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Karim Kellay, Thomas Normand and Marius Tucsnak

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# SHARP REACHABILITY RESULTS FOR THE HEAT EQUATION IN ONE SPACE DIMENSION

KARIM KELLAY, THOMAS NORMAND AND MARIUS TUCSNAK

This paper gives a complete characterization of the reachable space for a system described by the 1-dimensional heat equation with $L^2$ (with respect to time) Dirichlet boundary controls at both ends. More precisely, we prove that this space coincides with the sum of two spaces of analytic functions (of Bergman type). These results are then applied to give a complete description of the reachable space via inputs which are $n$-times differentiable functions of time. Moreover, we establish a connection between the norm in the obtained sum of Bergman spaces and the cost of null controllability in small time. Finally we show that our methods yield new complex analytic results on the sums of Bergman spaces in infinite sectors.

## 1. Introduction

Determining the reachable space of a controlled dynamical system is a major question in control theory. Knowledge of this set gives important information on our capability of acting on the state of a system and for safety verifications. This fundamental question is well understood for linear finite-dimensional systems (see Section 2 below for some background material) but much less is known for time-invariant linear infinite-dimensional systems (namely those governed by partial differential equations). Most of the known results in this context concern the case when the system is exactly controllable, which means, as stated below, that the reachable state coincides with the state space of the system. When the reachable space is a strict subspace of the state space, its description is generally far from complete. In this work we focus on a case which might look very elementary but which encompasses a rich structure: a system described by the heat equation in one space dimension with Dirichlet boundary control. The first results on this problem go back to the seminal paper of [Fattorini and Russell 1971] (see also [Ervedoza and Zuazua 2011] for first improvements by a different approach). More refined estimates have been obtained only during the last three years, see work of Martin, Rosier and Rouchon [Martin et al. 2016], Dardé and Ervedoza [2018] and Hartmann, Kellay and Tucsnak [Hartmann et al. 2020]. The results in the papers quoted above reveal surprising and deep connections between controllability and reachability theory for the heat equation and spaces of analytic or Gevrey-type functions and open the way towards new applications, namely for the control of nonlinear parabolic equations and for time optimal control problems (with point target) for the heat equation. The main contribution brought in by the present work

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consists in providing a complete characterization of the reachable space of the system described by the heat equation in one space dimension with Dirichlet boundary control. Our results, presented in terms of sums of classical Hilbert spaces of analytic functions, are sharp in the following sense: unlike the existing results quoted above, which assert that the reachable space is sandwiched between two spaces of analytic functions, we prove that this space *coincides* with the sum of two spaces of analytic functions. This main result is further applied in obtaining a complete characterization of the space of functions which can be reached by smooth (in a Sobolev scale) inputs and then in deriving an estimate for the cost of null controllability in small time. Note that very recently Orsoni [2021] gave an apparently different characterization of the reachable space for the same system. This result motivated the discussion in Section 7 below, where we obtain new connections, which might be of independent interest, concerning sums of possibly weighted Bergman spaces.

In the remaining part of this introduction we give an overview of the existing theory and we state the main results which will be proved in the following sections.

We consider the system

$$\begin{cases} \dfrac{\partial w}{\partial t}(t,x) = \dfrac{\partial^2 w}{\partial x^2}(t,x), & t \geqslant 0,\ x \in (0,\pi), \\ w(t,0) = u_0(t), \quad w(t,\pi) = u_\pi(t), & t \in [0,\infty), \\ w(0,x) = 0, & x \in (0,\pi), \end{cases} \tag{1-1}$$

which models the heat propagation in a rod of length $\pi$, controlled by prescribing the temperature at both ends. It is well known that for every $u_0, u_\pi \in L^2[0,\infty)$ the problem (1-1) admits a unique solution $w$ and that the restriction of this function to $(0,\infty) \times (0,\pi)$ is an analytic function. The *input-to-state maps* (briefly, input maps) $(\Phi_\tau)_{\tau \geqslant 0}$ are defined by

$$\Phi_\tau \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} = w(\tau,\cdot), \qquad \tau \geqslant 0,\ u_0, u_\pi \in L^2[0,\tau]. \tag{1-2}$$

*Determining the reachable space at instant* $\tau$ *of the system determined by the* 1*-dimensional heat equation with boundary control consists in determining* $\operatorname{Ran} \Phi_\tau$.[1] For a long time, this question was considered elusive, so efforts went first towards determining the largest possible subspaces of $\operatorname{Ran} \Phi_\tau$. As mentioned above, the beginnings of the research in this direction go back to [Fattorini and Russell 1971], where it is shown that $\operatorname{Ran} \Phi_\tau$ contains the space of continuous functions which are $2\pi$-periodic on $\mathbb{R}$, which extend holomorphically on the strip $|\operatorname{Im} z| < \frac{\pi}{2}$ and with all the derivatives of even order vanishing at $x = 0$ and $x = \pi$. This last condition is quite restrictive since it does not provide information on the reachability of very smooth functions (like polynomials) which are not vanishing at $x = 0$ and $x = \pi$. This lack of information has been partially filled in by [Schmidt 1986], where it was proved that particular types of smooth functions (in particular polynomials) are in the reachable space, independently of their boundary values.

[1] An alternative concept of reachable space which might seem natural involves, for every $w_0 \in W^{-1,2}(0,\pi)$, the operator $R_\tau\left(w_0; \left[\begin{smallmatrix} u_0 \\ u_\pi \end{smallmatrix}\right]\right) = \tilde{w}(\tau,\cdot)$, where $\tilde{w}$ satisfies the first two equations in (1-1) with the initial condition $\tilde{w}(0,\cdot) = w_0$. However, as explained in Remark 3.8 below, in the case of the system described by first two equations in (1-1), for every $\tau > 0$ we have $\operatorname{Ran} R_\tau = \operatorname{Ran} \Phi_\tau$

Interest in this fascinating question was revealed by the recent work [Martin et al. 2016]. To give a precise statement of the main recent contributions to this problem we need some notation which will be used in the remaining part of this work.

Given an open set $\Omega$ containing $(0,\pi)$, we denote by $\operatorname{Hol}(\Omega)$ the space of continuous functions on $(0,\pi)$ admitting a holomorphic extension to $\Omega$. Moreover we identify these functions with their holomorphic extensions to $\Omega$. The article [Martin et al. 2016] is, to our knowledge, the first work proving that $\operatorname{Ran}\Phi_\tau$ can be sandwiched between two spaces of analytic functions. More precisely, the main result in [Martin et al. 2016] asserts that

$$\operatorname{Hol}(\widetilde{D}) \subset \operatorname{Ran}\Phi_\tau \subset \operatorname{Hol}(D),$$

where $\widetilde{D}$ in the disk centered in $\frac{\pi}{2}$ and of diameter $\pi \mathrm{e}^{(2\mathrm{e})^{-1}}$ and

$$D = \{s = x + iy \in \mathbb{C} \mid |y| < x \text{ and } |y| < \pi - x\}. \tag{1-3}$$

This result was further improved in [Dardé and Ervedoza 2018], where it was shown that for every $\varepsilon > 0$ we have

$$\operatorname{Hol}(D_\varepsilon) \subset \operatorname{Ran}\Phi_\tau \subset \operatorname{Hol}(D),$$

where $D_\varepsilon$ is the set of those $s \in \mathbb{C}$ such that $\operatorname{dist}(s, D) < \varepsilon$. A significant advancement towards a characterization, in terms of Banach spaces of analytic functions, of $\operatorname{Ran}\Phi_\tau$ was obtained in [Hartmann et al. 2020]. In this work it was proved that

$$E^2(D) \subset \operatorname{Ran}\Phi_\tau \subset A^2(D), \tag{1-4}$$

where $E^2(D)$ and $A^2(D)$ are the Hardy–Smirnov and Bergman spaces on $D$, respectively. For the reader's convenience we give simplified definitions of these spaces, which are

$$E^2(D) = \left\{ f \in \operatorname{Hol}(D) \cap L^2(\partial D) \;\middle|\; \int_{\partial D} f(\zeta)\zeta^n \,\mathrm{d}z = 0 \text{ for all } n \geqslant 1 \right\}, \tag{1-5}$$

$$A^2(D) = \operatorname{Hol}(D) \cap L^2(D). \tag{1-6}$$

When these spaces are endowed, respectively, with norms

$$\|f\|^2_{E^2(D)} = \int_{\partial D} |f(\zeta)|^2 \,|\mathrm{d}\zeta|,$$

$$\|f\|^2_{A^2(D)} = \int_D |f(x+iy)|^2 \,\mathrm{d}x\,\mathrm{d}y,$$

they become Hilbert spaces. The main new result in this paper is a complete characterization of $\operatorname{Ran}\Phi_\tau$ in terms of the sum of two weighted Bergman spaces. A surprising consequence of this result is that although each one of these spaces depends on a parameter $\delta > 0$, their sum is independent of this parameter (first for $\delta$ small enough and then for all $\delta > 0$, see Theorem 1.1 and Proposition 1.2 below). To state this result we introduce the sets

