x}\\ L^{\mathbf{s}}_{z,v}\end{pmatrix}\cdot\begin{pmatrix}\tilde{x}_{\mathbf{s}}\\ v\end{pmatrix}+O_\nu\Big((\tilde{x}_{\mathbf{s}},v)^2\Big)\end{aligned}$$

and the same argument that we used to establish (7.3.37) yields that up to a $O_{\nu,L^2}(h^{\frac{3+d}{2}}\mathrm{e}^{-S(\mathbf{m})/h})$, the function X^h_0f satisfies

$$(7.3.43)\quad X^h_0f(x,v)=\Big(\frac{h}{2\pi}\Big)^{1/2}\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\int_{\gamma_1+\nu}^{1}k^{\mathbf{s}}(z)\begin{pmatrix}0&-\mathcal{V}_{\mathbf{s}}\\ \mathrm{Id}&0\end{pmatrix}\begin{pmatrix}L^{\mathbf{s}}_{z,x}\\ L^{\mathbf{s}}_{z,v}\end{pmatrix}\cdot\begin{pmatrix}\tilde{x}_{\mathbf{s}}\\ v\end{pmatrix}\mathrm{e}^{-\frac{\widetilde{W}^{\mathbf{m}}_z(x,v)}{h}}\mathrm{d}z\mathbf{1}_{B_0(\mathbf{s},r)}(x,v).$$

The conclusion follows from (7.3.42) and (7.3.43). □

Remark 7.3.5. *Since $P^*_h=-X^h_0+Q_h$, it is clear from (7.3.42) and (7.3.43) that*

$$P^*_hf^{\mathbf{m}}_{\nu,h}=\Big(\frac{h}{2\pi}\Big)^{1/2}\int_{\gamma_1+\nu}^{1}\overset{*}{\omega}{}^{\mathbf{m}}_{\nu,z}(x,v)\,\exp\Big[-\frac{1}{h}\widetilde{W}^{\mathbf{m}}_z(x,v)\Big]\mathrm{d}z+O_{\nu,L^2}\Big(h^{\frac{3+d}{2}}\mathrm{e}^{-\frac{S(\mathbf{m})}{h}}\Big)$$

with

$$\overset{*}{\omega}{}^{\mathbf{m}}_{\nu,z}(x,v)=-\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\Bigg[k^{\mathbf{s}}_\nu(z)\begin{pmatrix}0&-\mathcal{V}_{\mathbf{s}}\\ \mathrm{Id}&0\end{pmatrix}\begin{pmatrix}L^{\mathbf{s}}_{z;x}\\ L^{\mathbf{s}}_{z;v}\end{pmatrix}+\int_{\gamma_1+\nu}^{z}k^{\mathbf{s}}_\nu(\gamma)\,\mathrm{d}\gamma\begin{pmatrix}\partial_zL^{\mathbf{s}}_{z;x}\\ \partial_zL^{\mathbf{s}}_{z;v}\end{pmatrix}\Bigg]\cdot\begin{pmatrix}\tilde{x}_{\mathbf{s}}\\ v\end{pmatrix}\mathbf{1}_{\mathbf{j}^W(\mathbf{m})+B_0(0,r)}(x,v).$$

7.3.3 Choices of ℓ and k

Following the steps from [4, 36], we would like in view of Proposition 7.3.4 to find $(\ell^{\mathbf{s}})_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} \subset \mathbb{R}^{2d}$ satisfying (7.3.2) and (7.3.3) as well as some probability densities $(k_\nu^{\mathbf{s}})_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}$ on $[\gamma_1+\nu, 1]$ for which the leading term of $P_h f_{\nu,h}^{\mathbf{m}}$ vanishes, i.e such that

$$k_\nu^{\mathbf{s}}(z)\begin{pmatrix}0 & -\mathcal{V}_{\mathbf{s}}\\ \mathrm{Id} & 0\end{pmatrix}\begin{pmatrix}L_{z;x}^{\mathbf{s}}\\ L_{z;v}^{\mathbf{s}}\end{pmatrix} - \int_{\gamma_1+\nu}^{z} k_\nu^{\mathbf{s}}(\gamma)\,\mathrm{d}\gamma\begin{pmatrix}\partial_z L_{z;x}^{\mathbf{s}}\\ \partial_z L_{z;v}^{\mathbf{s}}\end{pmatrix} = 0, \qquad \forall \mathbf{s}\in\mathbf{j}(\mathbf{m}),\ \forall z\in[\gamma_1+\nu,1]. \tag{7.3.44}$$

As it will be more convenient to handle than the function $k_\nu^{\mathbf{s}}$, let us introduce the cumulative distribution function (CDF) on $[\gamma_1+\nu, 1]$ associated to $k_\nu^{\mathbf{s}}$:

$$K_\nu^{\mathbf{s}}(z) = \int_{\gamma_1+\nu}^{z} k_\nu^{\mathbf{s}}(\gamma)\,\mathrm{d}\gamma. \tag{7.3.45}$$

Lemma 7.3.6. *Recall the notations (7.1.11)-(7.1.12). If $(\ell^{\mathbf{s}})_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}$ is a family of vectors satisfying (7.3.2) and $(k_\nu^{\mathbf{s}})_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}$ is a family of probability densities on $[\gamma_1+\nu, 1]$ for which (7.3.44) holds true, then*

$$\mathcal{V}_{\mathbf{s}}\ell_v^{\mathbf{s}} = \tau_{\mathbf{s}}\ell_v^{\mathbf{s}} \qquad ; \qquad \ell_x^{\mathbf{s}} = -\sqrt{2|\tau_{\mathbf{s}}|}\,\ell_v^{\mathbf{s}} \tag{7.3.46}$$

(in particular, $\ell_x^{\mathbf{s}}$ satisfies (7.3.3)) and the function $K_\nu^{\mathbf{s}}$ defined in (7.3.45) is a CDF on $[\gamma_1+\nu, 1]$ satisfying the ODE

$$(K_\nu^{\mathbf{s}})'(z) - \frac{2\sqrt{2}}{\sqrt{|\tau_{\mathbf{s}}|}P(z)}K_\nu^{\mathbf{s}}(z) = 0. \tag{7.3.47}$$

Proof. Let $(\ell^{\mathbf{s}})_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}$ and $(k_\nu^{\mathbf{s}})_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}$ satisfying the hypotheses of the lemma. According to (7.2.8), (7.3.2) and (7.3.44), we have

$$-k_\nu^{\mathbf{s}}(z)\mathcal{V}_{\mathbf{s}}\ell_v^{\mathbf{s}} + 2\frac{K_\nu^{\mathbf{s}}(z)}{P(z)}\ell_x^{\mathbf{s}} = 0 \qquad \text{and} \qquad k_\nu^{\mathbf{s}}(z)\ell_x^{\mathbf{s}} + 4\frac{K_\nu^{\mathbf{s}}(z)}{P(z)}\ell_v^{\mathbf{s}} = 0 \tag{7.3.48}$$

from which we deduce that there exists $\sigma_{\mathbf{s}} < 0$ such that $\ell_x^{\mathbf{s}} = \sigma_{\mathbf{s}}\ell_v^{\mathbf{s}}$ and consequently, that $\ell_v^{\mathbf{s}}$ is an eigenvector of $\mathcal{V}_{\mathbf{s}}$ associated to its negative eigenvalue $\tau_{\mathbf{s}}$. Plugging these informations in (7.3.48), we obtain

$$|\tau_{\mathbf{s}}|k_\nu^{\mathbf{s}}(z) + 2\sigma_{\mathbf{s}}\frac{K_\nu^{\mathbf{s}}(z)}{P(z)} = 0 \qquad \text{and} \qquad \sigma_{\mathbf{s}}k_\nu^{\mathbf{s}}(z) + 4\frac{K_\nu^{\mathbf{s}}(z)}{P(z)} = 0$$

which yield $\sigma_{\mathbf{s}} = -\sqrt{2|\tau_{\mathbf{s}}|}$ and (7.3.47). ☐

Since the sign of $\ell^{\mathbf{s}}$ was fixed by Proposition 7.3.2 and $|\ell_v^{\mathbf{s}}|^2 = 1$, the choice of $\ell^{\mathbf{s}}$ is entirely determined by (7.3.46). Unfortunately, there is no CDF on $[\gamma_1+\nu, 1]$ satisfying (7.3.47). However, there exists a CDF on the whole segment $(\gamma_1, 1]$ solving (7.3.47), which up to renormalization is given by

$$K_0^{\mathbf{s}}(z) = \left(\frac{z-\gamma_1}{z-\gamma_2}\right)^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}} \qquad \text{i.e} \qquad k_0^{\mathbf{s}}(z) = \frac{\gamma_1-\gamma_2}{2\sqrt{|\tau_{\mathbf{s}}|}(z-\gamma_2)^2}\left(\frac{z-\gamma_1}{z-\gamma_2}\right)^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}-1}. \tag{7.3.49}$$

This leads to the introduction of the following CDF on $[\gamma_1+\nu, 1]$ which will be an approximate solution of (7.3.47) :

$$K_\nu^{\mathbf{s}}(z) = \frac{K_0^{\mathbf{s}}(z) - K_0^{\mathbf{s}}(\gamma_1+\nu)}{B_\nu^{\mathbf{s}}} \qquad \text{and} \qquad k_\nu^{\mathbf{s}}(z) = \big(K_\nu^{\mathbf{s}}\big)'(z) = \frac{k_0^{\mathbf{s}}(z)}{B_\nu^{\mathbf{s}}} \tag{7.3.50}$$

where

$$B_\nu^{\mathbf{s}} = K_0^{\mathbf{s}}(1) - K_0^{\mathbf{s}}(\gamma_1+\nu) = K_0^{\mathbf{s}}(1) + O\Big(\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}}\Big). \tag{7.3.51}$$

Lemma 7.3.7. *Recall the notation* (7.1.12) *and let* $(\ell^{\mathbf{s}})_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}$ *a family of vectors satisfying* (7.3.2), (7.3.46) *and whose signs are fixed by Proposition 7.3.2. Let also* $(k^{\mathbf{s}}_\nu)_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}$ *the probability densities on* $[\gamma_1+\nu,1]$ *defined in* (7.3.50). *Then for all* $\mathbf{s}\in\mathbf{j}(\mathbf{m})$ *and* $(x,v)\in B_0(\mathbf{s},r)$, *the prefactor from Proposition 7.3.4 satisfies*

$$\omega^{\mathbf{m}}_{\nu,z}(x,v)=O\Big(\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}}\Big)\begin{pmatrix}\partial_z L^{\mathbf{s}}_{z;x}\\ \partial_z L^{\mathbf{s}}_{z;v}\end{pmatrix}\cdot\begin{pmatrix}\tilde{x}_{\mathbf{s}}\\ v\end{pmatrix}.$$

Proof. By some computations similar to the ones we made in the proof of Lemma 7.3.6, we get that the choice of $(\ell^{\mathbf{s}})_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}$ implies that

$$\omega^{\mathbf{m}}_{\nu,z}(x,v)=\frac{\sqrt{|\tau_{\mathbf{s}}|}P(z)}{2\sqrt{2}}\left[k^{\mathbf{s}}_\nu(z)-\frac{2\sqrt{2}}{\sqrt{|\tau_{\mathbf{s}}|}P(z)}K_\nu(z)\right]\begin{pmatrix}\partial_z L^{\mathbf{s}}_{z;x}\\ \partial_z L^{\mathbf{s}}_{z;v}\end{pmatrix}\cdot\begin{pmatrix}\tilde{x}_{\mathbf{s}}\\ v\end{pmatrix}.$$

The term between brackets is exactly the one appearing in (7.3.47) so using (7.3.50) and the fact that K_0 is a solution of (7.3.47), we get

$$\omega^{\mathbf{m}}_{\nu,z}(x,v)=\frac{K^{\mathbf{s}}_0(\gamma_1+\nu)}{B^{\mathbf{s}}_\nu}\begin{pmatrix}\partial_z L^{\mathbf{s}}_{z;x}\\ \partial_z L^{\mathbf{s}}_{z;v}\end{pmatrix}\cdot\begin{pmatrix}\tilde{x}_{\mathbf{s}}\\ v\end{pmatrix}=O\Big(\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}}\Big)\begin{pmatrix}\partial_z L^{\mathbf{s}}_{z;x}\\ \partial_z L^{\mathbf{s}}_{z;v}\end{pmatrix}\cdot\begin{pmatrix}\tilde{x}_{\mathbf{s}}\\ v\end{pmatrix}$$

by (7.3.51) and the definition of $K^{\mathbf{s}}_0$. ☐

Proposition 7.3.8. *Recall the notation* (7.1.12) *and let* $f^{\mathbf{m}}_{\nu,h}$ *be the quasimode defined in* (7.3.26) *with* $(\ell^{\mathbf{s}})_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}$ *and* $(k^{\mathbf{s}}_\nu)_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}$ *satisfying the hypotheses from Lemma 7.3.7. Then*

$$\|P_h f^{\mathbf{m}}_{\nu,h}\|=h\,\mathrm{e}^{-\frac{S(\mathbf{m})}{h}}\|f^{\mathbf{m}}_{\nu,h}\|\bigg(O_\nu\Big(h^{\frac12}\Big)+O\Big(\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}}|\ln(\nu)|\Big)\bigg).$$

Proof. First notice that thanks to item a) from Hypothesis 7.2.5, one can apply a standard Laplace method to obtain with the notation (7.1.11)

$$\|f^{\mathbf{m}}_{\nu,h}\|^2=\frac{(2\pi h)^d}{\det(\mathcal{V}_{\mathbf{m}})^{1/2}}\Big(1+O(h)\Big).\tag{7.3.52}$$

Hence, according to Proposition 7.3.4, it is sufficient to show that

$$\big\|P_h f^{\mathbf{m}}_{\nu,h}-R^{\mathbf{m}}_{\nu,h}\big\|=h^{1+\frac{d}{2}}\,\mathrm{e}^{-\frac{S(\mathbf{m})}{h}}O\Big(\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}}|\ln(\nu)|\Big).\tag{7.3.53}$$