$$\Delta = \left\{ s \in \mathbb{C} \mid -\tfrac{\pi}{4} < \arg s < \tfrac{\pi}{4} \right\}, \qquad \tilde{\Delta} = \pi - \Delta, \tag{1-7}$$

the weight functions

$$\omega_{0,\delta}(s) = \frac{e^{\operatorname{Re}(s^2)/(2\delta)}}{\delta}, \qquad \delta > 0,\ s \in \Delta, \tag{1-8}$$

$$\omega_{\pi,\delta}(\tilde{s}) = \frac{e^{\operatorname{Re}[(\pi-\tilde{s})^2]/(2\delta)}}{\delta}, \quad \delta > 0,\ \tilde{s} \in \tilde{\Delta}, \tag{1-9}$$

and the weighted Bergman spaces $A^2(\Delta, \omega_{0,\delta})$ and $A^2(\tilde{\Delta}, \omega_{\pi,\delta})$:

$$A^2(\Delta, \omega_{0,\delta}) = \left\{ f \in \operatorname{Hol}(\Delta) \;\middle|\; \int_\Delta |f(x+iy)|^2\, \omega_{0,\delta}(x+iy)\, dx\, dy < \infty \right\},$$

$$A^2(\tilde{\Delta}, \omega_{\pi,\delta}) = \left\{ f \in \operatorname{Hol}(\tilde{\Delta}) \;\middle|\; \int_{\tilde{\Delta}} |f(x+iy)|^2\, \omega_{\pi,\delta}(x+iy)\, dx\, dy < \infty \right\}.$$

When endowed with the norms

$$\|f\|^2_{A^2(\Delta,\omega_{0,\delta})} = \int_\Delta |f(x+iy)|^2\, \omega_{0,\delta}(x+iy)\, dx\, dy, \quad f \in A^2(\Delta, \omega_{0,\delta}),$$

$$\|\tilde{f}\|^2_{A^2(\tilde{\Delta},\omega_{\pi,\delta})} = \int_{\tilde{\Delta}} |\tilde{f}(x+iy)|^2\, \omega_{\pi,\delta}(x+iy)\, dx\, dy, \quad \tilde{f} \in A^2(\tilde{\Delta}, \omega_{\pi,\delta}),$$

$A^2(\Delta, \omega_{0,\delta})$ and $A^2(\tilde{\Delta}, \omega_{\pi,\delta})$ become Hilbert spaces. An important role in the remaining part of this work will be played by the sum of the two spaces above, i.e., the space $X_\delta$ defined for every $\delta > 0$ by

$$X_\delta = \big\{\psi \in C(0,\pi) \mid \text{there exist } \varphi_0 \in A^2(\Delta, \omega_{0,\delta}) \text{ and } \varphi_\pi \in A^2(\tilde{\Delta}, \omega_{\pi,\delta}) \text{ such that } \psi = \varphi_0 + \varphi_\pi \text{ on } (0,\pi)\big\}, \tag{1-10}$$

which is endowed with the norm

$$\|\varphi\|_\delta = \inf\{\|\varphi_0\|_{A^2(\Delta,\omega_{0,\delta})} + \|\varphi_\pi\|_{A^2(\tilde{\Delta},\omega_{\pi,\delta})} \mid \varphi_0 + \varphi_\pi = \varphi,\ \ \varphi_0 \in A^2(\Delta, \omega_{0,\delta}),\ \ \varphi_\pi \in A^2(\tilde{\Delta}, \omega_{\pi,\delta})\}. \tag{1-11}$$

**Theorem 1.1.** *For every $\tau > 0$ we have*

$$\Phi_\tau \in \mathcal{L}(L^2([0,\tau]; \mathbb{C}^2); X_\tau). \tag{1-12}$$

*Moreover, there exists $\delta^* > 0$ such that for every $\tau > 0$ and every $\delta \in (0, \delta^*)$ we have*

$$\operatorname{Ran} \Phi_\tau = X_\delta, \quad \tau > 0. \tag{1-13}$$

According to a general property of control linear time-invariant systems which are null controllable in any time (see Proposition 3.4 below), it is known that $\operatorname{Ran} \Phi_\tau$ is independent of $\tau > 0$. This fact is one of the ingredients of the proof of Theorem 1.1, as shown in Section 4. The fact that the sum of the Bergman spaces in the right-hand side of (1-10) is independent of $\delta \in (0, \delta^*)$ seems to be a new result, which is strengthened by the following proposition, which will be proved in Section 7.

**Proposition 1.2.** *For each $\delta > 0$ let $X_\delta$ be the space defined in (1-10). Then*

$$X_t = X_\tau, \quad t,\ \tau > 0.$$

Putting together Theorem 1.1 and Proposition 1.2 we obtain:

**Corollary 1.3.** *With the notation in Theorem 1.1 we have*

$$\operatorname{Ran} \Phi_\tau = X_\delta, \qquad \tau,\ \delta > 0. \tag{1-14}$$

The remaining part of this work is organized as follows. In Section 2, in order to introduce the appropriate vocabulary, we state some necessary concepts from linear finite-dimensional systems theory. Section 3 is devoted to some background on infinite-dimensional well-posed linear systems, with emphasis on the concept of reachable space and on its general properties. Moreover, we end this section by showing that the system determined by the boundary-controlled heat equation fits the introduced abstract framework and we describe the corresponding output maps. Section 4 is devoted to the proof of the main result, which gives a complete characterization of the reachable space. In Section 5 this result is applied to give a complete characterization of the reachable space obtained via controls which are $n$ times differentiable and with derivatives up to order $n-1$ vanishing at the initial time. Section 6 provides a link, is of possible interest for improving the existing estimates on the control cost in small time, between the natural norm on the reachable space and the above-mentioned control cost. Finally, in Section 7 we discuss the implications of our results on sums of Bergman spaces on infinite sectors and we obtain a new proof of a different characterization of the reachable space, recently obtained in [Orsoni 2021].

***Notation.*** Throughout this paper, $\mathbb{N}$, $\mathbb{Z}$ stand for the sets of natural numbers (starting with 1) and integer numbers, respectively. We set $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$.

## 2. Some background on finite-dimensional linear time-invariant systems

In this section, in order to introduce several concepts and operators in a simple motivating case, we briefly give some well-known facts for linear time-invariant systems (LTIs) in the finite-dimensional case. For more details we refer to good introductory chapters on this classical subject, such as those of D'Azzo and Houpis [1975], Friedland [1986], Ionescu, Oară and Weiss [Ionescu et al. 1999], Kwakernaak and Sivan [1972], Maciejowski [1989], Rugh [1993] and Wonham [1974].

Let $U$ (the input space) and $X$ (the state space) be finite-dimensional inner product spaces, with $\dim X = n$. A finite-dimensional linear time-invariant control system with input space $U$ and state space $X$ is traditionally described by the equations

$$\dot{z}(t) = Az(t) + Bu(t), \qquad t \geqslant 0. \tag{2-1}$$

In the above equations $u \in L^2([0, \infty); U)$ is the *input function* and $z \in C([0, \infty); X)$ is the *state trajectory*, whereas $A$, $B$ are linear operators such that $A : X \to X$, and $B : U \to X$. By the variation of constants formula (sometimes called Duhamel's formula) equation (2-1) yields

$$z(t) = \mathrm{e}^{tA} z(0) + \int_0^t \mathrm{e}^{(t-\sigma)A} Bu(\sigma)\,\mathrm{d}\sigma, \qquad t \geqslant 0. \tag{2-2}$$

In the above formula we can notice the appearance of two families of operators $\mathbb{T} = (\mathbb{T}_t)_{t\geqslant 0}$ (the $C^0$ semigroup on $X$ generated by $A$) and $\Phi = (\Phi_t)_{t\geqslant 0}$ (the input to state maps) defined by

$$\mathbb{T}_t\varphi = \mathrm{e}^{tA}\varphi, \qquad t \geqslant 0,\ \varphi \in X, \tag{2-3}$$

$$\Phi_t u = \int_0^t e^{(t-\sigma)A} Bu(\sigma)\,\mathrm{d}\sigma, \quad t \geqslant 0,\ u \in L^2([0,\infty); U). \tag{2-4}$$

The two operator families above have important properties (which will be used below to define general well-posed control systems). More precisely:

- $\mathbb{T} = (\mathbb{T}_t)_{t\geqslant 0}$ is a $C^0$ semigroup of operators (briefly an *operator semigroup*) on $X$, which means that $\mathbb{T}_t \in \mathcal{L}(X)$ for every $t \geqslant 0$ and

$$\mathbb{T}_0\varphi = \varphi, \qquad \varphi \in X, \tag{2-5}$$

$$\mathbb{T}_{t+\tau} = \mathbb{T}_t\mathbb{T}_\tau, \quad t, \tau \geqslant 0, \tag{2-6}$$

$$\lim_{t\to 0+} \mathbb{T}_t\varphi = \varphi, \qquad \varphi \in X. \tag{2-7}$$

- For every $t \geqslant 0$ we have $\Phi_t \in \mathcal{L}(L^2([0,\infty); U), X)$ and

$$\Phi_{\tau+t}(u \underset{\tau}{\diamondsuit} v) = \mathbb{T}_t\Phi_\tau u + \Phi_t v, \quad t, \tau \geqslant 0, \tag{2-8}$$

where the $\tau$*-concatenation* of two signals $u$ and $v$, denoted by $u \underset{\tau}{\diamondsuit} v$, is the function

$$u \underset{\tau}{\diamondsuit} v = \begin{cases} u(t) & \text{for } t \in [0,\tau), \\ v(t-\tau) & \text{for } t \geqslant \tau. \end{cases} \tag{2-9}$$

Let us also note that, with the above notation, formula (2-2) can be rewritten

$$z(t) = \mathbb{T}_t z(0) + \Phi_t u, \quad t \geqslant 0. \tag{2-10}$$

**Definition 2.1.** Given a finite-dimensional LTI control system described by (2-1) and $\tau > 0$, the reachable space of this system at time $\tau > 0$ is the range $\operatorname{Ran}\Phi_\tau$ of the operator $\Phi_\tau$ defined in (2-4). The system is said to be controllable in time $\tau$ if $\operatorname{Ran}\Phi_\tau = X$.