Now, still using Proposition 7.3.4 as well as Minkowski's integral inequality and Lemma 7.3.7, we have

$$\begin{aligned}\big\|P_h f^{\mathbf{m}}_{\nu,h}-R^{\mathbf{m}}_{\nu,h}\big\|&\leq Ch^{1/2}\int_{\gamma_1+\nu}^1\bigg(\int_{\mathbf{j}^W(\mathbf{m})+B_0(0,r)}\omega^{\mathbf{m}}_{\nu,z}(x,v)^2\exp\Big[-\frac{2}{h}\widetilde{W}^{\mathbf{m}}_z(x,v)\Big]\mathrm{d}(x,v)\bigg)^{1/2}\mathrm{d}z\\&\leq Ch^{1/2}\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}}\int_{\gamma_1+\nu}^1\bigg(\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\int_{B_0(\mathbf{s},r)}\begin{pmatrix}\partial_z L^{\mathbf{s}}_{z;x}\\ \partial_z L^{\mathbf{s}}_{z;v}\end{pmatrix}\begin{pmatrix}\partial_z L^{\mathbf{s}}_{z;x}\\ \partial_z L^{\mathbf{s}}_{z;v}\end{pmatrix}^t\begin{pmatrix}\tilde{x}_{\mathbf{s}}\\ v\end{pmatrix}\cdot\begin{pmatrix}\tilde{x}_{\mathbf{s}}\\ v\end{pmatrix}\\&\qquad\qquad\times\exp\Big[-\frac{2}{h}\widetilde{W}^{\mathbf{m}}_z(x,v)\Big]\mathrm{d}(x,v)\bigg)^{1/2}\mathrm{d}z.\end{aligned}$$

With the notation (7.1.11), the change of variables

$$(y,w)=\Big(\frac{2}{h}\Big)^{1/2}\left[\mathcal{W}_{\mathbf{s}}+\begin{pmatrix}L^{\mathbf{s}}_{z;x}\\ L^{\mathbf{s}}_{z;v}\end{pmatrix}\begin{pmatrix}L^{\mathbf{s}}_{z;x}\\ L^{\mathbf{s}}_{z;v}\end{pmatrix}^t\right]^{1/2}(\tilde{x}_{\mathbf{s}},v)$$

then yields according to Lemma 7.3.1

$$\begin{aligned}\big\|P_h f^{\mathbf{m}}_{\nu,h}-R^{\mathbf{m}}_{\nu,h}\big\|&\leq Ch^{1+\frac{d}{2}}\mathrm{e}^{-\frac{S(\mathbf{m})}{h}}\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}}\\&\quad\times\int_{\gamma_1+\nu}^1\bigg(\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\int_{\mathbb{R}^{2d}}a_z a_z^t\begin{pmatrix}y\\ w\end{pmatrix}\cdot\begin{pmatrix}y\\ w\end{pmatrix}\mathrm{e}^{-\frac{(y,w)^2}{2}}\mathrm{d}(y,w)\bigg)^{1/2}\mathrm{d}z\end{aligned}\tag{7.3.54}$$

where
$$a_z = \left[\mathcal{W}_{\mathbf{s}} + \begin{pmatrix} L^{\mathbf{s}}_{z;x} \\ L^{\mathbf{s}}_{z;v} \end{pmatrix} \begin{pmatrix} L^{\mathbf{s}}_{z;x} \\ L^{\mathbf{s}}_{z;v} \end{pmatrix}^t\right]^{-1/2} \begin{pmatrix} \partial_z L^{\mathbf{s}}_{z;x} \\ \partial_z L^{\mathbf{s}}_{z;v} \end{pmatrix}.$$
Thanks to Proposition 7.6.4, we know that

(7.3.55)
$$(2\pi)^{-d}\int_{\mathbb{R}^{2d}} a_z a_z^t \begin{pmatrix} y \\ w \end{pmatrix} \cdot \begin{pmatrix} y \\ w \end{pmatrix} \mathrm{e}^{-\frac{(y,w)^2}{2}} \mathrm{d}(y,w) = |a_z|^2 = \left[\mathcal{W}_{\mathbf{s}} + \begin{pmatrix} L^{\mathbf{s}}_{z;x} \\ L^{\mathbf{s}}_{z;v} \end{pmatrix} \begin{pmatrix} L^{\mathbf{s}}_{z;x} \\ L^{\mathbf{s}}_{z;v} \end{pmatrix}^t\right]^{-1} \begin{pmatrix} \partial_z L^{\mathbf{s}}_{z;x} \\ \partial_z L^{\mathbf{s}}_{z;v} \end{pmatrix} \cdot \begin{pmatrix} \partial_z L^{\mathbf{s}}_{z;x} \\ \partial_z L^{\mathbf{s}}_{z;v} \end{pmatrix}.$$
Since by (7.2.8)
$$\left[\mathcal{W}_{\mathbf{s}} + \begin{pmatrix} L^{\mathbf{s}}_{z;x} \\ L^{\mathbf{s}}_{z;v} \end{pmatrix} \begin{pmatrix} L^{\mathbf{s}}_{z;x} \\ L^{\mathbf{s}}_{z;v} \end{pmatrix}^t\right]^{-1} \begin{pmatrix} \partial_z L^{\mathbf{s}}_{z;x} \\ \partial_z L^{\mathbf{s}}_{z;v} \end{pmatrix} = \frac{-8}{P(z)^{3/2}} \begin{pmatrix} \big(2|\tau_{\mathbf{s}}|\big)^{-1/2}(1-z)\,\ell^{\mathbf{s}}_v \\ (1+z)\,\ell^{\mathbf{s}}_v \end{pmatrix},$$
we get

(7.3.56)
$$|a_z|^2 = \frac{16}{P(z)^3}\Big(2(1+z)^2 - (1-z)^2\Big) = \frac{16}{P(z)^2}.$$

Putting together (7.3.54), (7.3.55), (7.3.56) and computing the integral in z, we obtain (7.3.53) so the proof is complete. □

7.4 Computation of the approximated small eigenvalues

Let us denote

(7.4.1)
$$\tilde f^{\mathbf{m}}_{\nu,h} = \frac{f^{\mathbf{m}}_{\nu,h}}{\|f^{\mathbf{m}}_{\nu,h}\|}$$

the renormalization of the quasimodes defined in (7.3.26) and satisfying the hypotheses of Proposition 7.3.8. The goal of this section is to compute the *approximated eigenvalues*

(7.4.2)
$$\tilde\lambda^{\mathbf{m}}_{\nu,h} := \langle P_h \tilde f^{\mathbf{m}}_{\nu,h}, \tilde f^{\mathbf{m}}_{\nu,h}\rangle = \langle Q_h \tilde f^{\mathbf{m}}_{\nu,h}, \tilde f^{\mathbf{m}}_{\nu,h}\rangle$$

as X_0^h is a skew-adjoint differential operator and $\tilde f^{\mathbf{m}}_{\nu,h}$ is real valued.
This will require to study the matrix

(7.4.3)
$$H^{\mathbf{s}}_\gamma = \begin{pmatrix} \mathcal{V}_{\mathbf{s}} + 2L_{\gamma,x}L^t_{\gamma,x} & L_{\gamma,x}L^t_{\gamma,v} & L_{\gamma,x}L^t_{\gamma,v} \\ L_{\gamma,v}L^t_{\gamma,x} & \frac12 + L_{\gamma,v}L^t_{\gamma,v} & 0 \\ L_{\gamma,v}L^t_{\gamma,x} & 0 & \frac12 + L_{\gamma,v}L^t_{\gamma,v} \end{pmatrix}$$

where we used the notation (7.1.11) and for shortness, we wrote $L_{\gamma,x}$ and $L_{\gamma,v}$ instead of $L^{\mathbf{s}}_{\gamma,x}$ and $L^{\mathbf{s}}_{\gamma,v}$.

Lemma 7.4.1. *For $\gamma \in [\gamma_1+\nu, 1]$, the matrix $H^{\mathbf{s}}_\gamma$ is positive definite.*

Proof. It suffices to notice that
$$H^{\mathbf{s}}_\gamma \begin{pmatrix} x \\ v \\ v' \end{pmatrix} \cdot \begin{pmatrix} x \\ v \\ v' \end{pmatrix} = \left[\mathcal{W}_{\mathbf{s}} + \begin{pmatrix} L^{\mathbf{s}}_{\gamma;x} \\ L^{\mathbf{s}}_{\gamma;v} \end{pmatrix} \begin{pmatrix} L^{\mathbf{s}}_{\gamma;x} \\ L^{\mathbf{s}}_{\gamma;v} \end{pmatrix}^t\right] \begin{pmatrix} x \\ v \end{pmatrix} \cdot \begin{pmatrix} x \\ v \end{pmatrix} + \left[\mathcal{W}_{\mathbf{s}} + \begin{pmatrix} L^{\mathbf{s}}_{\gamma;x} \\ L^{\mathbf{s}}_{\gamma;v} \end{pmatrix} \begin{pmatrix} L^{\mathbf{s}}_{\gamma;x} \\ L^{\mathbf{s}}_{\gamma;v} \end{pmatrix}^t\right] \begin{pmatrix} x \\ v' \end{pmatrix} \cdot \begin{pmatrix} x \\ v' \end{pmatrix}$$
and apply Lemma 7.3.1. □

In the spirit of Proposition 7.2.2 and with the notation (7.2.3), let us denote

(7.4.4)
$$Q_{y,h} = b_h^* \circ \mathrm{Op}_h(m_{y,h}\,\mathrm{Id}) \circ b_h.$$

For $\mathbf{m} \in \mathtt{U}^{(0)}\backslash\{\underline{\mathbf{m}}\}$, $\mathbf{s} \in \mathbf{j}(\mathbf{m})$, we also denote $\langle\cdot,\cdot\rangle_{\tilde r}$ the inner product on $L^2\big(B(\mathbf{s},\tilde r)\times\mathbb{R}^d_v\big)$.

Lemma 7.4.2. *Let* $\mathbf{s} \in \mathbf{j}(\mathbf{m})$ *for some* $\mathbf{m} \in \mathtt{U}^{(0)} \backslash \{\underline{\mathbf{m}}\}$ *and recall the notations* (7.1.11), (7.3.24) *and* (7.3.29). *Then for all* $\gamma \in [\gamma_1 + \nu, 1]$ *and* $y \in (0,1)$,

$$\Big\langle Q_{y,h}\Big(\tilde{\theta}^{\mathbf{s}}_{\gamma,h} \mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}}\Big), \tilde{\theta}^{\mathbf{s}}_{\gamma,h} \mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}}\Big\rangle_{\tilde{r}} = 2h^2 \mathrm{e}^{\frac{-2S(\mathbf{m})}{h}} (2\pi h)^d |\det \mathcal{V}_{\mathbf{s}}|^{-1/2} \frac{1}{(1+y)\Big(1 + \big(1 + 2|L^{\mathbf{s}}_{\gamma,v}|^2\big)y\Big)} |L^{\mathbf{s}}_{\gamma,v}|^2.$$

Proof. First, let us use the definition of $Q_{y,h}$ to write

$$\Big\langle Q_{y,h}\Big(\tilde{\theta}^{\mathbf{s}}_{\gamma,h} \mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}}\Big), \tilde{\theta}^{\mathbf{s}}_{\gamma,h} \mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}}\Big\rangle_{\tilde{r}} = \Big\langle \mathrm{Op}_h(m_{y,h}\,\mathrm{Id}) \circ b_h\Big(\tilde{\theta}^{\mathbf{s}}_{\gamma,h} \mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}}\Big), b_h\Big(\tilde{\theta}^{\mathbf{s}}_{\gamma,h} \mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}}\Big)\Big\rangle_{\tilde{r}}.$$

Using (7.3.34), we get

$$\Big\langle Q_{y,h}\Big(\tilde{\theta}^{\mathbf{s}}_{\gamma,h} \mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}}\Big), \tilde{\theta}^{\mathbf{s}}_{\gamma,h} \mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}}\Big\rangle_{\tilde{r}} = 2h^2 (2\pi h)^{-d/2} \frac{(y+1)^{d-2}}{(4y)^{\frac{d}{2}}} \mathrm{e}^{\frac{V(\mathbf{m})}{h}} |L^{\mathbf{s}}_{\gamma,v}|^2 \times \tag{7.4.5}$$

$$\int_{|x-\mathbf{s}|<\tilde{r},\, v,v' \in \mathbb{R}^d} \exp\Big[-\frac{1}{h}\Big(V(x) + \frac{v^2 + v'^2}{4} + \frac{y}{8}(v+v')^2 + \frac{(v-v')^2}{8y} + \frac{L^{\mathbf{s}}_{\gamma}(x,v)^2 + L^{\mathbf{s}}_{\gamma}(x,v')^2}{2}\Big)\Big] \mathrm{d}x \mathrm{d}v \mathrm{d}v'.$$

By the change of variables $x' = \phi_{\mathbf{s}}(x)$ and with the notation $\boldsymbol{\sigma}(\mathbf{m})$ from Definition C.0.7, the last integral becomes

$$\mathrm{e}^{\frac{-2\boldsymbol{\sigma}(\mathbf{m})}{h}} \int_{|\phi_{\mathbf{s}}^{-1}(x')-\mathbf{s}|<\tilde{r},\, v,v' \in \mathbb{R}^d} \exp\Big[-\frac{1}{2h} H^{\mathbf{s}}_{\gamma,y} \begin{pmatrix} x' \\ v \\ v' \end{pmatrix} \cdot \begin{pmatrix} x' \\ v \\ v' \end{pmatrix}\Big] |\det D_{x'}\phi_{\mathbf{s}}^{-1}| \,\mathrm{d}x' \mathrm{d}v \mathrm{d}v' \tag{7.4.6}$$

where using the notation (7.4.3),

$$H^{\mathbf{s}}_{\gamma,y} = \begin{pmatrix} \mathcal{V}_{\mathbf{s}} + 2L_{\gamma,x}L^t_{\gamma,x} & L_{\gamma,x}L^t_{\gamma,v} & L_{\gamma,x}L^t_{\gamma,v} \\ L_{\gamma,v}L^t_{\gamma,x} & \frac{(y+1)^2}{4y} + L_{\gamma,v}L^t_{\gamma,v} & \frac{y^2-1}{4y} \\ L_{\gamma,v}L^t_{\gamma,x} & \frac{y^2-1}{4y} & \frac{(y+1)^2}{4y} + L_{\gamma,v}L^t_{\gamma,v} \end{pmatrix} = H^{\mathbf{s}}_{\gamma} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{y^2+1}{4y} & \frac{y^2-1}{4y} \\ 0 & \frac{y^2-1}{4y} & \frac{y^2+1}{4y} \end{pmatrix} \tag{7.4.7}$$

is a positive-definite matrix uniformly in $(\gamma, y) \in [\gamma_1 + \nu, 1] \times (0,1)$ thanks to Lemma 7.4.1. Hence, $(H^{\mathbf{s}}_{\gamma,y})^{-1/2}$ exists and is $O_\nu(1)$ so by a standard Laplace method,

$$\int_{|\phi_{\mathbf{s}}^{-1}(x')-\mathbf{s}|<\tilde{r},\, v,v' \in \mathbb{R}^d} \exp\Big[-\frac{1}{2h} H^{\mathbf{s}}_{\gamma,y} \begin{pmatrix} x' \\ v \\ v' \end{pmatrix} \cdot \begin{pmatrix} x' \\ v \\ v' \end{pmatrix}\Big] |\det D_{x'}\phi_{\mathbf{s}}^{-1}| \,\mathrm{d}x' \mathrm{d}v \mathrm{d}v' = (2\pi h)^{3d/2} \det(H^{\mathbf{s}}_{\gamma,y})^{-1/2} \times \big(1 + O_\nu(h)\big)$$

$$= (2\pi h)^{3d/2} |\det \mathcal{V}_{\mathbf{s}}|^{-1/2} \frac{(4y)^{d/2}}{(1+y)^{d-1}\Big(1 + \big(1 + 2|L^{\mathbf{s}}_{\gamma,v}|^2\big)y\Big)} \big(1 + O_\nu(h)\big) \tag{7.4.8}$$

where we also used Lemma 7.6.3. The conclusion then follows from (7.4.5), (7.4.6) and (7.4.8). ☐

Lemma 7.4.3. *Recall the notation* (7.3.32) *and let* $\gamma_1 + \nu \leq z \leq \gamma < 1$. *For* $y \in [0,1)$, *we have*

$$\Gamma_z^{-1} \circ \Gamma_\gamma(y) \in [0,1)$$

and

$$Q_{y,h}\Big(\tilde{\theta}^{\mathbf{s}}_{\gamma,h} \mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}}\Big) = (\Gamma_z^{-1} \circ \Gamma_\gamma)'(y)\, Q_{\Gamma_z^{-1} \circ \Gamma_\gamma(y),h}\Big(\tilde{\theta}^{\mathbf{s}}_{z,h} \mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}}\Big)$$

on $B(\mathbf{s}, \tilde{r}) \times \mathbb{R}^d_v$.

Proof. First, notice that for all $\gamma \in [\gamma_1 + \nu, 1)$, the function $\Gamma_\gamma : [0,1) \to [\gamma, 1)$ is an increasing bijection whose inverse is given by

$$\Gamma_\gamma^{-1}(y) = \frac{y - \gamma}{1 - y\gamma} \tag{7.4.9}$$

so the first assertion follows from the hypothesis on z and γ. Now, by Lemma 7.3.3 applied with $Q_{y,h}$ instead of Q_h, we get using the notation (7.3.31) as well as (7.3.33) that on $B(\mathbf{s},\tilde r)\times\mathbb{R}^d_v$,

$$Q_{y,h}\Big(\tilde\theta^{\mathbf{s}}_{\gamma,h}\mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}}\Big)=-h\,\partial_y\mathscr{L}(\gamma,y)\,\mathrm{e}^{-\frac{\widetilde W_{\Gamma_\gamma(y)}(x,v)}{h}}\cdot\begin{pmatrix}\tilde x_{\mathbf{s}}\\ v\end{pmatrix}\tag{7.4.10}$$

(here we once again disregarded the fact that the linear form L_γ is twisted in x as $Q_{y,h}$ only acts in v). Thus, denoting $\partial_2\mathscr{L}(\gamma,\cdot)$ the derivative of $\mathscr{L}$ w.r.t its second argument and still using (7.3.33), we also have

$$\begin{aligned}Q_{y,h}\Big(\tilde\theta^{\mathbf{s}}_{\gamma,h}\mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}}\Big)&=-h\,\partial_y(L_{\Gamma_\gamma(y)})\,\mathrm{e}^{-\frac{\widetilde W_{\Gamma_\gamma(y)}(x,v)}{h}}\cdot\begin{pmatrix}\tilde x_{\mathbf{s}}\\ v\end{pmatrix}\\&=-h\,\partial_y\Big(\mathscr{L}\big(z,\Gamma_z^{-1}\circ\Gamma_\gamma(y)\big)\Big)\,\mathrm{e}^{-\frac{\widetilde W_{\Gamma_\gamma(y)}(x,v)}{h}}\cdot\begin{pmatrix}\tilde x_{\mathbf{s}}\\ v\end{pmatrix}\\&=-h\,(\Gamma_z^{-1}\circ\Gamma_\gamma)'(y)\,\partial_2\mathscr{L}\big(z,\Gamma_z^{-1}\circ\Gamma_\gamma(y)\big)\,\mathrm{e}^{-\frac{\widetilde W_{\Gamma_\gamma(y)}(x,v)}{h}}\cdot\begin{pmatrix}\tilde x_{\mathbf{s}}\\ v\end{pmatrix}\end{aligned}$$

so (7.4.10) with $Q_{\Gamma_z^{-1}\circ\Gamma_\gamma(y),h}$ and $\tilde\theta^{\mathbf{s}}_{z,h}$ yields the last statement. □

Proposition 7.4.4. *With the notations (7.1.11), (7.1.12), (7.3.49) and (7.4.2), we have for* $\mathbf{m}\in\mathtt{U}^{(0)}\backslash\{\underline{\mathbf{m}}\}$

$$\tilde\lambda^{\mathbf{m}}_{\nu,h}=h\,\tilde\varrho_{\nu,h}(\mathbf{m})\,\mathrm{e}^{\frac{-2S(\mathbf{m})}{h}}$$

with

$$\begin{aligned}\tilde\varrho_{\nu,h}(\mathbf{m})=&\frac{1}{\pi}\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\Big(\frac{2+\sqrt2}{2-\sqrt2}\Big)^{\frac{1}{\sqrt{|\tau_{\mathbf{s}}|}}}\Big(\frac{\det\mathcal{V}_{\mathbf{m}}}{|\det\mathcal{V}_{\mathbf{s}}|}\Big)^{1/2}\int_{\gamma_1\le z\le\gamma<1}k_0^{\mathbf{s}}(\gamma)k_0^{\mathbf{s}}(z)\ln\Big(2\,\frac{(1+z)(1+\gamma)}{1+3z+3\gamma+z\gamma}\Big)\,\mathrm{d}z\mathrm{d}\gamma\\&+O_\nu(h)+O\Big(\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}}\Big).\end{aligned}$$

Proof. As we mentionned at the begining of the section, since X_0^h is a skew-adjoint differential operator and $\tilde f^{\mathbf{m}}_{\nu,h}$ is real valued, we have

$$\langle P_h\tilde f^{\mathbf{m}}_{\nu,h},\tilde f^{\mathbf{m}}_{\nu,h}\rangle=\langle Q_h\tilde f^{\mathbf{m}}_{\nu,h},\tilde f^{\mathbf{m}}_{\nu,h}\rangle.$$

Now by Proposition 7.2.2, we get

$$\langle Q_hf^{\mathbf{m}}_{\nu,h},f^{\mathbf{m}}_{\nu,h}\rangle=\big\langle\mathrm{Op}_h(m_h\,\mathrm{Id})\big(b_hf^{\mathbf{m}}_{\nu,h}\big),b_hf^{\mathbf{m}}_{\nu,h}\big\rangle\tag{7.4.11}$$

and we saw through (7.3.36)-(7.3.40) that

$$b_hf^{\mathbf{m}}_{\nu,h}=\Big(\frac{h}{2\pi}\Big)^{1/2}\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\int_{\gamma_1+\nu}^1k_\nu^{\mathbf{s}}(\gamma)\mathrm{e}^{-\frac{\widetilde W^{\mathbf{m}}_\gamma}{h}}L^{\mathbf{s}}_{\gamma,v}\mathrm{d}\gamma\,\mathbf{1}_{|x-\mathbf{s}|<\tilde r}+O_\nu\Big(h^{\frac{3+d}{2}}\mathrm{e}^{-\frac{S(\mathbf{m})}{h}}\Big)\tag{7.4.12}$$

$$=(2\pi h)^{-1/2}\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\int_{\gamma_1+\nu}^1k_\nu^{\mathbf{s}}(\gamma)b_h\Big(\tilde\theta_{\gamma,h}\mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}}\Big)\mathrm{d}\gamma\,\mathbf{1}_{|x-\mathbf{s}|<\tilde r}+O_\nu\Big(h^{\frac{3+d}{2}}\mathrm{e}^{-\frac{S(\mathbf{m})}{h}}\Big)\tag{7.4.13}$$

Note that (7.4.12) also implies

$$b_hf^{\mathbf{m}}_{\nu,h}=O_\nu\Big(h^{\frac{1+d}{2}}\mathrm{e}^{-\frac{S(\mathbf{m})}{h}}\Big).\tag{7.4.14}$$

Combining the boundedness of $\mathrm{Op}_h(m_h\,\mathrm{Id})$ with (7.4.13)-(7.4.14) and using the notation (7.4.4), (7.4.11) becomes

$$\begin{aligned}\langle Q_h f^{\mathbf{m}}_{\nu,h}, f^{\mathbf{m}}_{\nu,h}\rangle &= (2\pi h)^{-1}\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\int_{[\gamma_1+\nu,1]^2} k^{\mathbf{s}}_\nu(\gamma)k^{\mathbf{s}}_\nu(z)\Big\langle Q_h\Big(\tilde\theta_{\gamma,h}\mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}}\Big),\tilde\theta_{z,h}\mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}}\Big\rangle_{\tilde r}\mathrm{d}\gamma\mathrm{d}z\\ &\quad+O_\nu\Big(h^{d+2}\mathrm{e}^{-\frac{2S(\mathbf{m})}{h}}\Big)\\ &= (2\pi h)^{-1}\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\int_0^1\int_{[\gamma_1+\nu,1]^2} k^{\mathbf{s}}_\nu(\gamma)k^{\mathbf{s}}_\nu(z)\Big\langle Q_{y,h}\Big(\tilde\theta_{\gamma,h}\mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}}\Big),\tilde\theta_{z,h}\mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}}\Big\rangle_{\tilde r}\mathrm{d}\gamma\mathrm{d}z\mathrm{d}y\\ &\quad+O_\nu\Big(h^{d+2}\mathrm{e}^{-\frac{2S(\mathbf{m})}{h}}\Big)\\ &= 2(2\pi h)^{-1}\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\int_0^1\int_{\gamma_1+\nu\le z\le\gamma<1} k^{\mathbf{s}}_\nu(\gamma)k^{\mathbf{s}}_\nu(z)\Big\langle Q_{y,h}\Big(\tilde\theta_{\gamma,h}\mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}}\Big),\tilde\theta_{z,h}\mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}}\Big\rangle_{\tilde r}\mathrm{d}z\mathrm{d}\gamma\mathrm{d}y\\ &\quad+O_\nu\Big(h^{d+2}\mathrm{e}^{-\frac{2S(\mathbf{m})}{h}}\Big)\end{aligned}\tag{7.4.15}$$

where for the last equation we used the fact that $Q_{y,h}$ is self-adjoint. Applying Lemma 7.4.3 together with the change of variables $\tilde y = \Gamma_z^{-1}\circ\Gamma_\gamma(y)$, we get that (7.4.15) yields

$$\begin{aligned}&\langle Q_h f^{\mathbf{m}}_{\nu,h}, f^{\mathbf{m}}_{\nu,h}\rangle + O_\nu\Big(h^{d+2}\mathrm{e}^{-\frac{2S(\mathbf{m})}{h}}\Big) =\\ &2(2\pi h)^{-1}\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\int_{\gamma_1+\nu\le z\le\gamma<1}\int_{\Gamma_z^{-1}(\gamma)}^1 k^{\mathbf{s}}_\nu(\gamma)k^{\mathbf{s}}_\nu(z)\Big\langle Q_{\tilde y,h}\Big(\tilde\theta_{z,h}\mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}}\Big),\tilde\theta_{z,h}\mathrm{e}^{-\frac{W^{\mathbf{m}}(x,v)}{h}}\Big\rangle_{\tilde r}\mathrm{d}\tilde y\mathrm{d}z\mathrm{d}\gamma\end{aligned}$$

which by Lemma 7.4.2 is further equal to

$$\frac{2}{\pi}h(2\pi h)^d\mathrm{e}^{\frac{-2S(\mathbf{m})}{h}}\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}|\det\mathcal{V}_{\mathbf{s}}|^{-1/2}\int_{\gamma_1+\nu\le z\le\gamma<1}\int_{\Gamma_z^{-1}(\gamma)}^1\frac{k^{\mathbf{s}}_\nu(\gamma)k^{\mathbf{s}}_\nu(z)|L^{\mathbf{s}}_{z,v}|^2}{(1+\tilde y)\Big(1+\big(1+2|L^{\mathbf{s}}_{z,v}|^2\big)\tilde y\Big)}\mathrm{d}\tilde y\mathrm{d}z\mathrm{d}\gamma.\tag{7.4.16}$$

By partial fraction decomposition, the $\tilde y$-integral becomes

$$\begin{aligned}\int_{\Gamma_z^{-1}(\gamma)}^1\frac{1}{(1+\tilde y)\Big(1+\big(1+2|L^{\mathbf{s}}_{z,v}|^2\big)\tilde y\Big)}\mathrm{d}\tilde y &= \frac{1}{2|L^{\mathbf{s}}_{z,v}|^2}\int_{\Gamma_z^{-1}(\gamma)}^1\frac{1+2|L^{\mathbf{s}}_{z,v}|^2}{1+\big(1+2|L^{\mathbf{s}}_{z,v}|^2\big)\tilde y}-\frac{1}{1+\tilde y}\mathrm{d}\tilde y\\ &= \frac{1}{2|L^{\mathbf{s}}_{z,v}|^2}\ln\left(\frac{\big(1+|L^{\mathbf{s}}_{z,v}|^2\big)\big(1+\Gamma_z^{-1}(\gamma)\big)}{1+\big(1+2|L^{\mathbf{s}}_{z,v}|^2\big)\Gamma_z^{-1}(\gamma)}\right)\end{aligned}\tag{7.4.17}$$

and using (7.3.4)-(7.3.5) as well as (7.4.9), the quantity in the logarithm from (7.4.17) simplifies as follows :

$$\begin{aligned}\frac{\big(1+|L^{\mathbf{s}}_{z,v}|^2\big)\big(1+\Gamma_z^{-1}(\gamma)\big)}{1+\big(1+2|L^{\mathbf{s}}_{z,v}|^2\big)\Gamma_z^{-1}(\gamma)} &= \frac{\big(P(z)+(1-z)^2\big)(1-z)(1+\gamma)}{P(z)(1-\gamma z)+(3z^2+2z+3)(\gamma-z)}\\ &= 2\,\frac{(1+z)^2(1-z)(1+\gamma)}{(1-z^2)(1+3z+3\gamma+z\gamma)}\\ &= 2\,\frac{(1+z)(1+\gamma)}{1+3z+3\gamma+z\gamma}.\end{aligned}\tag{7.4.18}$$