The result below shows that, within the very simple framework considered in this section, the reachable space and the controllability property do not depend on the time $\tau > 0$. More precisely, the following result, known as the *Kalman rank condition* for controllability holds:

**Theorem 2.2.** *We have, for every* $\tau > 0$,

$$\operatorname{Ran}\Phi_\tau = \operatorname{Ran}\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}. \tag{2-11}$$

*Moreover, the pair* $(A, B)$ *is controllable if and only if*

$$\operatorname{rank}\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} = n. \tag{2-12}$$

For exactly controllable systems there exist controls steering the state trajectory $z$ of (2-1) from any initial state $z_0$ to any final state $z_1$. Among these controls there is one of "minimal energy". To state these facts in a precise manner we need the concept of the controllability Gramian $R_\tau$ of the pair $(A, B)$, which is defined by

$$R_\tau = \Phi_\tau \Phi_\tau^*, \qquad \tau \geqslant 0.$$

It is easily checked that $R_\tau \in \mathcal{L}(X)$ can alternatively be written as

$$R_\tau = \int_0^\tau \mathrm{e}^{tA} B B^* \mathrm{e}^{tA^*} \, \mathrm{d}t,$$

and that $(A, B)$ is controllable if and only if $R_\tau$ is a positive operator. Moreover, we have:

**Proposition 2.3.** *Suppose that $(A, B)$ is controllable and let $z_0, z_1 \in X$, $\tau > 0$. If*

$$u = \Phi_\tau^* R_\tau^{-1}(z_1 - \mathbb{T}_\tau z_0), \tag{2-13}$$

*then the corresponding state trajectory $z$ of (2-1) satisfies $z(0) = z_0$ and $z(\tau) = z_1$. Moreover, among all the inputs $v \in L^2([0, \tau]; U)$ for which $z(0) = z_0$ and $z(\tau) = z_1$, $u$ is the unique one that has minimal $L^2([0, \tau]; U)$ norm.*

**Remark 2.4.** From (2-11) it follows that $\operatorname{Ran} \Phi_\tau$ contains $\operatorname{Ran} B$ and it is invariant under $A$, and thus under $\mathbb{T}_t$ for every $t \geqslant 0$. Denoting by $\tilde{A}$ and $\widetilde{\mathbb{T}}$ the restrictions of $A$ and of $\mathbb{T}$ to $\operatorname{Ran} \Phi_\tau$, the above facts imply that the input maps of $(\tilde{A}, B)$ coincide with those of $(A, B)$. We thus have that $(\Phi, \widetilde{\mathbb{T}})$ (alternatively described by $(\tilde{A}, B)$) is an exactly controllable LTI system with state space $\operatorname{Ran} \Phi_\tau$ and input space $U$.

**Remark 2.5.** Noting that, given $\tau > 0$, the space $L^\infty([0, \tau]; U)$ can be seen as a subspace of $L^2([0, \infty); U)$ and having in mind that controls which can be effectively applied are generally bounded functions in time, a natural question is the characterization of the set $\operatorname{Ran} \Phi_\tau^\infty$, where $\Phi_\tau^\infty$ is the restriction of $\Phi_\tau$ to $L^\infty([0, \tau]; U)$. It turns out that, in the simple case considered in this section, $\operatorname{Ran} \Phi_\tau^\infty$ coincides with $\operatorname{Ran} \Phi_\tau$. Indeed, let $\eta \in \operatorname{Ran} \Phi_\tau$. As seen in Remark 2.4, the system $(\Phi, \widetilde{\mathbb{T}})$ is exactly controllable. Thus, the minimal $L^2([0, \tau]; U)$ control $\tilde{u}$ steering its state trajectory from 0 at $t = 0$ to $\eta$ at $t = \tau$ is given, according to Proposition 2.3, by

$$\tilde{u} = \Phi_\tau^* \widetilde{R}_\tau^{-1} \eta,$$

where $\widetilde{R}_\tau$ is the controllability Gramian in time $\tau$ of the system $(\tilde{A}, B)$. The control $\tilde{u}$ above clearly steers the state trajectory of the original system $(A, B)$ from the null state at $t = 0$ to the state $\eta$ at $t = \tau$ and $\tilde{u}$ obviously extends to an analytic function from $\mathbb{C}$ to $U$. We have thus shown the stronger property that $\operatorname{Ran} \Phi_\tau$ coincides with the range of the restriction of $\Phi_\tau$ to signals which can be extended to analytic functions from $\mathbb{C}$ to $U$.

## 3. Reachable space for well-posed linear control systems

We begin this section by stating some basic definitions and properties (mostly without proofs but with appropriate references) of well-posed linear control time-invariant systems, with emphasis on the concept of reachability and on properties of the reachable space. These systems provide a framework to generalize some of the concepts in classical linear control theory to infinite-dimensional systems. In particular, all finite-dimensional control LTI systems are well-posed in the sense of the definition below, which is general enough to include some basic examples of systems governed by partial differential equations (such as the heat, Schrödinger and wave equations) with boundary control. We generally do not give proofs and we refer to [Weiss 1989] (where these systems have been introduced under the name of *abstract control systems*) and [Tucsnak and Weiss 2009, Chapters 2,3,4; 2014] for more information, including detailed proofs. In the last part of this section we also provide proofs for two results which are quite simple but which play an important role in this paper.

Let $U$ (the input space) and $X$ (the state space) be Hilbert spaces (possibly infinite-dimensional). From a system-theoretic viewpoint the simplest way to define a linear well-posed time-invariant system in a possibly infinite-dimensional setting is to introduce families of operators, inspired by those in (2-3), (2-4) and sharing some of their properties.

**Definition 3.1.** Let $U$ and $Y$ be Hilbert spaces. A *well-posed linear control system* is a couple $(\mathbb{T}, \Phi)$ of families of operators such that

(1) $\mathbb{T} = (\mathbb{T}_t)_{t\geqslant 0}$ is an operator semigroup on $X$, i.e., it satisfies conditions (2-5)-(2-7);

(2) $\Phi = (\Phi_t)_{t\geqslant 0}$ is a family of bounded linear operators from $L^2([0,\infty); U)$ to $X$ such that (2-8) holds for every $u, v \in L^2([0,\infty); U)$ and all $\tau, t \geqslant 0$.

It follows from the above definition that $\Phi$ is *causal*, i.e., the state does not depend on the future input: $\Phi_\tau \Pi_\tau = \Phi_\tau$ for all $\tau \geqslant 0$, where $\Pi_\tau$ stands for the orthogonal projection from $L^2([0,\infty); U)$ onto $L^2[0,\tau); U)$. Moreover, it can be shown that the above properties imply that the map

$$(t, u) \mapsto \Phi_t u$$

is continuous from $[0,\infty) \times L^2([0,\infty); U)$ to $X$.

From a PDEs viewpoint, the above definition is, in general, not easy to use. In most of the cases encountered in applications, an infinite-dimensional system is described by evolution partial differential equations with appropriate boundary conditions (some of them being the boundary controls), and thus by partial differential and trace operators. To describe such a system in the terms of Definition 3.1 one needs to define a notion of solution of the considered PDE system and to prove appropriate existence and uniqueness results for these solutions (including the "correct" choices for $X$ and $U$) and allowing one to define the families of operators $(\mathbb{T}_t)$ and $(\Phi_t)$. In general there are no explicit formulas for these families of operators. However, as shown at the end of this section, such formulas are available for the system described by the first two equations in (1-1), so this viewpoint is quite convenient in our case.

As in the finite-dimensional case, the *reachable space* of a well-posed control system at time $\tau > 0$ is defined as $\operatorname{Ran} \Phi_\tau$. Unlike the finite-dimensional case, in this more general framework there is no simple characterization of the reachable space. Moreover, this space depends in general on $\tau$ and for most systems described by partial differential equations we have only a small amount of information on the reachable space. Another difference with respect to the finite-dimensional case is that the range $\operatorname{Ran} \Phi_\tau^\infty$ of the restriction of $\Phi_\tau$ to $L^\infty([0,\tau]; U)$ is in general a strict subset of $\operatorname{Ran} \Phi_\tau$. We also note that, given

$\tau > 0$, the reachable space $\operatorname{Ran} \Phi_\tau$ can be endowed with the norm induced from $L^2([0, \tau]; U)$, which is

$$\|\eta\|_{\operatorname{Ran} \Phi_\tau} = \inf_{\substack{u \in L^2([0,\tau];U) \\ \Phi_\tau u = \psi}} \|u\|_{L^2([0,\tau];U)}, \qquad \eta \in \operatorname{Ran} \Phi_\tau. \tag{3-1}$$

As mentioned in the Introduction, the concept of reachable set plays an essential role in control theory. It appears, in particular, in the definition of the main three controllability concepts used in infinite-dimensional system theory.