Putting together (7.4.16), (7.4.17), (7.4.18) and using (7.3.52), we get

$$\begin{aligned}&\langle P_h\tilde f^{\mathbf{m}}_{\nu,h},\tilde f^{\mathbf{m}}_{\nu,h}\rangle + O_\nu\Big(h^2\mathrm{e}^{-\frac{2S(\mathbf{m})}{h}}\Big) =\\ &\qquad\frac{h}{\pi}\mathrm{e}^{\frac{-2S(\mathbf{m})}{h}}\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\left(\frac{\det\mathcal{V}_{\mathbf{m}}}{|\det\mathcal{V}_{\mathbf{s}}|}\right)^{1/2}\int_{\gamma_1+\nu\le z\le\gamma<1}k^{\mathbf{s}}_\nu(\gamma)k^{\mathbf{s}}_\nu(z)\ln\left(2\,\frac{(1+z)(1+\gamma)}{1+3z+3\gamma+z\gamma}\right)\mathrm{d}z\mathrm{d}\gamma.\end{aligned}\tag{7.4.19}$$

Now, the function $1+3z+3\gamma+z\gamma$ is non-negative on $[\gamma_1,1]^2$ and vanishes only at (γ_1,γ_1). Moreover, we have by Taylor expansion that

$$1+3z+3\gamma+z\gamma \geq \frac{|(\gamma,z)-(\gamma_1,\gamma_1)|}{C} \geq \max\left(\frac{z-\gamma_1}{C},\frac{\gamma-\gamma_1}{C}\right)$$

for $(\gamma,z)\in[\gamma_1,1]^2$ close enough to (γ_1,γ_1) and thus

$$\ln\left(2\,\frac{(1+z)(1+\gamma)}{1+3z+3\gamma+z\gamma}\right) = O\big(|\ln(z-\gamma_1)|\big)$$

holds as well as

$$\ln\left(2\,\frac{(1+z)(1+\gamma)}{1+3z+3\gamma+z\gamma}\right) = O\big(|\ln(\gamma-\gamma_1)|\big).$$

Besides, by (7.3.50) and (7.3.51), we have

$$k^{\mathbf{s}}_{\nu}(z) = \left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)^{\frac{-1}{2\sqrt{|\tau_{\mathbf{s}}|}}} k^{\mathbf{s}}_0(z)\left(1+O\left(\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}}\right)\right) \tag{7.4.20}$$

with $k^{\mathbf{s}}_0(z) = O\left(|z-\gamma_1|^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}-1}\right)$ on $[\gamma_1,1]$. Consequently, the integral

$$\int_{\gamma_1\leq z\leq\gamma<1} k^{\mathbf{s}}_0(\gamma)k^{\mathbf{s}}_0(z)\ln\left(2\,\frac{(1+z)(1+\gamma)}{1+3z+3\gamma+z\gamma}\right)\mathrm{d}z\mathrm{d}\gamma$$

exists and we have

$$\begin{aligned}\int_{\gamma_1+\nu\leq z\leq\gamma<1} k^{\mathbf{s}}_0(\gamma)k^{\mathbf{s}}_0(z)\ln\left(2\,\frac{(1+z)(1+\gamma)}{1+3z+3\gamma+z\gamma}\right)\mathrm{d}z\mathrm{d}\gamma + O\left(\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}}\right)\\ = \int_{\gamma_1\leq z\leq\gamma<1} k^{\mathbf{s}}_0(\gamma)k^{\mathbf{s}}_0(z)\ln\left(2\,\frac{(1+z)(1+\gamma)}{1+3z+3\gamma+z\gamma}\right)\mathrm{d}z\mathrm{d}\gamma.\end{aligned} \tag{7.4.21}$$

Combining (7.4.19), (7.4.20) and (7.4.21), we get the announced result. □

7.5 Proof of the main results

We now introduce a series of results which will enable us to go from the approximated eigenvalues of P_h to the actual ones.

Lemma 7.5.1. *Let $\mathbf{m}\in\mathtt{U}^{(0)}\backslash\{\underline{\mathbf{m}}\}$. Using the notations (7.1.12), (7.4.1) and (7.4.2), we have*

i) $\|P_h\tilde{f}^{\mathbf{m}}_{\nu,h}\| = \sqrt{h\tilde{\lambda}^{\mathbf{m}}_{\nu,h}}\left(O_\nu\left(h^{\frac{1}{2}}\right)+O\left(\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}}|\ln(\nu)|\right)\right)$

ii) $\|P_h^*\tilde{f}^{\mathbf{m}}_{\nu,h}\| = \sqrt{h\tilde{\lambda}^{\mathbf{m}}_{\nu,h}}\left(O_\nu\left(h^{\frac{1}{2}}\right)+O\big(|\ln(\nu)|\big)\right).$

Proof. The first item is an immediate consequence of Propositions 7.3.8 and 7.4.4. The second one can be obtained similarly using Remark 7.3.5 and mimicking the proof of Proposition 7.3.8 after noticing that

$$\overset{*}{\omega}{}^{\mathbf{m}}_{\nu,z}(x,v) = O(1)\begin{pmatrix}\partial_z L^{\mathbf{s}}_{z;x}\\ \partial_z L^{\mathbf{s}}_{z;v}\end{pmatrix}\cdot\begin{pmatrix}\tilde{x}_{\mathbf{s}}\\ v\end{pmatrix}.$$

□

Lemma 7.5.2. *For $\mathbf{m}$ and $\mathbf{m}'$ two distinct elements of $\mathtt{U}^{(0)}$, we have*

i) $\langle P_h\tilde{f}^{\mathbf{m}}_{\nu,h},\tilde{f}^{\mathbf{m}'}_{\nu,h}\rangle = O_\nu\left(h^\infty\sqrt{\tilde{\lambda}^{\mathbf{m}}_{\nu,h}\tilde{\lambda}^{\mathbf{m}'}_{\nu,h}}\right)$

ii) There exists $c>0$ such that $\langle\tilde{f}^{\mathbf{m}}_{\nu,h},\tilde{f}^{\mathbf{m}'}_{\nu,h}\rangle = O(\mathrm{e}^{-c/h})$

Proof. The proof is a straightforward adaptation of the one of Lemma 5.5 in [36], even though the operator P_h and the quasimodes $(\tilde{f}^{\mathbf{m}}_{\nu,h})_{\mathbf{m}}$ differ from the ones of this reference. We recall the main steps for the reader's convenience.

i) : The idea is to use (7.3.27), the fact that P_h is local in x, Hypothesis 7.2.5 and the support properties of $\nabla\theta^{\mathbf{m}}_{\nu,h}$ and $\nabla\chi_{\mathbf{m}}$ to show that

$$\left|\left\langle P_h f^{\mathbf{m}}_{\nu,h}, f^{\mathbf{m}'}_{\nu,h}\right\rangle\right| \leq \left\langle \mathrm{Op}_h(m_h\,\mathrm{Id})(\theta^{\mathbf{m}}_{\nu,h}(\partial_v\chi_{\mathbf{m}})\mathrm{e}^{-W^{\mathbf{m}}/h})\,,\, b_h f_{\mathbf{m}',h}\right\rangle = O_\nu\left(h^\infty \mathrm{e}^{-\frac{S(\mathbf{m})+S(\mathbf{m}')}{h}}\right)$$

by (7.4.14). We can then conclude with (7.3.52).

ii) : It is shown in [36] (proof of Lemma 5.5) that when $V(\mathbf{m}) = V(\mathbf{m}')$, the supports of $f^{\mathbf{m}}_{\nu,h}$ and $f^{\mathbf{m}'}_{\nu,h}$ do not meet. Thus we can suppose that $V(\mathbf{m}) > V(\mathbf{m}')$ and in that case, using once again (7.3.27) and Hypothesis 7.2.5, we show that

$$\left\langle f^{\mathbf{m}}_{\nu,h}, f^{\mathbf{m}'}_{\nu,h}\right\rangle = \int_{E(\mathbf{m})+B(0,\varepsilon')} \theta^{\mathbf{m}}_{\nu,h}\theta^{\mathbf{m}'}_{\nu,h}\chi_{\mathbf{m}}\chi_{\mathbf{m}'}\mathrm{e}^{-\frac{2V-V(\mathbf{m})-V(\mathbf{m}')+v^2}{2h}}\mathrm{d}(x,v) = O\left(\mathrm{e}^{-\frac{V(\mathbf{m})-V(\mathbf{m}')}{2h}}\right)$$

so the conclusion immediately follows from (7.3.52). ☐

In order to go from quasimodes to functions that actually belong to the generalized eigenspace associated to the small eigenvalues of P_h, let us now consider the operator

$$\Pi_0 = \frac{1}{2i\pi}\int_{|z|=ch}(z-P_h)^{-1}\mathrm{d}z$$

introduced in [39]. Using the resolvent estimates from Theorem 7.1.2, the following is established in [39] :

Proposition 7.5.3. *The operator Π_0 is a projector on the generalized eigenspace associated to the small eigenvalues of P_h and satisfies $\|\Pi_0\| = O(1)$.*

Lemma 7.5.4. *Using the notations (7.1.12), (7.4.1) and (7.4.2), for any $\mathbf{m}\in\mathtt{U}^{(0)}$, we have*

$$\|(1-\Pi_0)\tilde{f}^{\mathbf{m}}_{\nu,h}\| = \sqrt{\tilde{\lambda}_{\mathbf{m},h}}\left(O_\nu(1) + O\left(h^{-1/2}\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}}\,|\ln(\nu)|\right)\right).$$

Proof. We simply recall the proof from [26] : we write

$$\begin{aligned}(1-\Pi_0)\tilde{f}^{\mathbf{m}}_{\nu,h} &= \frac{1}{2i\pi}\int_{|z|=ch}\left(z^{-1}-(z-P_h)^{-1}\right)\tilde{f}^{\mathbf{m}}_{\nu,h}\mathrm{d}z\\ &= \frac{-1}{2i\pi}\int_{|z|=ch} z^{-1}(z-P_h)^{-1}P_h\tilde{f}^{\mathbf{m}}_{\nu,h}\mathrm{d}z.\end{aligned}$$

We can then conclude using Lemma 7.5.1 and the resolvent estimate from Theorem 7.1.2. ☐

Lemma 7.5.5. *Recall the notations (7.1.12), (7.4.1) and (7.4.2). The family $\left(\Pi_0\tilde{f}^{\mathbf{m}}_{\nu,h}\right)_{\mathbf{m}\in\mathtt{U}^{(0)}}$ is almost orthonormal : there exists $c>0$ such that*

$$\langle \Pi_0\tilde{f}^{\mathbf{m}}_{\nu,h}, \Pi_0\tilde{f}^{\mathbf{m}'}_{\nu,h}\rangle = \delta_{\mathbf{m},\mathbf{m}'} + O_\nu(\mathrm{e}^{-c/h}).$$

In particular, it is a basis of the space $\mathrm{Ran}\,\Pi_0$.
Moreover, we have

$$\langle P_h\Pi_0\tilde{f}^{\mathbf{m}}_{\nu,h}, \Pi_0\tilde{f}^{\mathbf{m}'}_{\nu,h}\rangle = \delta_{\mathbf{m},\mathbf{m}'}\tilde{\lambda}^{\mathbf{m}}_{\nu,h} + \sqrt{\tilde{\lambda}^{\mathbf{m}}_{\nu,h}\tilde{\lambda}^{\mathbf{m}'}_{\nu,h}}\left(O_\nu\left(\sqrt{h}\right) + O\left(\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}}\,|\ln(\nu)|^2\right)\right).$$

Proof. The proof is the same as the one of Proposition 4.10 in [26]. It suffices to write

$$\langle\Pi_0\tilde{f}^{\mathbf{m}}_{\nu,h}, \Pi_0\tilde{f}^{\mathbf{m}'}_{\nu,h}\rangle = \langle\tilde{f}^{\mathbf{m}}_{\nu,h}, \tilde{f}^{\mathbf{m}'}_{\nu,h}\rangle + \langle\tilde{f}^{\mathbf{m}}_{\nu,h}, (\Pi_0-1)\tilde{f}^{\mathbf{m}'}_{\nu,h}\rangle + \langle(\Pi_0-1)\tilde{f}^{\mathbf{m}}_{\nu,h}, \Pi_0\tilde{f}^{\mathbf{m}'}_{\nu,h}\rangle$$

as well as

$$\langle P_h\Pi_0\tilde{f}^{\mathbf{m}}_{\nu,h}, \Pi_0\tilde{f}^{\mathbf{m}'}_{\nu,h}\rangle = \langle P_h\tilde{f}^{\mathbf{m}}_{\nu,h}, \tilde{f}^{\mathbf{m}'}_{\nu,h}\rangle + \langle(\Pi_0-1)\tilde{f}^{\mathbf{m}}_{\nu,h}, P_h^*\tilde{f}^{\mathbf{m}'}_{\nu,h}\rangle + \langle\Pi_0P_h\tilde{f}^{\mathbf{m}}_{\nu,h}, (\Pi_0-1)\tilde{f}^{\mathbf{m}'}_{\nu,h}\rangle.$$

and use all the previous results of this section together with Proposition 7.4.4. ☐

Let us re-label the local minima $\mathbf{m}_1, \dots, \mathbf{m}_{n_0}$ so that $(S(\mathbf{m}_j))_{j=1,\dots,n_0}$ is non increasing in j. For shortness, we will now denote

$$\tilde{f}_j = \tilde{f}_{\nu,h}^{\mathbf{m}_j} \qquad \text{and} \qquad \tilde{\lambda}_j = \tilde{\lambda}_{\nu,h}^{\mathbf{m}_j}$$

which still depend on ν and h. Note in particular that according to Proposition 7.4.4, $\tilde{\lambda}_j = O_\nu(\tilde{\lambda}_k)$ whenever $1 \leq j \leq k \leq n_0$. We also denote $(\tilde{u}_j)_{j=1,\dots,n_0}$ the orthogonalization by the Gram-Schmidt procedure of the family $(\Pi_0 \tilde{f}_j)_{j=1,\dots,n_0}$ and

$$u_j = \frac{\tilde{u}_j}{\|\tilde{u}_j\|}.$$

In this setting and with our previous results, we get the following (see [26], Proposition 4.12 for a proof).