**Definition 3.2.** Let $\tau > 0$ and let the pair $(\mathbb{T}, \Phi)$ define a well-posed control LTI system.

- The pair $(\mathbb{T}, \Phi)$ is *exactly controllable in time* $\tau$ if $\operatorname{Ran} \Phi_\tau = X$.
- $(\mathbb{T}, \Phi)$ is *approximately controllable in time* $\tau$ if $\operatorname{Ran} \Phi_\tau$ is dense in $X$.
- The pair $(\mathbb{T}, \Phi)$ is *null controllable in time* $\tau$ if $\operatorname{Ran} \Phi_\tau \supset \operatorname{Ran} \mathbb{T}_\tau$.

**Remark 3.3.** Let $(\mathbb{T}, \Phi)$ be a well-posed linear LTI control system which is approximately controllable in time $\tau$. It is not difficult to check that for every $\eta \in \operatorname{Ran} \Phi_\tau$ there exists a unique $\psi \in X$ such that $\eta = \Phi_\tau \Phi_\tau^* \psi$. Moreover, we have

$$\|\eta\|_{\operatorname{Ran} \Phi_\tau} = \|\Phi_\tau^* \psi\|_{L^2([0,\tau];U)}.$$

The above facts imply that $\operatorname{Ran} \Phi_\tau$ endowed with the norm (3-1) is a Banach space. Indeed, let $(\eta_k)_{k \in \mathbb{N}} \subset \operatorname{Ran} \Phi_\tau$ be a Cauchy sequence with respect to the norm (3-1). For each $k \in \mathbb{N}$ let $\psi_k$ be the unique vector in $X$ such that $\eta_k = \Phi_\tau \Phi_\tau^* \psi_k$. Then for every $k, l \in \mathbb{N}$ we have

$$\eta_k - \eta_l = \Phi_\tau \Phi_\tau^* (\psi_k - \psi_l), \qquad \|\eta_k - \eta_l\|_{\operatorname{Ran} \Phi_\tau} = \|\Phi_\tau^* (\psi_k - \psi_l)\|_{L^2([0,\tau];U)}.$$

It follows that $(\Phi_\tau^* \psi_k)$ is a Cauchy sequence in $L^2([0, \tau]; U)$ such that

$$\lim_{k \to \infty} \|\Phi_\tau^* \psi_k - v\|_{L^2([0,\tau];U)} = 0$$

for some $v \in L^2([0, \tau]; U)$. Setting $\eta = \Phi_\tau v$ we see that $\|\eta_k - \eta\|_{\operatorname{Ran} \Phi_\tau} \to 0$, and thus we obtain the desired conclusion.

We continue this section with two results on well-posed control LTI systems which are null controllable in any time $\tau > 0$. Although the first one is classical, see [Fattorini 1978], we give a very short proof below, following essentially [Seidman 1979].

**Proposition 3.4.** *Assume that the well-posed control LTI system* $(\mathbb{T}, \Phi)$ *is null controllable in any positive time. Then* $\operatorname{Ran} \Phi_\tau$ *does not depend on* $\tau > 0$.

*Proof.* Let $0 < \tau < t$. The inclusion $\operatorname{Ran} \Phi_\tau \subset \operatorname{Ran} \Phi_t$ is easy to establish. Indeed, let $\eta \in \operatorname{Ran} \Phi_\tau$ and $\tilde{u}$ be a control such that $\Phi_\tau \tilde{u} = \eta$. Let $u = 0 \underset{t-\tau}{\diamond} \tilde{u}$ (for the notation $\underset{t-\tau}{\diamond}$ see (2-9)). Then, according to (2-8)

$$\Phi_t u = \Phi_{(t-\tau)+\tau} u = \mathbb{T}_\tau \Phi_{t-\tau} 0 + \Phi_\tau \tilde{u} = \eta,$$

and thus we have shown that $\operatorname{Ran} \Phi_\tau \subset \operatorname{Ran} \Phi_t$.

To establish the inclusion $\operatorname{Ran}\Phi_t \subset \operatorname{Ran}\Phi_\tau$, take $\eta \in \operatorname{Ran}\Phi_t$ and $u \in L^2([0,\tau];U)$ such that $\Phi_t u = \eta$. Setting

$$\tilde{u}(\sigma) = u(\sigma + t - \tau), \quad \sigma \in [0,\tau],$$

we remark that $u = u \underset{t-\tau}{\diamond} \tilde{u}$. Consequently, applying again (2-8), it follows that

$$\eta = \Phi_t u = \Phi_{(t-\tau)+\tau}(u \underset{t-\tau}{\diamond} \tilde{u}) = \mathbb{T}_\tau \Phi_{t-\tau} u + \Phi_\tau \tilde{u}.$$

Since the system is null controllable in any time, we have $\operatorname{Ran}\Phi_\tau \supset \operatorname{Ran}\mathbb{T}_\tau$. This, combined with the above formula, implies that $\eta \in \operatorname{Ran}\Phi_\tau$, so that indeed we have $\operatorname{Ran}\Phi_t \subset \operatorname{Ran}\Phi_\tau$, which ends the proof. □

**Remark 3.5.** If the pair $(A, B)$ determines the well-posed control LTI system $(\mathbb{T}, \Phi)$ then for every $z_0 \in X$ the Cauchy problem

$$\dot{z}(t) = Az(t) + Bu(t), \quad z(0) = z_0,$$

admits a unique solution $z \in C([0,\infty); X))$ satisfying

$$z(\tau) = \mathbb{T}_\tau z_0 + \Phi_\tau u, \quad \tau > 0,\ z_0 \in X,\ u \in L^2([0,\infty); U).$$

Moreover, if the system is null controllable in time $\tau$ (i.e., $\operatorname{Ran}\Phi_\tau \supset \operatorname{Ran}\mathbb{T}_\tau$), the last formula implies that for every $z_0 \in X$ the set described by $z(\tau)$ when $u$ takes all possible values in $L^2([0,\tau];U)$ coincides with $\operatorname{Ran}\Phi_\tau$.

The following result in this section, although simple, does not seem to have been explicitly stated in the literature.

**Proposition 3.6.** *Let* $(\mathbb{T}, \Phi)$ *be a well-posed control LTI system which is null controllable in any positive time let* $\tau > 0$ *and let* $g : (0,\tau] \to (0,\infty)$ *be a continuous and bounded function. Define*

$$\mathcal{U}_{\tau,g} = \left\{ u \in L^2([0,\tau];U) \,\middle|\, \left(t \mapsto \frac{u(t)}{g(t)}\right) \in L^2([0,\tau];U) \right\}. \tag{3-2}$$

*Then for every* $\tau > 0$ *we have* $\Phi_\tau(\mathcal{U}_{\tau,g}) = \operatorname{Ran}\Phi_\tau$.

*Proof.* Since the inclusion $\Phi_\tau(\mathcal{U}_{\tau,g}) \subset \operatorname{Ran}\Phi_\tau$ is obvious, we only need to check that $\operatorname{Ran}\Phi_\tau \subset \Phi_\tau(\mathcal{U}_{\tau,g})$. To this aim we note that according to Proposition 3.4 we have $\operatorname{Ran}\Phi_{\tau/2} = \operatorname{Ran}\Phi_\tau$; thus for every $\eta \in \operatorname{Ran}\Phi_\tau$ there exists $u \in L^2([0,\infty);U)$ such that $\Phi_{\tau/2} u = \eta$. Setting $\tilde{u} = 0 \underset{\tau/2}{\diamond} u$ and applying (2-8) it follows that

$$\Phi_\tau \tilde{u} = \Phi_{\tau/2+\tau/2}(0 \underset{\tau/2}{\diamond} u) = \mathbb{T}_{\tau/2}\Phi_{\tau/2} 0 + \Phi_{\tau/2} u = \eta.$$

Moreover, since $\tilde{u} = 0$ on $[0,\tau/2]$, we have that $\tilde{u} \in \mathcal{U}_{\tau,g}$, so that $\eta \in \Phi_\tau(\mathcal{U}_{\tau,g})$, which ends the proof. □

We end this section by stating the known fact (this goes back to [Fattorini and Russell 1974]; see also [Tucsnak and Weiss 2009, Proposition 10.7.1]) that the two first equations of (1-1) determine a well-posed control LTI system, with appropriate choices for $X$ and $U$, which is null controllable in any positive time. Moreover, we give an expression of the input maps which has already been used in [Hartmann et al. 2020].