Lemma 7.5.6. *With the notations (7.1.12), (7.4.1) and (7.4.2), for all $1 \leq j, k \leq n_0$, it holds*

$$\langle P_h u_j, u_k \rangle = \delta_{j,k} \tilde{\lambda}_j + \sqrt{\tilde{\lambda}_j \tilde{\lambda}_k} \Big(O_\nu\big(\sqrt{h}\big) + O\big(\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}} |\ln(\nu)|^2\big)\Big).$$

In order to compute the small eigenvalues of P_h, let us now consider the restriction $P_h|_{\operatorname{Ran}\Pi_0} : \operatorname{Ran}\Pi_0 \to \operatorname{Ran}\Pi_0$. We denote $\hat{u}_j = u_{n_0-j+1}$, $\hat{\lambda}_j = \tilde{\lambda}_{n_0-j+1}$ and $\mathcal{M}$ the matrix of $P_h|_{\operatorname{Ran}\Pi_0}$ in the orthonormal basis $(\hat{u}_1, \dots, \hat{u}_{n_0})$. Since $\hat{u}_{n_0} = u_1 = \tilde{f}_1$, we have

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}' & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{where} \qquad \mathcal{M}' = \big(\langle P_h \hat{u}_j, \hat{u}_k \rangle\big)_{1 \leq j,k \leq n_0 - 1}$$

and it is sufficient to study the spectrum of $\mathcal{M}'$. We will also denote $\{\hat{S}_1 < \dots < \hat{S}_p\}$ the set $\{S(\mathbf{m}_j)\,;\, 2 \leq j \leq n_0\}$ and for $1 \leq k \leq p$, E_k the subspace of $L^2(\mathbb{R}^{2d})$ generated by $\{\hat{u}_r\,;\, S(\mathbf{m}_r) = \hat{S}_k\}$. Finally, we set $\varpi_k = \mathrm{e}^{-(\hat{S}_k - \hat{S}_{k-1})/h}$ for $2 \leq k \leq p$ and $\varepsilon_j(\varpi) = \prod_{k=2}^{j} \varpi_k = \mathrm{e}^{-(\hat{S}_j - \hat{S}_1)/h}$ for $2 \leq j \leq p$ (with the convention $\varepsilon_1(\varpi) = 1$). In view of Proposition 7.4.4, let us also denote

$$\tilde{\varrho}_0(\mathbf{m}) = \frac{1}{\pi} \sum_{\mathbf{s} \in \mathbf{j}(\mathbf{m})} \left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right)^{\frac{1}{\sqrt{|\tau_{\mathbf{s}}|}}} \left(\frac{\det \mathcal{V}_{\mathbf{m}}}{|\det \mathcal{V}_{\mathbf{s}}|}\right)^{1/2} \int_{\gamma_1 \leq z \leq \gamma < 1} k_0^{\mathbf{s}}(\gamma) k_0^{\mathbf{s}}(z) \ln\left(2 \frac{(1+z)(1+\gamma)}{1+3z+3\gamma+z\gamma}\right) \mathrm{d}z \mathrm{d}\gamma$$

and

$$\hat{\lambda}_j^0 = h\, \tilde{\varrho}_0(\mathbf{m}_{n_0-j+1})\, \mathrm{e}^{\frac{-2S(\mathbf{m}_{n_0-j+1})}{h}}.$$

Lemma 7.5.7. *With the above notations, the matrix $\mathcal{M}'$ satisfies*

$$h^{-1} \mathrm{e}^{2\hat{S}_1/h} \mathcal{M}' = \Omega(\varpi) \Big(M_0^{\#} + O_\nu\big(\sqrt{h}\big) + O\big(\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}} |\ln(\nu)|^2\big)\Big) \Omega(\varpi)$$

with

$$M_0^{\#} = \operatorname{diag}\Big(\tilde{\varrho}_0(\mathbf{m}_{n_0-j+1})\,;\, 1 \leq j \leq n_0 - 1\Big)$$

and

$$\Omega(\varpi) = \operatorname{diag}\big(\varepsilon_1(\varpi)\mathrm{Id}_{E_1}, \dots, \varepsilon_p(\varpi)\mathrm{Id}_{E_p}\big).$$

In particular, for all $\nu > 0$, there exists $h_0 > 0$ such that for all $0 < h < h_0$,

$$h^{-1} \mathrm{e}^{2\hat{S}_1/h} \mathcal{M}' = \Omega(\varpi) \Big(M_0^{\#} + O\big(\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}} |\ln(\nu)|^2\big)\Big) \Omega(\varpi).$$

Remark 7.5.8. *In the words of Definition A.1 from [26], the last Lemma implies that for all $\nu > 0$, there exists $h_0 > 0$ such that for all $0 < h < h_0$,*

$$h^{-1} \mathrm{e}^{2\hat{S}_1/h} \mathcal{M}' \textit{ is a } \Big((E_k)_k\,,\, \varpi\,,\, \nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}} |\ln(\nu)|^2\Big)\textit{-graded matrix.}$$

Proof. According to Lemma 7.5.6 and Proposition 7.4.4, we can decompose $\mathcal{M}' = \mathcal{M}'_1 + \mathcal{M}'_2$ with

$$\mathcal{M}'_1 = \mathrm{diag}(\hat{\lambda}^0_j\,;\, 1 \leq j \leq n_0-1) \qquad \text{and} \qquad \mathcal{M}'_2 = \left(\sqrt{\hat{\lambda}_j\hat{\lambda}_k}\Big[O_\nu\Big(\sqrt{h}\Big)+O\Big(\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}}\,|\ln(\nu)|^2\Big)\Big]\right)_{1\leq j,k\leq n_0-1}.$$

It then suffices to notice that $M_0^{\#} = h^{-1}\mathrm{e}^{2\hat{S}_1/h}\Omega(\varpi)^{-1}\mathcal{M}'_1\Omega(\varpi)^{-1}$ and that

$$h^{-1}\mathrm{e}^{2\hat{S}_1/h}\Omega(\varpi)^{-1}\mathcal{M}'_2\Omega(\varpi)^{-1} = O_\nu\Big(\sqrt{h}\Big) + O\Big(\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}}\,|\ln(\nu)|^2\Big)$$

where we still used Proposition 7.4.4. □

Proof of Theorem 7.1.3. According to Remark 7.5.8, it now suffices to combine the result of Lemma 7.5.7 with Theorem A.4 from [26] which gives a description of the spectrum of graded matrices. We get that for all $\nu > 0$, there exists $h_0 > 0$ such that for all $0 < h < h_0$,

$$h^{-1}\mathrm{e}^{2S(\mathbf{m})/h}\lambda(\mathbf{m}, h) - \tilde{\varrho}_0(\mathbf{m}) = O\Big(\nu^{\frac{1}{2\sqrt{|\tau_{\mathbf{s}}|}}}\,|\ln(\nu)|^2\Big)$$

and the result is proven. □

Proof of Corollaries 7.1.4 and 7.1.5. With the notations from Theorem 7.1.3, it is shown in [39], section 4 with the use of PT-Symmetry arguments and a quantitative version of the Gearhart-Prüss Theorem, that there exist $c > 0$ and some projectors $(\Pi_j)_{1\leq j\leq n_0}$ which are $O(1)$ and such that

- $\Pi_1 = \mathbb{P}_1$
- $\Pi_j\Pi_k = \delta_{j,k}\Pi_j$
- $\mathbb{P}_k = \sum_{\{j\,;\,S(\mathbf{m}_j)\geq S(\mathbf{m}_k)\}}\Pi_j$
- $\mathrm{e}^{-tP_h/h} = \sum_{j=1}^{n_0}\mathrm{e}^{-t\lambda(\mathbf{m}_j,h)/h}\Pi_j + O(\mathrm{e}^{-ct}) \qquad$ for $t \geq 0$ and h small enough.

Corollary 7.1.4 directly follows, while the proof of Corollary 7.1.5 is then an easy adaptation of the one of Corollary 1.6 from [4]. (Note that our notations t_k^- and t_k^+ differ from that in [4]). □

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7.6 Appendix

7.6.1 Structure of the collision operator

The aim of this section is to show Proposition 7.2.2 and Corollary 7.2.3. For a, b two symbols, we denote $a\#b$ the symbol of $\mathrm{Op}_h(a)\circ\mathrm{Op}_h(b)$. We start by showing that Q_h defined in (7.1.7) is a pseudo-differential operator :

Lemma 7.6.1. *One has $\Pi_h = \mathrm{Op}_h(\varpi_h)$ with $\varpi_h \in S^{1/2}(1)$ given by*

$$\varpi_h(v,\eta) = 2^d\mathrm{e}^{-\frac{v^2+4\eta^2}{2h}}.$$

Proof of Lemma 7.6.1. First, notice that the distributional kernel of Π_h is $\mu_h(v)\mu_h(v')$. Using the formula (7.1.8) to compute the symbol of a pseudo-differential operator from its distributional kernel, we get

$$\mathcal{F}_{h,v'}\Big(\mu_h(v+v'/2)\mu_h(v-v'/2)\Big)(v,\eta) = 2^d\mathrm{e}^{-\frac{v^2+4\eta^2}{2h}}$$

which is clearly in $S^{1/2}(1)$ as $\mathrm{e}^{-\frac{v^2+4\eta^2}{2}} \in S^0(1)$. □

Proof of Proposition 7.2.2. Let us first check that $m_h \in S^{1/2}\big(\langle v,\eta\rangle^{-2}\big)$. We have

$$m_h(v,\eta) = \tilde{m}(h^{-1/2}v, h^{-1/2}\eta) \qquad \text{and} \qquad \tilde{m}(v,\eta) = \check{m}\Big(\frac{v^2}{2} + 2\eta^2\Big) \tag{7.6.1}$$

with

$$\tilde{m}(v,\eta) = 2\int_0^1 (y+1)^{d-2}\mathrm{e}^{-y\left(\frac{v^2}{2}+2\eta^2\right)}\mathrm{d}y \qquad \text{and} \qquad \check{m}(\mu) = 2\int_0^1 (y+1)^{d-2}\mathrm{e}^{-y\mu}\mathrm{d}y.$$

One can then check using integration by parts that for all $k \in \mathbb{N}$, there exists C_k such that $|\partial_\mu^k \check{m}(\mu)| \leq C_k\langle\mu\rangle^{-k-1}$ from which we deduce using (7.6.1) that $\tilde{m} \in S^0\big(\langle v,\eta\rangle^{-2}\big)$. Thus, still using (7.6.1), for $\alpha \in \mathbb{N}^{2d}$, there exists C_α such that

$$|\partial^\alpha m_h(v,\eta)| = h^{-|\alpha|/2}\big|\partial^\alpha\tilde{m}(h^{-1/2}v, h^{-1/2}\eta)\big| \leq C_\alpha h^{-|\alpha|/2}\big\langle h^{-1/2}v, h^{-1/2}\eta\big\rangle^{-2} \leq C_\alpha h^{-|\alpha|/2}\langle v,\eta\rangle^{-2},$$

so m_h indeed belongs to $S^{1/2}\big(\langle v,\eta\rangle^{-2}\big)$. Using symbolic calculus and Lemma 7.6.1, one could then simply check that

$$(-i\eta^t + v^t/2)\#(m_h\,\mathrm{Id})\#(i\eta + v/2) = h(1-\varpi_h) \tag{7.6.2}$$

but let us explain how the suitable m_h (i.e the one solving (7.6.2)) was found. Since $(i\eta + v/2)$ and its conjugate are both polynomials of degree 1, we compute

$$\begin{aligned}(-i\eta^t + v^t/2)\#(m_h\,\mathrm{Id})\#(i\eta + v/2) = \Big(\eta^2{+}\frac{v^2}{4}\Big)m_h \\ -\frac{h}{2}\big(dm_h + v\cdot\partial_v m_h + \eta\cdot\partial_\eta m_h\big) + \frac{h^2}{4}\Big(\Delta_v + \frac{1}{4}\Delta_\eta\Big)m_h.\end{aligned} \tag{7.6.3}$$

Let us look for solutions under the form $m_h(v,\eta) = u_h(v,\eta)\mathrm{e}^{\frac{v^2+4\eta^2}{2h}}$. In that case,

$$\partial_v m_h = \mathrm{e}^{\frac{v^2+4\eta^2}{2h}}\Big(\partial_v u_h + \frac{u_h}{h}v\Big) \qquad \text{and} \qquad \Delta_v m_h = \Big(\Delta_v u_h + \frac{2v}{h}\cdot\partial_v u_h + \frac{d}{h}u_h + \frac{v^2}{h^2}u_h\Big)\mathrm{e}^{\frac{v^2+4\eta^2}{2h}}$$

so

$$\frac{h^2}{4}\Delta_v m_h - \frac{h}{2}v\cdot\partial_v m_h = \Big(\frac{h^2}{4}\Delta_v u_h + \frac{hd}{4}u_h - \frac{v^2}{4}u_h\Big)\mathrm{e}^{\frac{v^2+4\eta^2}{2h}}.$$

Similarly, we compute

$$\frac{h^2}{16}\Delta_\eta m_h - \frac{h}{2}\eta\cdot\partial_\eta m_h = \Big(\frac{h^2}{16}\Delta_\eta u_h + \frac{hd}{4}u_h - \eta^2 u_h\Big)\mathrm{e}^{\frac{v^2+4\eta^2}{2h}}$$

so according to (7.6.3), (7.6.2) becomes

$$\frac{h^2}{4}\Big(\Delta_v u_h + \frac{1}{4}\Delta_\eta u_h\Big) = h\Big(\mathrm{e}^{-\frac{v^2+4\eta^2}{2h}} - 2^d\mathrm{e}^{-\frac{v^2+4\eta^2}{h}}\Big).$$

Applying the semiclassical Fourier transform on $\mathbb{R}^{2d}$, this yields

$$-\frac{1}{4}\Big(v^{*2} + \frac{\eta^{*2}}{4}\Big)\mathcal{F}_h u_h = h(\pi h)^d\Big(\mathrm{e}^{-\frac{4v^{*2}+\eta^{*2}}{8h}} - \mathrm{e}^{-\frac{4v^{*2}+\eta^{*2}}{16h}}\Big) = -\frac{(\pi h)^d}{4}\Big(v^{*2} + \frac{\eta^{*2}}{4}\Big)\int_1^2 \mathrm{e}^{-s\frac{4v^{*2}+\eta^{*2}}{16h}}\mathrm{d}s$$

where (v^*,η^*) denotes the dual variable of (v,η). Hence

$$\mathcal{F}_h u_h(v^*,\eta^*) = (\pi h)^d\int_1^2 \mathrm{e}^{-s\frac{4v^{*2}+\eta^{*2}}{16h}}\mathrm{d}s$$

and applying the inverse semiclassical Fourier transform, we get

$$u_h(v,\eta) = 2^d\int_1^2 s^{-d}\mathrm{e}^{-\frac{v^2+4\eta^2}{sh}}\mathrm{d}s.$$