**Proposition 3.7.** *The first two equations in (1-1) determine a well-posed control LTI system with state space $X = W^{-1,2}(0,\pi)$ and input space $U = \mathbb{C}^2$. Moreover the corresponding family $\Phi$ of input maps is given by*

$$\left(\Phi_\tau \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix}\right)(x) = \int_0^\tau \frac{\partial K_0}{\partial x}(\tau-\sigma, x) u_0(\sigma)\,\mathrm{d}\sigma + \int_0^\tau \frac{\partial K_\pi}{\partial x}(\tau-\sigma, x) u_\pi(\sigma)\,\mathrm{d}\sigma, \quad \tau > 0,\ u_0,\ u_\pi \in L^2[0,\tau],\ x \in (0,\pi), \tag{3-3}$$

*where*

$$K_0(\sigma, x) = -\sqrt{\frac{1}{\pi\sigma}} \sum_{m\in\mathbb{Z}} \mathrm{e}^{-(x+2m\pi)^2/(4\sigma)}, \quad \sigma > 0,\ x \in [0,\pi], \tag{3-4}$$

$$K_\pi(\sigma, x) = K_\pi(\sigma, \pi - x), \quad \sigma > 0,\ x \in [0,\pi]. \tag{3-5}$$

*Finally, the considered system is null controllable in any time $\tau > 0$.*

**Remark 3.8.** For every $w_0 \in W^{-1,2}(0,\pi)$ and $\tau > 0$, let $R_\tau(w_0; \cdot)$ be defined by

$$R_\tau\left(w_0; \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix}\right) = \tilde{w}(\tau, \cdot), \quad \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} \in L^2([0,\tau]; \mathbb{C}^2),$$

where $\tilde{w}$ satisfies the first two equations in (1-1) with the initial condition $\tilde{w}(0,\cdot) = w_0$. The null controllability of the system determined by the first two equations in (1-1) implies, according to Remark 3.5, that for every $\tau > 0$ and $w_0 \in W^{-1,2}(0,\pi)$ we have $\operatorname{Ran} R_\tau = \operatorname{Ran} \Phi_\tau$, where $\Phi_\tau$ is defined in (1-2).

## 4. Proof of Theorem 1.1

As already mentioned, the proof of Theorem 1.1 uses in an essential manner results from [Hartmann et al. 2020]. More precisely, the main ingredient of this proof is a result which is not explicitly stated in [Hartmann et al. 2020], but which is implicitly proved in this reference. To make this clear, we give its precise statement and we describe the main steps of the proof.

**Proposition 4.1.** *For $\tau > 0$ let $\mathcal{U}_{\tau,1/2}$ be the set defined in (3-2) with $g(t) = \sqrt{t}$. Then there exists $\delta^* > 0$ such that*

$$\Phi_\tau(\mathcal{U}_{\tau,1/2}) = A^2(\Delta, \omega_{0,\tau}) + A^2(\tilde{\Delta}, \omega_{\pi,\tau}), \quad \tau \in (0,\delta^*), \tag{4-1}$$

*where the weights $\omega_{0,\tau}$ and $\omega_{\pi,\tau}$ were introduced in (1-8) and (1-9), respectively.*

To prove the above result we introduce some notation and state some results from [Hartmann et al. 2020]. We first introduce the families of operators

$$(P_\tau f)(s) = \int_0^\tau \frac{s\mathrm{e}^{-s^2/(4(\tau-\sigma))}}{2\sqrt{\pi}(\tau-\sigma)^{3/2}} f(\sigma)\sqrt{\sigma}\,\mathrm{d}\sigma, \quad \tau \geqslant 0,\ f \in L^2[0,\tau],\ s \in \Delta, \tag{4-2}$$

$$(Q_\tau g)(s) = \int_0^\tau \frac{(\pi-s)\mathrm{e}^{-(\pi-s)^2/(4(\tau-\sigma))}}{2\sqrt{\pi}(\tau-\sigma)^{3/2}} g(\sigma)\sqrt{\sigma}\,\mathrm{d}\sigma, \quad \tau \geqslant 0,\ g \in L^2[0,\tau],\ s \in \tilde{\Delta}, \tag{4-3}$$

where the sets $\Delta$ and $\tilde{\Delta}$ were introduced in (1-7). Each of the two operators above can be seen as the input maps of a system governed by the boundary-controlled heat equation on a half-line. Looking, for instance, to $(P_\tau)_{\tau\geqslant 0}$ and setting

$$w_l(t,x) = (P_t f)(x), \qquad t \geqslant 0,\ x \geqslant 0,$$

we have that

$$\begin{cases} \dfrac{\partial w_l}{\partial t}(t,x) = \dfrac{\partial^2 w_l}{\partial x^2}(t,x), & t \geqslant 0,\ x \geqslant 0,\\ w_l(t,0) = \sqrt{t}\, f(t), & t \in [0,\infty),\\ w_l(0,x) = 0, & x \geqslant 0. \end{cases}$$

Using results of Aikawa, Hayashi and Saitoh [Aikawa et al. 1990] it was shown in [Hartmann et al. 2020, Theorem 2.2 and Corollary 2.3] that the following result holds:

**Lemma 4.2.** *For every $\tau > 0$ the operator*

$$\begin{bmatrix} P_\tau & 0 \\ 0 & Q_\tau \end{bmatrix}$$

*is bounded and invertible from $L^2([0,\pi]))^2$ onto $A^2(\Delta,\omega_{0,\tau}) \times A^2(\tilde{\Delta},\omega_{\pi,\tau})$. Moreover,*

$$\left\| \begin{bmatrix} P_\tau & 0 \\ 0 & Q_\tau \end{bmatrix} \right\|_{\mathcal{L}((L^2([0,\pi]))^2, A^2(\Delta,\omega_{0,\tau})\times A^2(\tilde{\Delta},\omega_{\pi,\tau}))} = 1, \qquad \tau > 0.$$

For $\tau > 0$ we introduce the family of operators $(M_\tau)_{\tau>0}$ defined by

$$M_\tau \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} = \begin{bmatrix} g_0(u_0) \\ g_\pi(u_\pi) \end{bmatrix}, \qquad u_0,\ u_\pi \in L^2[0,\tau], \tag{4-4}$$

where

$$g_0(u_0)(s) = \int_0^\tau \frac{\partial K_0}{\partial s}(\sigma,s)\sqrt{\sigma}\, u_0(\sigma)\,\mathrm{d}\sigma, \qquad u_0 \in L^2[0,\tau],\ s \in \Delta, \tag{4-5}$$

$$g_\pi(u_\pi)(s) = \int_0^\tau \frac{\partial K_\pi}{\partial s}(\sigma,s)\sqrt{\sigma}\, u_\pi(\sigma)\,\mathrm{d}\sigma, \qquad u_\pi \in L^2[0,\tau],\ s \in \tilde{\Delta}, \tag{4-6}$$

and the kernels $K_0$ and $K_\pi$ were introduced in (3-4) and (3-5), respectively. Comparing the above formulas with (3-3) we see that

$$g_0(u_0)(s) + g_\pi(u_\pi)(s) = \Phi_\tau \begin{bmatrix} v_0 \\ v_\pi \end{bmatrix}(s), \qquad s \in D, \tag{4-7}$$

where

$$v_0(t) = \sqrt{t}\, u_0(t), \qquad v_\pi(t) = \sqrt{t}\, u_\pi(t), \qquad t \in [0,\tau],$$

and $D$ was defined in (1-3).

Another important estimate proved in [Hartmann et al. 2020] is:

**Lemma 4.3.** *For every $\tau > 0$ the operator $M_\tau$ defined in (4-4) is bounded from $(L^2([0,\pi]))^2$ to $A^2(\Delta,\omega_{0,\tau}) \times A^2(\tilde{\Delta},\omega_{\pi,\tau})$. Moreover,*

$$\lim_{\tau\to 0+} \left\| M_\tau - \begin{bmatrix} P_\tau & 0 \\ 0 & Q_\tau \end{bmatrix} \right\|_{\mathcal{L}((L^2([0,\pi]))^2, A^2(\Delta,\omega_{0,\tau})\times A^2(\tilde{\Delta},\omega_{\pi,\tau}))} = 0.$$

We are now in a position to prove Proposition 4.1.

*Proof of Proposition 4.1.* Let $\varphi \in A^2(\Delta, \omega_0) + A^2(\tilde{\Delta}, \omega_\pi)$, so that there exist $\varphi_0 \in A^2(\Delta, \omega_0)$ and $\varphi_\pi \in A^2(\tilde{\Delta}, \omega_\pi)$ such that $\varphi = \varphi_0 + \varphi_\pi$. By combining Lemmas 4.2 and 4.3 it follows that there exists $\delta^* > 0$ such that the operator $M_\tau$ is bounded and invertible from $L^2([0, \pi]))^2$ onto $A^2(\Delta, \omega_0) \times A^2(\tilde{\Delta}, \omega_\pi)$ for every $\tau \in (0, \delta^*)$. According to the definition (4-4) it follows that for every $\tau \in (0, \delta^*)$ there exist $\tilde{u}_0, \tilde{u}_\pi \in L^2[0, \tau]$ such that

$$\int_0^\tau \frac{\partial K_0}{\partial s}(\sigma, s)\sqrt{\sigma}\, \tilde{u}_0(\sigma)\, d\sigma = \varphi_0(s), \qquad \tau \in (0, \delta^*),\ s \in \Delta,$$

$$\int_0^\tau \frac{\partial K_\pi}{\partial s}(\sigma, s)\sqrt{\sigma}\, \tilde{u}_\pi(\sigma)\, d\sigma = \varphi_\pi(s), \qquad \tau \in (0, \delta^*), s \in \tilde{\Delta}.$$

The last two formulas, combined with (3-3) imply that

$$\Phi_\tau \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} = \varphi_0 + \varphi_\pi = \varphi, \qquad \tau \in (0, \delta^*),$$

where $u_0(t) = \sqrt{t}\, \tilde{u}_0(t)$ and $u_\pi(t) = \sqrt{t}\, \tilde{u}_\pi(t)$ for $t \in [0, \tau]$. The conclusion (4-1) follows now from the obvious fact that $u_0, u_\pi$ lie in $\mathcal{U}_{\tau,1/2}$. □

Finally, we give below the proof of Theorem 1.1.

*Proof.* The fact that $\Phi_\tau$ is bounded from $(L^2[0, \tau])^2$ to $A^2(\Delta, \omega_{0,\tau}) + A^2(\tilde{\Delta}, \omega_{\pi,\tau})$ is shown in the proof of Proposition 2.1 from [Hartmann et al. 2020], but, for the sake of completeness, we make this clear below.

For $u_0, u_\pi \in L^2[0, \tau]$ we note that from (3-3) it follows that

$$\left(\Phi_\tau \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix}\right)(x) = \int_0^{\tau/2} \frac{\partial K_0}{\partial x}(\tau - \sigma, x) u_0(\sigma)\, d\sigma + \int_0^{\tau/2} \frac{\partial K_\pi}{\partial x}(\tau - \sigma, x) u_\pi(\sigma)\, d\sigma + \int_0^\tau \frac{\partial K_0}{\partial x}(\tau - \sigma, x) \tilde{u}_0(\sigma)\sqrt{\sigma}\, d\sigma + \int_0^\tau \frac{\partial K_\pi}{\partial x}(\tau - \sigma, x) \tilde{u}_\pi(\sigma)\sqrt{\sigma}\, d\sigma, \tag{4-8}$$

where, for $\gamma \in \{0, \pi\}$, we define

$$\tilde{u}_\gamma(\sigma) := \begin{cases} 0 & \text{if } \sigma \in [0, \tau/2], \\ u_\gamma(\sigma)/\sqrt{\sigma} & \text{if } \sigma \in [\tau/2, \tau]. \end{cases} \tag{4-9}$$

It can be checked by direct calculations that $(\partial K_\gamma/\partial s)(\tau - \sigma, \cdot) \in L^2(\partial D)$ for $\sigma \in [0, \tau/2]$, where $\partial D$ is the boundary of the open set $D$ defined in (1-3). Hence, by the Cauchy–Schwarz inequality, the operator $\Phi_{\tau,1}$, defined by

$$\Phi_{\tau,1}\left(\begin{bmatrix} u_0 \\ u_\pi \end{bmatrix}\right)(s) = \int_0^{\tau/2} \frac{\partial K_0}{\partial s}(\tau - \sigma, s) u_0(\sigma)\, d\sigma + \int_0^{\tau/2} \frac{\partial K_\pi}{\partial s}(\tau - \sigma, s) u_\pi(\sigma)\, d\sigma, \tag{4-10}$$

is linear and bounded from $(L^2[0, \tau])^2$ to the Hardy–Smirnov space $E^2(D)$ defined in (1-5). On the other hand it was shown in [Hartmann et al. 2020] that we have $E^2(D) \subset X_\tau$, where

$$X_\tau = A^2(\Delta, \omega_{0,\tau}) + A^2(\tilde{\Delta}, \omega_{\pi,\tau})$$

with continuous embedding so that the operator defined in (4-10) is linear and bounded from $(L^2[0,\tau])^2$ to $X_\tau$. The fact that the operator $\Phi_{\tau,2}$ defined by

$$\Phi_{\tau,2}\left(\begin{bmatrix} u_0 \\ u_\pi \end{bmatrix}\right)(s) = \int_0^\tau \frac{\partial K_0}{\partial s}(\tau-\sigma, s)\tilde{u}_0(\sigma)\sqrt{\sigma}\,\mathrm{d}\sigma + \int_0^\tau \frac{\partial K_\pi}{\partial s}(\tau-\sigma, s)\tilde{u}_\pi(\sigma)\sqrt{\sigma}\,\mathrm{d}\sigma, \tag{4-11}$$

with $\tilde{u}_0$ and $\tilde{u}_\pi$ defined in (4-9), is linear and bounded from $(L^2[0,\tau])^2$ to $X_\tau$ follows easily from Lemma 4.3. Putting together the above estimates and the fact that

$$\Phi_\tau = \Phi_{\tau,1} + \Phi_{\tau,2},$$

we have thus proved (1-12).

The main assertion of Theorem 1.1 which says that the operator $\Phi_\tau$ is onto can now be obtained by putting together several of our previous results. Indeed, let $\delta^*$ be the constant in Proposition 4.1. By combining Propositions 3.6 and 4.1 it follows that

$$\operatorname{Ran}\Phi_\delta = A^2(\Delta, \omega_{0,\delta}) + A^2(\tilde{\Delta}, \omega_{\pi,\delta}), \qquad \delta \in (0, \delta^*).$$

On the other hand, we know from Propositions 3.4 and 3.7 that $\operatorname{Ran}\Phi_\tau = \operatorname{Ran}\Phi_\delta$ for every $\tau, \delta > 0$, and thus we obtain the conclusion (1-13). □

## 5. Reachable space with smooth inputs

It can be seen as an obvious consequence of Theorem 1.1 (but it can also be checked in a more elementary manner) that the well-posed control LTI system determined by (1-1), with state space $X = W^{-1,2}(0,\pi)$ and input space $U = L^2[0,\pi]$, is not exactly controllable in any time $\tau > 0$. A natural question is: can this system be seen as an exactly controllable one by choosing a different state space (and possibly a class of smoother input functions)? By analogy with Remark 2.4 (valid for finite-dimensional LTI systems) and based on our main result in Theorem 1.1, a candidate for the new state space is $X_\delta = \operatorname{Ran}\Phi_\tau$. Indeed, in our case it is easily checked that $\mathbb{T}_t(\operatorname{Ran}\Phi_\tau) \subset \operatorname{Ran}\Phi_\tau$ for every $t \geqslant 0$ and $\tau > 0$ and that the family $\widetilde{\mathbb{T}} = \mathbb{T}_{|\operatorname{Ran}\Phi_\tau}$ satisfies the semigroup properties (2-5) and (2-6). Moreover, it is clear that the pair $(\widetilde{\mathbb{T}}, \Phi)$ satisfies (2-8). It follows that the only condition still to be checked in order to prove that $(\widetilde{\mathbb{T}}, \Phi)$ is a well-posed exactly controllable system, with state space $X_\delta$ and input space $U$, is the strong continuity property of the semigroup $\widetilde{\mathbb{T}}$ on $X_\delta$. This seems a difficult question. The recently developed theory on $C^0$-semigroups on spaces of analytic functions (see, for instance, [Gal and Gal 2017]) could provide a good track for exploring this question.

Motivated by applications to nonlinear problems, [Martin et al. 2016; Laurent and Rosier 2018] study a controllability concept for the heat equation which is quite different of the exact controllability introduced in Definition 3.2. More precisely, given $\tau > 0$, the main results in [Martin et al. 2016; Laurent and Rosier 2018] assert that there exist controls $u_0$, $u_\pi$ having a Gevrey-type regularity on $[0,\tau]$ which steer the solution of (1-1) to any state which can be holomorphically extended to a ball in $\mathbb{C}$ which is centered at $\frac{\pi}{2}$ and of diameter large enough. We give below a result in the same direction, with less regularity for both the target states and the input signals. More precisely, our result below gives a complete characterization of the states which can be reached by inputs lying in

$$W_{\mathrm{L}}^{n,2}(0,\tau) := \left\{ v \in W^{n,2}(0,\tau) \;\middle|\; v(0) = \cdots = \frac{\mathrm{d}^{n-1} v}{\mathrm{d}t^{n-1}}(0) = 0 \right\} \tag{5-1}$$

for some $n \in \mathbb{N}$ and $\tau > 0$. Moreover, we set $W_{\mathrm{L}}^{0,2}(0,\tau) := L^2[0,\tau]$.