Consequently,

$$m_h(v,\eta) = 2^d \int_1^2 s^{-d} \mathrm{e}^{-\frac{v^2+4\eta^2}{2h}(\frac{2}{s}-1)} \mathrm{d}s$$

and we find the final expression of m_h by substituting $y = \frac{2}{s} - 1$. □

Proof of Corollary 7.2.3. By symbolic calculus, we just have to check that $g_h = (-i\eta^t + v^t/2)\#(m_h \,\mathrm{Id})$. Since the symbol on the left hand side is a polynomial of degree 1, we have

$$(-i\eta^t + v^t/2)\#(m_h \,\mathrm{Id}) = m_h(-i\eta^t + v^t/2) - \frac{h}{2}\Big(\partial_v^t - \frac{i}{2}\partial_\eta^t\Big) m_h.$$

Now

$$-\frac{h}{2}\partial_v^t m_h(v,\eta) = \int_0^1 y(y+1)^{d-2} \mathrm{e}^{-\frac{y}{h}\left(\frac{v^2}{2}+2\eta^2\right)} \mathrm{d}y\, v^t$$

so we easily get

$$m_h(v,\eta)\,\frac{v^t}{2} - \frac{h}{2}\partial_v^t m_h(v,\eta) = \int_0^1 (y+1)^{d-1} \mathrm{e}^{-\frac{y}{h}\left(\frac{v^2}{2}+2\eta^2\right)} \mathrm{d}y\, v^t.$$

One can show similarly that

$$-i m_h(v,\eta)\,\eta^t + \frac{ih}{4}\partial_\eta^t m_h(v,\eta) = -2i\int_0^1 (y+1)^{d-1} \mathrm{e}^{-\frac{y}{h}\left(\frac{v^2}{2}+2\eta^2\right)} \mathrm{d}y\, \eta^t$$

which is enough to conclude. □

7.6.2 Bilinear algebra

Lemma 7.6.2. *Let $L(x,v) = L_x \cdot x + L_v \cdot v$ a linear form on $\mathbb{R}^{2d}$ and recall the notation (7.1.11). Then for any $\mathbf{s} \in \mathtt{U}^{(1)}$, the matrix $\mathcal{W}_\mathbf{s} + \nabla L\, \nabla L^t$ is positive definite if and only if*

$$-\mathcal{V}_\mathbf{s}^{-1} L_x \cdot L_x - L_v^2 > \frac{1}{2}. \tag{7.6.4}$$

Moreover, its determinant is

$$2^{-2d} \det \mathcal{V}_\mathbf{s}\, \big(1 + 2\mathcal{V}_\mathbf{s}^{-1} L_x \cdot L_x + 2L_v^2\big).$$

Proof. First notice that since $\mathbf{s} \in \mathtt{U}^{(1)}$ and $\mathcal{W}_\mathbf{s} + \nabla L\, \nabla L^t \geq \mathcal{W}_\mathbf{s}$, the matrix $\mathcal{W}_\mathbf{s} + \nabla L\, \nabla L^t$ has at most one negative eigenvalue, so it is sufficient to show that its determinant is positive if and only if (7.6.4) holds. The rest of the proof is inspired by [4] (Lemma 3.3). We have

$$\det\Big(\mathcal{W}_\mathbf{s} + \nabla L\, \nabla L^t\Big) = \det \mathcal{W}_\mathbf{s} \det\Big(\mathrm{Id} + \mathcal{W}_\mathbf{s}^{-1} \nabla L\, \nabla L^t\Big) = 2^{-2d} \det \mathcal{V}_\mathbf{s} \det\Big(\mathrm{Id} + \mathcal{W}_\mathbf{s}^{-1} \nabla L\, \nabla L^t\Big)$$

and since $\det \mathcal{V}_\mathbf{s} < 0$, it only remains to show that

$$(7.6.4) \iff \det\Big(\mathrm{Id} + \mathcal{W}_\mathbf{s}^{-1} \nabla L\, \nabla L^t\Big) < 0.$$

Now it is easy to see that

$$\Big(\mathrm{Id} + \mathcal{W}_\mathbf{s}^{-1} \nabla L\, \nabla L^t\Big)|_{\nabla L^\perp} = \mathrm{Id} \quad \text{and} \quad \Big(\mathrm{Id} + \mathcal{W}_\mathbf{s}^{-1} \nabla L\, \nabla L^t\Big)\nabla L \cdot \nabla L = \big(1 + 2\mathcal{V}_\mathbf{s}^{-1} L_x \cdot L_x + 2L_v^2\big)|\nabla L|^2.$$

Hence, $\det\big(\mathrm{Id} + \mathcal{W}_\mathbf{s}^{-1} \nabla L\, \nabla L^t\big) = 1 + 2\mathcal{V}_\mathbf{s}^{-1} L_x \cdot L_x + 2L_v^2$ which is negative if and only if (7.6.4) holds true. □

Lemma 7.6.3. *Recall the notations (7.1.11) and (7.4.7). For $\gamma \in [\gamma_1 + \nu, 1]$ and $y \in (0,1)$, we have*

$$\det H^\mathbf{s}_{\gamma,y} = \frac{(1+y)^{2d-2}}{(4y)^d}\Big(1 + \big(1 + 2|L^\mathbf{s}_{\gamma,v}|^2\big)y\Big)^2 |\det \mathcal{V}|. \tag{7.6.5}$$

Proof. We drop some exponents and indexes $\mathbf{s}$ in the notations for shortness. Let us begin by writing

$$(7.6.6)\ H_{\gamma,y} = \begin{pmatrix} \mathcal{V} & 0 & 0 \\ 0 & \frac{(y+1)^2}{4y} & \frac{y^2-1}{4y} \\ 0 & \frac{y^2-1}{4y} & \frac{(y+1)^2}{4y} \end{pmatrix} \left[\mathrm{Id} + \begin{pmatrix} \mathcal{V}^{-1} & 0 & 0 \\ 0 & 1 & \frac{1-y}{1+y} \\ 0 & \frac{1-y}{1+y} & 1 \end{pmatrix} \begin{pmatrix} L_{\gamma,x} & L_{\gamma,x} \\ L_{\gamma,v} & 0 \\ 0 & L_{\gamma,v} \end{pmatrix} \begin{pmatrix} L^t_{\gamma,x} & L^t_{\gamma,v} & 0 \\ L^t_{\gamma,x} & 0 & L^t_{\gamma,v} \end{pmatrix} \right].$$

Clearly, the determinant of the first factor is $(4y)^{-d}(y+1)^{2d} \det \mathcal{V}$. Denoting

$$\tilde{H}_{\gamma,y} = \begin{pmatrix} \mathcal{V}^{-1} & 0 & 0 \\ 0 & 1 & \frac{1-y}{1+y} \\ 0 & \frac{1-y}{1+y} & 1 \end{pmatrix} \begin{pmatrix} L_{\gamma,x} & L_{\gamma,x} \\ L_{\gamma,v} & 0 \\ 0 & L_{\gamma,v} \end{pmatrix} \begin{pmatrix} L^t_{\gamma,x} & L^t_{\gamma,v} & 0 \\ L^t_{\gamma,x} & 0 & L^t_{\gamma,v} \end{pmatrix},$$

it is also clear that $\tilde{H}_{\gamma,y}$ has rank 2, so it has at most 2 non zero eigenvalues. Besides, using Lemma 7.3.1, one can easily check that

$$\tilde{H}_{\gamma,y} \begin{pmatrix} (1+y)\mathcal{V}^{-1}L_{\gamma,x} \\ L_{\gamma,v} \\ L_{\gamma,v} \end{pmatrix} = \frac{-2}{1+y}\Big(1 + \big(1 + |L^{\mathbf{s}}_{\gamma,v}|^2\big)y\Big) \begin{pmatrix} (1+y)\mathcal{V}^{-1}L_{\gamma,x} \\ L_{\gamma,v} \\ L_{\gamma,v} \end{pmatrix}$$

and

$$\tilde{H}_{\gamma,y} \begin{pmatrix} 0 \\ L_{\gamma,v} \\ -L_{\gamma,v} \end{pmatrix} = \frac{2y|L_{\gamma,v}|^2}{1+y} \begin{pmatrix} 0 \\ L_{\gamma,v} \\ -L_{\gamma,v} \end{pmatrix}.$$

Hence, the determinant of the second factor from (7.6.6) is

$$-(1+y)^{-2}\Big(1 + \big(1 + 2|L^{\mathbf{s}}_{\gamma,v}|^2\big)y\Big)^2$$

and we get (7.6.5). □

7.6.3 Multivariate gaussian moment

Using the formulas of the first moments of the one dimensional gaussian, we easily establish the following.

Proposition 7.6.4. *If A is a real symmetric matrix, then*

$$\int_{\mathbb{R}^{d'}} Ax \cdot x\, \mathrm{e}^{-\frac{x^2}{2}}\, \mathrm{d}x = (2\pi)^{d'/2}\, \mathrm{Tr}(A).$$

Annexe A

Espaces de symboles analytiques

On présente ici (en anglais) quelques notions et notations d'analyse microlocale semiclassique qui servent tout au long de cette thèse. Ces dernières sont principalement issues de [47, chapitre 4].

We will denote $\Xi \in \mathbb{R}^{d'}$ the dual variable of $X \in \mathbb{R}^{d'}$ and consider the space of semiclassical symbols

$$S^\kappa\big(\langle(X,\Xi)\rangle^k\big) = \big\{a_h \in \mathcal{C}^\infty(\mathbb{R}^{2d'})\,;\, \forall \alpha \in \mathbb{N}^{2d'}, \exists\, C_\alpha > 0 \textit{ such that } |\partial^\alpha a_h(X,\Xi)| \leq C_\alpha h^{-\kappa|\alpha|}\langle(X,\Xi)\rangle^k\big\}$$

where $k \in \mathbb{R}$ and $\kappa \in [0, 1/2]$. Note that those symbols are allowed to depend on h ; however, in order to shorten the notations, we will drop the index h in the rest of the paper when dealing with semiclassical symbols. Given a symbol $a \in S^\kappa(\langle(X,\Xi)\rangle^k)$, we define the associated semiclassical pseudo-differential operator for the Weyl quantization acting on functions $u \in \mathcal{S}(\mathbb{R}^{d'})$ by

$$\mathrm{Op}_h(a)u(X) = (2\pi h)^{-d'} \int_{\mathbb{R}^{d'}} \int_{\mathbb{R}^{d'}} \mathrm{e}^{\frac{i}{h}(X-X')\cdot\Xi} a\Big(\frac{X+X'}{2}, \Xi\Big) u(X')\, \mathrm{d}X' \mathrm{d}\Xi$$

where the integrals may have to be interpreted as oscillating integrals. We will denote $\Psi^\kappa(\langle(X,\Xi)\rangle^k)$ the set of such operators. For the study of linear Boltzmann equations, we will work with $X = (x,v) \in \mathbb{R}^{2d}$ and $\Xi = (\xi,\eta) \in \mathbb{R}^{2d}$. We also need to introduce the notion of analytic symbols. For our purpose, we almost always consider symbols that do not depend on the variable ξ.

Definition A.0.1. *For $\tau > 0$, let us introduce the set*

$$\Sigma_\tau = \{z \in \mathbb{C}\,;\, |\mathrm{Im}\, z| < \tau\}^d \subset \mathbb{C}^d.$$

For $k \in \mathbb{R}$, we denote $S^0_\tau(\langle(x,v,\eta)\rangle^k)$ the space of symbols $a_h \in S^0(\langle(x,v,\eta)\rangle^k)$ independent of ξ such that :

(i) *For all $(x,v) \in \mathbb{R}^{2d}$, $a_h(x,v,\cdot)$ is analytic on Σ_τ*

(ii) *For all $\beta \in \mathbb{N}^{2d}$, there exists $C_\beta > 0$ such that $|\partial^\beta_{(x,v)} a_h| \leq C_\beta \langle(x,v,\eta)\rangle^k$ on $\mathbb{R}^{2d} \times \Sigma_\tau$.*

We will also use the notation $a_h = O_{S^0_\tau(\langle(x,v,\eta)\rangle^k)}(h^N)$ to say that for all $\alpha \in \mathbb{N}^{3d}$, there exists $C_{\alpha,N}$ such that $|\partial^\alpha a_h| \leq C_{\alpha,N}\, h^N \langle(x,v,\eta)\rangle^k$ on $\mathbb{R}^{2d} \times \Sigma_\tau$.

Here again, we will drop the index h in the notations of analytic symbols. Using the Cauchy-Riemann equations, we see that item (i) from Definition A.0.1 implies that for all $\beta \in \mathbb{N}^{2d}$ and $(x,v) \in \mathbb{R}^{2d}$, the functions $\partial^\beta_{(x,v)} a(x,v,\cdot)$ are also analytic on Σ_τ. Besides, the Cauchy formula implies that for any $\tilde\tau < \tau$, $\alpha \in \mathbb{N}^d$ and $\beta \in \mathbb{N}^{2d}$, there exists $C_{\alpha,\beta}$ such that

$$|\partial^\alpha_\eta \partial^\beta_{(x,v)} a| \leq C_{\alpha,\beta} \langle(x,v,\eta)\rangle^k \qquad \text{on } \mathbb{R}^{2d} \times \Sigma_{\tilde\tau}$$

i.e up to taking τ smaller, item (ii) from Definition A.0.1 can be extended to $\beta \in \mathbb{N}^{3d}$. Let us introduce a notion of expansion where the coefficients are allowed to depend on h : we will say that

$$a \sim_h \sum_{j \geq 0} h^j a_j \tag{A.0.1}$$

in $S^0(\langle(x,v,\eta)\rangle^k)$ (resp. in $S^0_\tau(\langle(x,v,\eta)\rangle^k)$) if $(a_j)_{j\geq 0}\subset S^0(\langle(x,v,\eta)\rangle^k)$ (resp. $(a_j)_{j\geq 0}\subset S^0_\tau(\langle(x,v,\eta)\rangle^k)$) is a family of symbols which may depend on h and are such that for all $N\in\mathbb{N}$,

$$a-\sum_{j=0}^{N-1}h^j a_j = O_{S^0(\langle(x,v,\eta)\rangle^k)}(h^N) \qquad \big(\text{resp. } O_{S^0_\tau(\langle(x,v,\eta)\rangle^k)}(h^N)\big)$$

Finally, we also have the usual notion of classical expansion for a symbol : $a\sim\sum_{j\geq 0}h^j a_j$ in $S^0(\langle(x,v,\eta)\rangle^k)$ (resp. in $S^0_\tau(\langle(x,v,\eta)\rangle^k)$) means that $a\sim_h\sum_{j\geq 0}h^j a_j$ in $S^0(\langle(x,v,\eta)\rangle^k)$ (resp. in $S^0_\tau(\langle(x,v,\eta)\rangle^k)$) and the $(a_j)_{j\geq 0}$ are independent of h.
We now extend these notions to matrix valued symbols : if

$$M=(m_{p,q})_{\substack{1\leq p\leq n_1\\1\leq q\leq n_2}}$$

is a matrix of functions such that each $m_{p,q}\in S^\kappa(\langle(x,v,\eta)\rangle^k)$ (resp. $m_{p,q}\in S^0_\tau(\langle(x,v,\eta)\rangle^k)$), we say that $M\in\mathcal{M}_{n_1,n_2}\big(S^\kappa(\langle(x,v,\eta)\rangle^k)\big)$ $\big(\text{resp. } M\in\mathcal{M}_{n_1,n_2}\big(S^0_\tau(\langle(x,v,\eta)\rangle^k)\big)\big)$ and we denote

$$\mathrm{Op}_h(M)=\Big(\mathrm{Op}_h(m_{p,q})\Big)_{\substack{1\leq p\leq n_1\\1\leq q\leq n_2}}.$$

The notation

$$M=O_{\mathcal{M}_{n_1,n_2}\left(S^0(\langle(x,v,\eta)\rangle^k)\right)}(h^N) \qquad \big(\text{resp. } M=O_{\mathcal{M}_{n_1,n_2}\left(S^0_\tau(\langle(x,v,\eta)\rangle^k)\right)}(h^N)\big)$$

means that for all $(p,q)\in[\![1,n_1]\!]\times[\![1,n_2]\!]$, the symbol $m_{p,q}$ is $O_{S^0(\langle(x,v,\eta)\rangle^k)}(h^N)$ (resp. $O_{S^0_\tau(\langle(x,v,\eta)\rangle^k)}(h^N)$). Furthermore, the notions of expansions $M\sim_h\sum_{n\geq 0}h^n M_n$ and $M\sim\sum_{n\geq 0}h^n M_n$ in $\mathcal{M}_{n_1,n_2}\big(S^0(\langle(x,v,\eta)\rangle^k)\big)$ $\big(\text{resp. } \mathcal{M}_{n_1,n_2}\big(S^0_\tau(\langle(x,v,\eta)\rangle^k)\big)\big)$ are straightforward adaptations of the ones for scalar symbols.

Annexe B

Méthode de Laplace

On donne ici un énoncé précis ainsi qu'une démonstration (tous deux en anglais) de la méthode de Laplace qu'on utilise pour obtenir des approximations d'intégrales dépendant de h.

Proposition B.0.1. *Let $x_0 \in \mathbb{R}^d$, K a compact neighborhood of x_0 and $\varphi \in \mathcal{C}^\infty(K)$ such that x_0 is a non degenerate minimum of φ and its only global minimum on K. Denote $H \in \mathcal{M}_d(\mathbb{R})$ the Hessian of φ at x_0.*

- *If a_h is a function bounded uniformly in h on K such that*
$$a_h = O\Big((x-x_0)^{2n}\Big),$$
then
$$h^{-d/2}\int_K a_h(x)\mathrm{e}^{-\frac{\varphi(x)-\varphi(x_0)}{h}}\mathrm{d}x = O(h^n).$$
- *If $a_h \sim \sum_{j\geq 0} h^j a_j$ in $\mathcal{C}^\infty(K)$, then the integral*
$$\frac{\det(H)^{1/2}}{(2\pi h)^{d/2}}\int_K a_h(x)\mathrm{e}^{-\frac{\varphi(x)-\varphi(x_0)}{h}}\mathrm{d}x$$
admits a classical expansion whose first term is given by $a_0(x_0)$.

Proof. We begin by applying the Morse Lemma to φ : there exists a hypercube $C \subset \mathbb{R}^d$ centered at 0 and $\phi : H^{-1/2}C \to \phi(H^{-1/2}C)$ a smooth diffeomorphism sending 0 on x_0, whose differential at 0 is the identity and such that for all $y \in H^{-1/2}C$,
$$\varphi\big(\phi(y)\big) - \varphi(x_0) = \frac{1}{2}Hy\cdot y.$$
By assumption, there exists $\delta > 0$ such that
$$\int_{K\setminus\phi(H^{-1/2}C)} a_h(x)\mathrm{e}^{-\frac{\varphi(x)-\varphi(x_0)}{h}}\mathrm{d}x = O(\mathrm{e}^{-\delta/h})$$
so it is sufficient to work with
$$\frac{\det(H)^{1/2}}{(2\pi h)^{d/2}}\int_{H^{-1/2}C} a_h(\phi(y))\mathrm{e}^{-\frac{Hy\cdot y}{2h}}|\det D_y\phi|\,\mathrm{d}y. \tag{B.0.1}$$
After a new change of variables, (B.0.1) becomes
$$(2\pi h)^{-d/2}\int_C \tilde{a}_h(z)\mathrm{e}^{-\frac{z^2}{2h}}\mathrm{d}z$$
where $\tilde{a}_h(z) = a_h(\phi(H^{-1/2}z))|\det D_{H^{-1/2}z}\phi|$. We can already conclude for the first item thanks to the change of variables $z = \sqrt{h}w$.

For the second item, thanks to the assumption on a_h, we have $\tilde{a}_h \sim \sum_{j\geq 0} h^j \tilde{a}_j$ in $\mathcal{C}^\infty(C)$ with $\tilde{a}_j(z) = a_j(\phi(H^{-1/2}z))|\det D_{H^{-1/2}z}\phi|$. Thus, with the notation (5.3.4), one easily gets

$$(2\pi h)^{-d/2}\int_C \tilde{a}_h(z)\mathrm{e}^{-\frac{z^2}{2h}}\mathrm{d}z \sim_h \sum_{j\geq 0} h^j (2\pi h)^{-d/2}\int_C \tilde{a}_j(z)\mathrm{e}^{-\frac{z^2}{2h}}\mathrm{d}z.$$

Using the fact that, by the definition of C, for all $\beta \in \mathbb{N}^d\backslash(2\mathbb{N})^d$, we have $\int_C z^\beta \mathrm{e}^{-\frac{z^2}{2h}}\mathrm{d}z = 0$, we get with another change of variables and a Taylor expansion :

$$\begin{aligned} h^{-d/2}\int_C \tilde{a}_j(z)\mathrm{e}^{-\frac{z^2}{2h}}\mathrm{d}z &= \int_{\sqrt{h}w\in C} \tilde{a}_j\big(\sqrt{h}w\big)\mathrm{e}^{-\frac{w^2}{2}}\mathrm{d}w \\ &= \sum_{|\beta|=0}^{N-1} \frac{\partial^{2\beta}\tilde{a}_j(0)}{(2\beta)!} h^{|\beta|}\int_{\sqrt{h}w\in C} w^{2\beta}\mathrm{e}^{-\frac{w^2}{2}}\mathrm{d}w + O(h^N). \end{aligned}$$

Thus $(2\pi h)^{-d/2}\int_C \tilde{a}_j(x)\mathrm{e}^{-\frac{x^2}{2h}}\mathrm{d}x$ admits a classical expansion whose first term is $\tilde{a}_j(0) = a_j(x_0)$ and the conclusion follows. □

Annexe C

Labeling et cas générique

On présente ici (en anglais) du matériel issu de [36] qui permet d'introduire les constructions fondamentales liées aux potentiels associés aux équations que l'on étudie ; mais aussi d'énoncer précisément l'Hypothèse de non dégénérescence C.0.8 qui apparait régulièrement dans cette thèse. De façon à ce que ces constructions s'appliquent aussi bien à W qu'à $V/2$ lorsque $W(x,v) = V(x)/2 + v^2/4$, elles sont ici faites pour un potentiel général qu'on note Y. Les objets définis sont ainsi notés avec des exposants ou indices Y, notations que l'on omettra dans la pratique lorsqu'il n'y aura pas d'ambiguité quant au potentiel auquel on les applique.

Let us consider $d' \in \mathbb{N}^*$ and a smooth Morse function Y on $\mathbb{R}^{d'}$ bounded from below, having at least two local minima and such that $|\nabla Y| \geq 1/C$ outside of a compact. Recall that by [30, Lemma 3.14], it implies that $Y(X) \geq |X|/C$ outside of a compact. We also denote

$$\mathcal{U}^{(k),Y} \text{ the critical points of } Y \text{ of index } k. \tag{C.0.1}$$

For shortness, we will write "CC" instead of "connected component".

Lemma C.0.1. *If $X \in \mathcal{U}^{(1),Y}$, then there exists $r_0 > 0$ such that for all $0 < r < r_0$, X has a connected neighborhood U_r in $B(X,r)$ such that $U_r \cap \{Y < Y(X)\}$ has exactly 2 CCs.*

Proof. Let $X \in \mathcal{U}^{(1),Y}$; according to the Morse Lemma, there exists a connected neighborhood U_r of X, $r' > 0$ and $\varphi : U_r \to B(0,r')$ a smooth diffeomorphism such that

$$Y \circ \varphi^{-1} = Y(X) + \frac{1}{2}\langle \mathrm{Hess}_X Y \cdot, \cdot \rangle.$$

Besides, it is easy to see that

$$U_r \cap \{Y < Y(X)\} = \varphi^{-1}\big(\{y \in B(0,r') \,;\, \langle \mathrm{Hess}_X Y\, y, y\rangle < 0\}\big)$$

and $\{y \in B(0,r') \,;\, \langle \mathrm{Hess}_X Y\, y, y\rangle < 0\}$ has exactly 2 CCs. □

Lemma C.0.2. *Let $X \in \mathbb{R}^{d'}$ and suppose there exists $r_0 > 0$ such that for every neighborhood U of X in $B(X, r_0)$, the set $U \cap \{Y < Y(X)\}$ is not connected. Then $X \in \mathcal{U}^{(1),Y}$.*

Proof. Let us first assume by contradiction that $\nabla Y(X) \neq 0$. One can then use the implicit function theorem to show that X has a neighborhood U in $B(X, r_0)$ such that $U \cap \{Y < Y(X)\}$ is connected, which is absurd so X is a critical point of Y. It is clear that $X \notin \mathcal{U}^{(0),Y}$ so let us assume again by contradiction that $X \in \mathcal{U}^{(k),Y}$ with $k \geq 2$. Then using the Morse Lemma as in the proof of Lemma C.0.1, we would once again get that X has a neighborhood U in $B(X, r_0)$ such that $U \cap \{Y < Y(X)\}$ has the same number of CCs as $\{y \in B(0,r) \,;\, \langle \mathrm{Hess}_X Y\, y, y\rangle < 0\}$ which is connected since $k \geq 2$. Hence X has to be in $\mathcal{U}^{(1),Y}$. □

In view of the result from Lemma C.0.1 and following the approach from [18, 23], we give the following definition :

Definition C.0.3. *a) We say that $X \in \mathcal{U}^{(1),Y}$ is a separating saddle point and we denote $X \in \mathtt{V}^{(1),Y}$ if for every $r > 0$ small enough, the two CCs of $U_r \cap \{Y < Y(X)\}$ are contained in different CCs of $\{Y < Y(X)\}$.*

b) We say that $\sigma \in \mathbb{R}$ is a separating saddle value if $\sigma \in Y(\mathtt{V}^{(1),Y})$.

c) Finally, we say that a set $E \subset \mathbb{R}^{d'}$ is critical if there exists $\sigma \in Y(\mathtt{V}^{(1),Y})$ such that E is a CC of $\{Y < \sigma\}$ satisfying $\partial E \cap \mathtt{V}^{(1),Y} \neq \emptyset$.

Lemma C.0.4. *Let $\mathbf{m}$, $\mathbf{m}'$ two distinct local minima of Y. The real number*

$$\sigma = \sup\left\{a \in \mathbb{R}\,;\, \mathbf{m} \text{ and } \mathbf{m}' \text{ are in two different CCs of } \{Y < a\}\right\}$$

is well defined and $\{Y < \sigma\}$ has at least 2 CCs $\Omega \ni \mathbf{m}$ and $\Omega' \ni \mathbf{m}'$. Moreover, σ is a separating saddle value and Ω (as well as Ω') is critical.

Proof. We can assume that $Y(\mathbf{m}) \leq Y(\mathbf{m}')$ so taking $a := \inf_{\mathcal{A}} Y$ where $\mathcal{A}$ is a well chosen annulus centered in $\mathbf{m}'$, we see that

$$\left\{a \in \mathbb{R}\,;\, \mathbf{m} \text{ and } \mathbf{m}' \text{ are in two different CCs of } \{Y < a\}\right\} \neq \emptyset \tag{C.0.2}$$

and it is then clear that σ is well defined. Besides, if $(\sigma_n)_{n\geq 1}$ is an increasing sequence in the set from (C.0.2) that converges towards σ and $\gamma : [0,1] \to \mathbb{R}^{d'}$ is a continuous path linking $\mathbf{m}$ and $\mathbf{m}'$, then

$$\gamma([0,1]) \cap \left(\mathbb{R}^{d'} \backslash \{Y < \sigma\}\right) = \bigcap_{n\geq 1} \gamma([0,1]) \cap \left(\mathbb{R}^{d'} \backslash \{Y < \sigma_n\}\right)$$

is non empty by compactness so we can consider $\Omega \ni \mathbf{m}$ and $\Omega' \ni \mathbf{m}'$ two different CCs of $\{Y < \sigma\}$. To prove that σ is a separating saddle value, we will actually show that there exists a CC of $\{Y < \sigma\}$ that we denote Ω'' which is not Ω and satisfies $\overline{\Omega} \cap \overline{\Omega''} \neq \emptyset$. Assume by contradiction that there exists $\varepsilon > 0$ such that $(\Omega + B(0,\varepsilon))\backslash\overline{\Omega}$ is included in $\{Y \geq \sigma\}$. In that case, the points of $(\Omega + B(0,\varepsilon))\backslash\overline{\Omega}$ on which Y takes the value σ are local minima of Y which is a Morse function, so there are finitely many such points. Thus, up to taking ε smaller, we can assume that

$$\Gamma := \mathrm{dist}(\cdot, \Omega)^{-1}(\{\varepsilon\}) \subseteq \{Y > \sigma\}.$$

Hence there exists $\delta > 0$ such that the minimum of Y on Γ is $\sigma + \delta$. Since any continuous path linking $\mathbf{m}$ and $\mathbf{m}'$ has to cross Γ, $\mathbf{m}$ and $\mathbf{m}'$ are in two different CCs of $\{Y < \sigma + \delta/2\}$. This contradicts the maximality of σ and proves the existence of Ω''. Hence, Lemma C.0.2 implies that $\overline{\Omega} \cap \overline{\Omega''} \subseteq \mathcal{U}^{(1),Y}$ and then $\overline{\Omega} \cap \overline{\Omega''} \subseteq \mathtt{V}^{(1),Y}$ follows obviously from the definition of $\mathtt{V}^{(1),Y}$. ☐

Thanks to Lemma C.0.4, we know that $\mathtt{V}^{(1),Y} \neq \emptyset$. Let us then denote $\sigma_2 > \cdots > \sigma_N$ where $N \geq 2$ the different separating saddle values of Y and for convenience we set $\sigma_1 = +\infty$. We call *labeling* of the minima of Y any injection $l : \mathcal{U}^{(0),Y} \to [\![1, N]\!] \times \mathbb{N}^*$. If $l(\mathbf{m}) = (k, j)$, we denote for shortness $\mathbf{m} = \mathbf{m}_{k,j}$. We are going to introduce the usual labeling of the minima for a potential Y (see for instance [18, 23, 26]). We adopt a slightly unusual point of view in order to facilitate the establishment of the correspondence between the constructions for W and the ones for $V/2$ that we will state later on. For $\sigma \in \mathbb{R} \cup \{+\infty\}$, let us denote $\mathcal{C}_\sigma^Y$ the set of all the CCs of $\{Y < \sigma\}$. Given a labeling l of the minima, we denote for $k \in [\![1, N]\!]$

$$\mathtt{U}_k^{(0),Y} = l^{-1}([\![1, k]\!] \times \mathbb{N}^*) \cap \{Y < \sigma_k\}$$

and we say that the labeling is *adapted* to the separating saddle values if for all $k \in [\![1, N]\!]$, each element of $l^{-1}(\{k\} \times \mathbb{N}^*)$ is a global minimum of Y restricted to some CC of $\{Y < \sigma_k\}$ and the map

$$T_k^Y : \mathtt{U}_k^{(0),Y} \to \mathcal{C}_{\sigma_k}^Y \tag{C.0.3}$$

sending $\mathbf{m} \in \mathtt{U}_k^{(0),Y}$ on the element of $\mathcal{C}_{\sigma_k}^Y$ to which it belongs is bijective. In particular, $l^{-1}(\{k\} \times \mathbb{N}^*)$ is contained in $\mathtt{U}_k^{(0),Y}$.