To state the main result in this section we introduce, for each $n \in \mathbb{Z}_+$ and $\tau > 0$, the space $X_{n,\tau}$ defined by $X_{0,\tau} := X_\tau$ and

$$X_{n,\tau} := \left\{ \psi \in X_\tau \;\middle|\; \frac{\mathrm{d}^{2k}\psi}{\mathrm{d}s^{2k}} \in X_\tau \text{ for } k = 0, 1, \dots, n \right\}, \qquad n \in \mathbb{N}, \tag{5-2}$$

where the family of Banach spaces $(X_\delta)_{\delta>0}$ was defined in (1-10) and (1-11). Note that $W_{\mathrm{L}}^{n,2}(0,\tau)$ and $X_{n,\tau}$ are Banach spaces when endowed with the norms

$$\|v\|_{W_{\mathrm{L}}^{n,2}(0,\tau)} = \left\| \frac{\mathrm{d}^n v}{\mathrm{d}t^n} \right\|_{L^2[0,\tau]}, \qquad v \in W_{\mathrm{L}}^{n,2}(0,\tau),$$

$$\|\psi\|_{X_{n,\tau}}^2 = \sum_{k=1}^{n} \left\| \frac{\mathrm{d}^{2k}\psi}{\mathrm{d}s^{2k}} \right\|_{X_{n,\tau}}^2, \qquad \psi \in X_{n,\tau}.$$

**Proposition 5.1.** *Let $n \in \mathbb{Z}_+$. Then for every $\tau > 0$ the restriction of $\Phi_\tau$ to the space $W_{\mathrm{L}}^{n,2}(0,\tau)$ introduced in (5-1) is a linear bounded operator from $W_{\mathrm{L}}^{n,2}(0,\tau)$ onto $X_{n,\tau}$, where the Banach space $X_{n,\tau}$ was defined in (5-2).*

*Proof.* The fact that for every $\tau > 0$ we have $\Phi_\tau \in \mathcal{L}(W_{\mathrm{L}}^{0,2}(0,\tau); X_{0,\tau})$ was proven in Theorem 1.1. For every $n \in \mathbb{N}$ and $u_0,\; u_\pi \in W_{\mathrm{L}}^{n,2}(0,\tau)$ we define

$$z(t,\cdot) := \Phi_t \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix}, \qquad \text{with } t \in [0,\tau].$$

By applying Lemma 2.1 from [Tucsnak and Weiss 2015] it follows that $z \in C^n([0,\tau]; W^{-1,2}(0,\pi))$ and

$$\frac{\partial^{2k} z}{\partial x^{2k}}(\tau,x) = \frac{\partial^k z}{\partial t^k}(\tau,x), \qquad k \in \{1,\dots,n\},\; x \in (0,\pi),$$

$$\frac{\partial^k z}{\partial t^k}(\tau,x) = \Phi_\tau \begin{bmatrix} \mathrm{d}^k u_0/\mathrm{d}t^k \\ \mathrm{d}^k u_\pi/\mathrm{d}t^k \end{bmatrix}(x), \quad k \in \{1,\dots,n\},\; x \in (0,\pi).$$

The two relations above combined with the fact, following from Theorem 1.1, that the maps

$$\begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} \mapsto \Phi_\tau \begin{bmatrix} \mathrm{d}^k u_0/\mathrm{d}t^k \\ \mathrm{d}^k u_\pi/\mathrm{d}t^k \end{bmatrix}, \qquad k \in \{0,\dots,n\},\; u_0, u_\pi \in W_{\mathrm{L}}^{n,2}(0,\tau),$$

are bounded from $W_{\mathrm{L}}^{n,2}(0,\tau)$ into $X_\tau$, yield that indeed $\Phi_\tau \in \mathcal{L}(W_{\mathrm{L}}^{n,2}(0,\tau); X_{n,\tau})$.

To show that $\Phi_\tau$ maps $W_{\mathrm{L}}^{n,2}(0,\tau)$ onto $X_{n,\tau}$ we begin by noticing that, according to Theorem 1.1, this holds for $n = 0$. For $n \in \mathbb{N}$ we remark that, by using again Theorem 1.1, for every $\psi \in X_{n,\tau}$ there exist

$v_0, v_\pi \in L^2[0,\tau]$ with the solution $w$ of

$$\begin{cases} \dfrac{\partial w}{\partial t}(t,x) = \dfrac{\partial^2 w}{\partial x^2}(t,x), & t \geqslant 0,\ x \in (0,\pi), \\ w(t,0) = v_0(t), \quad w(t,\pi) = v_\pi(t), & t \in [0,\infty), \\ w(0,x) = 0, & x \in (0,\pi), \end{cases} \tag{5-3}$$

satisfies

$$w(\tau,x) = \frac{\mathrm{d}^{2n}\psi}{\mathrm{d}x^{2n}}(x), \quad x \in (0,\pi). \tag{5-4}$$

Consider the functions $\tilde{v}_0$ and $\tilde{v}_\pi$ defined by

$$\tilde{v}_0(t) = \int_0^t v_0(\sigma)\,\mathrm{d}\sigma, \quad \tilde{v}_\pi(t) = \int_0^t v_\pi(\sigma)\,\mathrm{d}\sigma, \quad t \in [0,\tau]$$

for $n = 1$ and

$$\tilde{v}_0(t) = \int_0^t \int_0^{\sigma_1} \int_0^{\sigma_2} \cdots \int_0^{\sigma_{n-1}} v_{0,n}(\sigma_n)\,\mathrm{d}\sigma_n\,\mathrm{d}\sigma_{n-1}\ldots\mathrm{d}\sigma_1, \quad t \in [0,\tau],$$

$$\tilde{v}_\pi(t) = \int_0^t \int_0^{\sigma_1} \int_0^{\sigma_2} \cdots \int_0^{\sigma_{n-1}} v_\pi(\sigma_n)\,\mathrm{d}\sigma_n\,\mathrm{d}\sigma_{n-1}\ldots\mathrm{d}\sigma_1, \quad t \in [0,\tau]$$

for $n \geqslant 2$. We clearly have

$$\tilde{v}_0,\ \tilde{v}_\pi \in W_{\mathrm{L}}^{n,2}(0,\tau) \tag{5-5}$$

and

$$\frac{\mathrm{d}^n \tilde{v}_0}{\mathrm{d}t^n} = v_0, \quad \frac{\mathrm{d}^n \tilde{v}_\pi}{\mathrm{d}t^n} = v_\pi. \tag{5-6}$$

Let $\tilde{w}$ be the solution of the initial and boundary value problem

$$\begin{cases} \dfrac{\partial \tilde{w}}{\partial t}(t,x) = \dfrac{\partial^2 \tilde{w}}{\partial x^2}(t,x), & t \in [0,\tau],\ x \in (0,\pi), \\ \tilde{w}(t,0) = \tilde{v}_0(t), \quad \tilde{w}(t,\pi) = \tilde{v}_\pi(t), & t \in [0,\tau], \\ \tilde{w}(0,x) = 0, & x \in (0,\pi). \end{cases} \tag{5-7}$$

Thanks to (5-5) and (5-6) and using again Lemma 2.1 from [Tucsnak and Weiss 2015] we have

$$\tilde{w} \in C([0,\tau];\, W^{2n-1,2}(0,\pi)) \cap W^{n,2}([0,\tau];\, W^{-1,2}(0,\pi)),$$

and $\partial^n \tilde{w}/\partial t^n = w$, where $w$ is the solution of (5-3)–(5-4). It follows that

$$\frac{\partial^n \tilde{w}}{\partial t^n}(\tau,x) = \frac{\mathrm{d}^{2n}\psi}{\mathrm{d}x^{2n}}(x), \quad x \in (0,\pi).$$

The above relation and the first equation in (5-7) imply that

$$\frac{\partial^{2n} \tilde{w}}{\partial x^{2n}}(\tau,x) = \frac{\mathrm{d}^{2n}\psi}{\mathrm{d}x^{2n}}(x), \quad x \in (0,\pi).$$

It follows that

$$\tilde{w}(\tau,x) = \psi(x) + P(x), \quad x \in (0,\pi), \tag{5-8}$$

where $P$ is a polynomial of degree $2n-1$. The last formula can alternatively be written

$$\Phi_\tau \begin{bmatrix} \tilde{v}_0 \\ \tilde{v}_\pi \end{bmatrix} = \psi + P. \tag{5-9}$$

On the other hand, according to Theorem 2 of Laroche, Martin, and Rouchon [Laroche et al. 2000] (see also [Martin et al. 2016]) there exist $\xi_0,\ \xi_\pi \in C^\infty([0,\tau])$ such that

$$\Phi_\tau \begin{bmatrix} \xi_0 \\ \xi_\pi \end{bmatrix} = P, \tag{5-10}$$

$$\frac{\mathrm{d}^k \xi_0}{\mathrm{d}t^k}(0) = \frac{\mathrm{d}^k \xi_\pi}{\mathrm{d}t^k}(0) = 0, \quad k \in \mathbb{Z}_+. \tag{5-11}$$

Finally, setting

$$u_0 = \tilde{v}_0 - \xi_0 \quad u_\pi = \tilde{v}_\pi - \xi_\pi \tag{5-12}$$

and using (5-9) and (5-10) it follows that $u_0, u_\pi \in W_{\mathrm{L}}^{n,2}(0,\tau)$ and they satisfy

$$\Phi_\tau \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} = \psi, \tag{5-13}$$

which ends the proof. □

Let us now introduce, for each $n \in \mathbb{Z}_+$ and open set $\Omega \subset \mathbb{C}$, the spaces

$$A^{0,2}(\Omega) := A^2(\Omega), \qquad E^{0,2}(\Omega) := E^2(\Omega)$$

and

$$A^{n,2}(\Omega) := \left\{ \psi \in A^2(\Omega) \;\middle|\; \frac{\mathrm{d}^{2k}\psi}{\mathrm{d}s^{2k}} \in A^2(\Omega) \text{ for } k = 1, \dots, n \right\}, \quad n \geqslant 1, \tag{5-14}$$

$$E^{n,2}(\Omega) := \left\{ \psi \in E^2(\Omega) \;\middle|\; \frac{\mathrm{d}^{2k}\psi}{\mathrm{d}s^{2k}} \in E^2(\Omega) \text{ for } k = 1, \dots, n \right\}, \quad n \geqslant 1,$$

where the Bergman and Hardy–Smirnov spaces $A^2(\Omega)$ and $E^2(\Omega)$ were introduced in (1-6) and (1-5), respectively.