Lemma C.0.5. *There exists a labeling of the minima of Y adapted to its separating saddle values.*

Proof. We are going to show that the usual labeling procedure introduced in [23] that we call l is adapted to the separating saddle values. We begin by taking

$$N_1 = \#\{CCs \text{ of } \{Y < \sigma_1\}\} = 1,$$

$E_{1,1} = \mathbb{R}^{d'}$ is the only CC of $\{Y < \sigma_1\}$ and $\mathbf{m}_{1,1} = \underline{\mathbf{m}}$ a global minimum of Y (chosen arbitrarily if there are several). The other minima will have a label in $[\![2, N]\!] \times \mathbb{N}^*$. We now proceed by induction : suppose that there exists $1 \le k < N$ and $\mathcal{W}_k \subseteq \mathcal{U}^{(0),Y}$ such that the values of l on $\mathcal{W}_k$ have already been fixed in $[\![1, k]\!] \times \mathbb{N}^*$ and the values of l on $\mathcal{U}^{(0),Y} \backslash \mathcal{W}_k$ (if not empty) will be fixed later in $[\![k+1, N]\!] \times \mathbb{N}^*$. This way, we already know the set $l^{-1}(\{k\} \times \mathbb{N}^*)$ as well as the fact that $\mathtt{U}_k^{(0),Y} = \mathcal{W}_k \cap \{Y < \sigma_k\}$. Suppose moreover that each element of $l^{-1}(\{k\} \times \mathbb{N}^*)$ is a global minimum of Y restricted to some CC of $\{Y < \sigma_k\}$ and that $T_k^Y : \mathtt{U}_k^{(0),Y} \to \mathcal{C}_{\sigma_k}^Y$ is bijective.
There exists $\mathbf{s}_{k+1} \in \mathtt{V}^{(1),Y}$ such that $Y(\mathbf{s}_{k+1}) = \sigma_{k+1}$ and $\mathbf{s}_{k+1}$ is in the closure of two distinct CCs of $\{Y < \sigma_{k+1}\}$ according to Lemma C.0.1 and Definition C.0.3. Assume by contradiction that these two CCs each contain an element $\mathbf{m}$ and $\mathbf{m}'$ that are both in $\mathcal{W}_k$. Then there exists a continuous path linking $\mathbf{m}$ and $\mathbf{m}'$ included in $\{Y \le \sigma_{k+1}\} \subset \{Y < \sigma_k\}$ which contradicts the bijectivity from the induction hypothesis. Hence $\{Y < \sigma_{k+1}\}$ has $N_{k+1} \ge 1$ CCs in which no minimum has been labeled yet. We denote these CCs $E_{k+1,1}, \cdots, E_{k+1,N_{k+1}}$ and for $1 \le j \le N_{k+1}$, we label $\mathbf{m}_{k+1,j}$ a global minimum of $Y|_{E_{k+1,j}}$ (chosen arbitrarily if there are several). Setting $\mathcal{W}_{k+1} = \mathcal{W}_k \cup \{\mathbf{m}_{k+1,j}\,;\, 1 \le j \le N_{k+1}\}$, the possible remaining minima in $\mathcal{U}^{(0),Y} \backslash \mathcal{W}_{k+1}$ will have a label in $[\![k+2, N]\!] \times \mathbb{N}^*$. By construction, each element of $l^{-1}(\{k+1\} \times \mathbb{N}^*)$ is a global minimum of a CC of $\{Y < \sigma_{k+1}\}$ and it is clear that the map $T_{k+1}^Y : \mathtt{U}_{k+1}^{(0),Y} \to \mathcal{C}_{\sigma_{k+1}}^Y$ is surjective. It is also clear that the $(E_{k+1,j})_{1 \le j \le N_{k+1}}$ each contain at most one element from $\mathcal{W}_{k+1}$. If E is another element of $\mathcal{C}_{\sigma_{k+1}}^Y$, then it does not contain any $\mathbf{m}_{k+1,j}$ and since E is contained in a CC of $\{Y < \sigma_k\}$, it cannot contain two elements of $\mathcal{W}_k$ by the bijectivity from the induction hypothesis. Thus we have shown the induction step.

Let us now check that once we have treated the case $k = N$, all the local minima have been labeled, that is $\mathcal{W}_N = \mathcal{U}^{(0),Y}$. Assume by contradiction that there exists $\mathbf{m} \in \mathcal{U}^{(0),Y} \backslash \mathcal{W}_N$. We can apply Lemma C.0.4 to $\mathbf{m}$ and $\underline{\mathbf{m}}$ and we get that there exists $2 \le k \le N$ such that $\mathbf{m}$ is in a CC of $\{Y < \sigma_k\}$. By bijectivity of T_k^Y, this CC also contains some $\mathbf{m}' \in \mathcal{W}_k$. Once again, we can apply Lemma C.0.4 to $\mathbf{m}$ and $\mathbf{m}'$ to get a separating saddle value $\sigma < \sigma_k$ such that $\mathbf{m} \in \{Y < \sigma\}$. Continuing this process, we would get by induction that $\mathbf{m} \in \{Y < \sigma_N\}$ and then find a separating saddle value $\sigma < \sigma_N$, which is absurd. Hence, we have constructed a labeling l of the minima of Y which is adapted to the separating saddle values. □

Lemma C.0.6. *Under an adapted labeling of the minima of Y, for any $2 \le k \le N$, the elements of $T_k^Y\big(l^{-1}(\{k\} \times \mathbb{N}^*)\big)$ are critical.*

Proof. Let $\mathbf{m}_{k,j} \in l^{-1}(\{k\} \times \mathbb{N}^*)$. There exists a CC of $\{Y < \sigma_{k-1}\}$ that we call E which is such that $T_k^Y(\mathbf{m}_{k,j}) \subseteq E$ and E contains some $\mathbf{m}_{k',j'} \in E$ for $1 \le k' \le k-1$ and $j' \in \mathbb{N}^*$ by bijectivity of T_{k-1}^Y. Therefore, $\mathbf{m}_{k',j'}$ and $\mathbf{m}_{k,j}$ are in the same CC of $\{Y < \sigma_{k-1}\}$ but are not both in $T_k^Y(\mathbf{m}_{k,j})$ this time by bijectivity of T_k^Y. Applying Lemma C.0.4 to $\mathbf{m}_{k',j'}$ and $\mathbf{m}_{k,j}$, we obtain a separating saddle value $\tilde{\sigma}$ which is the maximal real number such that $\mathbf{m}_{k',j'}$ and $\mathbf{m}_{k,j}$ are in two different CCs of $\{Y < \tilde{\sigma}\}$. Therefore we get $\tilde{\sigma} = \sigma_k$ so $T_k^Y(\mathbf{m}_{k,j})$ is one of the CCs of $\{Y < \tilde{\sigma}\}$ called Ω and Ω' in Lemma C.0.4 and which are critical. □

Definition C.0.7. *Recall the notation* (C.0.1) *and Definition C.0.3. Given an adapted labeling $(\mathbf{m}_{k,j})_{k,j}$, we can now define the following mappings :*

- $E^Y : \mathcal{U}^{(0),Y} \longrightarrow \mathcal{P}(\mathbb{R}^{d'})$
  $\mathbf{m}_{k,j} \longmapsto T_k^Y(\mathbf{m}_{k,j})$
  where T_k^Y is the map defined in (C.0.3).
- $\mathbf{j}^Y : \mathcal{U}^{(0),Y} \to \mathcal{P}\big(\mathtt{V}^{(1),Y} \cup \{\mathbf{s}_1\}\big)$
  given by $\mathbf{j}^Y(\mathbf{m}_{1,1}) = \mathbf{s}_1$ where $\mathbf{s}_1$ is a fictive saddle point such that $Y(\mathbf{s}_1) = \sigma_1 = +\infty$; and for $2 \le k \le N$, $\mathbf{j}^Y(\mathbf{m}_{k,j}) = \partial E^Y(\mathbf{m}_{k,j}) \cap \mathtt{V}^{(1),Y}$ which is not empty according to Lemma C.0.6 and included in $\{Y = \sigma_k\}$.

- $\boldsymbol{\sigma}^Y : \mathcal{U}^{(0),Y} \to Y(\mathtt{V}^{(1),Y}) \cup \{\sigma_1\}$
  $\mathbf{m} \mapsto Y(\mathbf{j}^Y(\mathbf{m}))$
  where we allow ourselves to identify the set $Y(\mathbf{j}^Y(\mathbf{m}))$ and its unique element in $Y(\mathtt{V}^{(1),Y}) \cup \{\sigma_1\}$.
- $S^Y : \mathcal{U}^{(0),Y} \longrightarrow]0,+\infty]$
  $\mathbf{m} \longmapsto \boldsymbol{\sigma}^Y(\mathbf{m}) - Y(\mathbf{m})$.

We can now state the so-called "generic assumption" which appears in many works dealing with the study of semiclassical operators associated to some potential.

Hypothèse de non dégénérescence C.0.8. *Let $(\mathbf{m}_{k,j})_{k,j}$ a labeling adapted to Y. For all $\mathbf{m} \in \mathcal{U}^{(0)}$, we have*

a) $\mathbf{m}$ *is the only global minimum of $Y|_{E^Y(\mathbf{m})}$*

b) for any $\mathbf{m}' \in \mathcal{U}^{(0)} \backslash \{\mathbf{m}\}$, the sets $\mathbf{j}^Y(\mathbf{m})$ and $\mathbf{j}^Y(\mathbf{m}')$ do not intersect.

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Metastabilité de processus non locaux

Résumé : On étudie dans cette thèse des équations de Boltzmann inhomogènes, linéaires, dans un régime de basse température et en présence d'une force extérieure dérivant d'un potentiel. On s'intéresse plus particulièrement au spectre près de 0 des opérateurs associés dont on cherche à fournir une description précise. Cette dernière nous permet de récupérer des informations avancées sur le comportement en temps long des solutions avec notamment des résultats quantitatifs de retour à l'équilibre et de metastabilité.
On commence par traiter le cas d'opérateurs de collisions de type "relaxation douce" qui se trouvent être des opérateurs pseudo-différentiels présentant de bonnes propriétés microlocales. L'approche adoptée consiste à utiliser et adapter à notre cadre non local des constructions de quasimodes (fonctions propres approchées) récemment développées pour l'étude d'opérateurs différentiels de type Fokker-Planck. Ces dernières reposent en partie sur des estimations de résolvante que l'on obtient via des méthodes hypocoercives. On établit alors la description du spectre désirée dans le cas d'un potentiel de Morse satisfaisant une hypothèse générique.
À travers un exemple relativement simple d'opérateur semiclassique, elliptique et non local, on montre ensuite que cette hypothèse peut être relaxée et les constructions mises en place dans le cas d'un potentiel de Morse général.
Enfin, on s'intéresse à l'opérateur de Boltzmann de "relaxation linéaire" qui correspond au modèle BGK. Ce dernier s'avère également être un opérateur pseudo-différentiel mais présente de "mauvaises" propriétés microlocales qui font échouer les méthodes employées dans le cas de la relaxation douce. On surmonte ces difficultés en introduisant une superposition de quasimodes gaussiens inspirée des constructions précédentes et grâce à laquelle on parvient là encore à récupérer une formule d'Eyring-Kramers pour le spectre de cet opérateur.

Mots-clés : Équations aux dérivées partielles, Analyse spectrale, Analyse semiclassique

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Metastability of non local processes

Abstract : In this thesis, we study some inhomogeneous linear Boltzmann equations in a low temperature regime and in the presence of an external force deriving from a potential. We are particularly interested in the spectrum near 0 of the associated operators of which we aim to give an accurate description. Such a description enables us to obtain some precise information on the long time behavior of the solutions with in particular some quantitative results of return to equilibrium and metastability.
We start by treating the case of collision operators of "mild relaxation" type which are pseudo-differential operators presenting some nice microlocal properties. The approach that we adopt consists in using and adapting to our non local framework some quasimodal (approximated eigenfunctions) constructions recently developed for the study of Fokker-Planck type differential operators. These partly rely on some resolvent estimates obtained via hypocoercive methods. We then establish the desired description of the spectrum in the case of a Morse potential satisfying a generic hypothesis.
Through a fairly simple example of semiclassical elliptic and non local operator, we then show that this generic hypothesis can be relaxed and that the constructions can be done in the case of a general Morse potential.
Finally, we consider the "linear relaxation" Boltzmann operator which corresponds to the BGK model. It also appears to be a pseudo-differential operator but presenting some "bad" microlocal properties causing the failure of the methods used for the mild relaxation. We overcome these difficulties by introducing a superposition of gaussian quasimodes inspired by the previous constructions and thanks to which we are here again able to obtain an Eyring-Kramers formula for the spectrum of this operator.

Keywords : Partial differential equations, Spectral analysis, Semiclassical analysis

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