By combining Proposition 5.1 above with Proposition 1.1 and Theorem 1.3 in [Hartmann et al. 2020] it follows that:

**Corollary 5.2.** *Given $n \in \mathbb{Z}_+$ we have*

$$E^{n,2}(D) \subset \Phi_\tau(W_{\mathrm{L}}^{n,2}(0,\tau)) \subset A^{n,2}(D), \qquad \tau > 0,$$

*where $D$ is the open set defined in (1-3).*

**Remark 5.3.** As already mentioned, the main result in [Martin et al. 2016] asserts that functions which are analytic in a ball centered at $\frac{\pi}{2}$ and of a radius large enough are reachable by controls lying in the Gevrey class $G^2([0,\tau])$. Note that the Gevrey class $G^\gamma([0,\tau])$ of order $\gamma > 1$ is defined as the set of all functions $g \in C^\infty([0,\tau])$ such that for every $n \in \mathbb{Z}_+$ we have $\|f^{(n)}\|_\infty \leqslant A_f R_f^n (n!)^\gamma$ for some positive constants $A_f$, $R_f$. We conjecture that this result can be strengthened to the following "analytic" version of Proposition 5.1: for every $\tau > 0$ and $\psi \in \operatorname{Hol}(\widetilde{D})$, where $\widetilde{D} \subset \mathbb{C}$ is an open set containing $\overline{D}$, there exist $u_0, u_\pi \in G^2([0, \tau])$ such that

$$\Phi_\tau \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} = \psi.$$

A possible approach in proving this conjecture could consist in applying Proposition 5.1 with $n \to \infty$. This approach would require appropriate estimates of the derivatives, up to order $n-1$, of the controls $u_0, u_\pi$ constructed in Proposition 5.1. Obtaining such estimates is for now an open question.

## 6. Reachable space and the cost of null controllability

In this section we describe an application of the results and methods developed above in order to obtain estimates for the cost of null controllability in small time for the system determined by the two first equations in (1-1). We begin by stating in the general context introduced in Section 3, the definition of the cost of null controllability. To this aim, let $X$ and $U$ be Hilbert spaces and let $(\mathbb{T}, \Phi)$ be a well-posed control LTI system with state space $X$ and input space $U$ (in the sense of Definition 3.1). Assuming that the system $(\mathbb{T}, \Phi)$ is null controllable in some time $\tau > 0$ (According to Definition 3.2 this means that $\operatorname{Ran} \Phi_\tau \supset \operatorname{Ran} \mathbb{T}_\tau$.), *the cost of null controllability in time* $\tau$ is the number $c_\tau$ defined by

$$c_\tau = \sup_{\|\psi\|_X \leqslant 1} \|\mathbb{T}_\tau \psi\|_{\operatorname{Ran} \Phi_\tau}, \tag{6-1}$$

where the norm $\|\cdot\|_{\operatorname{Ran} \Phi_\tau}$ was defined in (3-1).

For systems which are null controllable in every time $\tau > 0$ we clearly have that $\limsup_{\tau \to 0+} c_\tau = +\infty$. A question of interest in this case is to estimate the blow-up rate of $c_\tau$ when $\tau \to 0$. For finite-dimensional LTI systems the question was first investigated in [Seidman 1988; Seidman and Yong 1996], where it was shown that, as $\tau$ tends to zero, $c_\tau$ behaves like $1/\tau^{k+1/2}$, where $k \in \mathbb{Z}_+$ is the smallest integer such that

$$\operatorname{Ran} \begin{bmatrix} B & AB & A^2 B & \cdots & A^k B \end{bmatrix} = X.$$

In the case of the system determined by the two first equations in (1-1), which is null controllable in any positive time (see Proposition 3.7), the study of the behavior of the cost of null controllability when $\tau \to 0+$ began with the classical work [Fattorini and Russell 1971] and continued with a series of papers including, with successive improvements, [Güichal 1985; Miller 2004; Tenenbaum and Tucsnak 2007; Lissy 2015; Dardé and Ervedoza 2019]. As far as we know, the most precise lower bound for $c_\tau$ when $\tau \to 0+$ is

$$\limsup_{\tau \to 0+} \tau \log c_\tau \leqslant \tfrac{1}{4} \kappa_0 \pi^2, \tag{6-2}$$

where $\kappa_0$ is a constant approximately equal to 0.6966. This was proved in [Dardé and Ervedoza 2019].

**Remark 6.1.** As far as we know, in the case of the heat equation with boundary control at both ends, there is no specific study of the lower bound of $c_\tau$ when $\tau \to 0+$. This is probably due to the fact that it is commonly accepted that any lower bound for the cost of null controllability for the case when $u_\pi$ in (1-1) is equal to zero yields a lower bound for $c_\tau$ by "symmetry" arguments (this is, for instance, implicitly claimed in [Dardé and Ervedoza 2019]). Accepting this claim yields, using the best known estimates for the one-sided control (see [Lissy 2015]), that

$$\liminf_{\tau\to 0+} \tau \log c_\tau \geqslant \tfrac{1}{8}\pi^2. \tag{6-3}$$

Since we did not find any obvious argument for deriving (6-3) from the results in [Lissy 2015], we give a proof of this fact at the end of this paper. More precisely, (6-3) follows from Corollary 8.4 in Section 8 below and Theorem 1.1 in [Lissy 2015].

In this section we prove that for small $\tau$ the constant $c_\tau$ is smaller than a constant $d_\tau$, which is simply defined in terms of the semigroup $\mathbb{T}$ and of the norm of the space $X_\tau$ defined in (1-10). Whether this estimate can lead to an improvement of the $\kappa_0$ in (6-2) is an open question, to be treated in a forthcoming work.

To give a precise statement of the main result in this section we note that for every $\tau > 0$ and $\psi \in W^{-1,2}(0,\pi)$, the function $x \mapsto (\mathbb{T}_\tau\psi)(x)$ clearly extends to a function which is holomorphic on $\mathbb{C}$, so that, according to Corollary 3.6 in [Hartmann et al. 2020] (or just using the null controllability of $(\mathbb{T}, \Phi)$ combined with Theorem 1.1) we have that $\mathbb{T}_\tau\psi \in X_\tau$. We can thus define, for each $\tau > 0$, the constant

$$d_\tau = \sup_{\|\psi\|_{W^{-1,2}(0,\pi)}\leqslant 1} \|\mathbb{T}_\tau\psi\|_\tau, \tag{6-4}$$

where $\mathbb{T}$ is the heat semigroup and the norm $\|\cdot\|_\tau$ was defined in (1-11).

**Proposition 6.2.** *With the above notation we have*

$$\limsup_{\tau\to 0+} \frac{c_\tau}{d_\tau} \leqslant 1. \tag{6-5}$$

*Proof.* We have seen above that $\mathbb{T}_\tau\psi$ lies in $X_\tau$ for every $\psi \in W^{-1,2}(0,\pi)$ and $\tau > 0$ so that we have

$$(\mathbb{T}_\tau\psi)(x) = \varphi_0(x) + \varphi_\pi(x), \qquad x \in (0,\pi), \tag{6-6}$$

where $\varphi_0 \in A^2(\Delta, \omega_{0,\tau})$ and $\varphi_\pi \in A^2(\tilde{\Delta}, \omega_{\pi,\tau})$ depend on both $\psi$ and $\tau$.

On the other hand, recall the operators $P_\tau$, $Q_\tau$ and $M_\tau$ defined in (4-2), (4-3) and (4-4), respectively. Using Lemmas 4.2 and 4.3 it follows that there exists $\delta^* > 0$ such that $M_\tau$ is invertible for every $\tau \in (0, \delta^*)$ and

$$\|M_\tau^{-1}\|_{\mathcal{L}(A^2(\Delta,\omega_{0,\tau})\times A^2(\tilde{\Delta},\omega_{\pi,\tau}),(L^2([0,\tau]))^2)} \leqslant \frac{1}{1-\gamma_\tau},$$

where for every $\tau > 0$ we have set

$$\gamma_\tau = \left\| M_\tau - \begin{bmatrix} P_\tau & 0 \\ 0 & Q_\tau \end{bmatrix} \right\|_{\mathcal{L}((L^2([0,\pi]))^2, A^2(\Delta,\omega_{0,\tau})\times A^2(\tilde{\Delta},\omega_{\pi,\tau}))}. \tag{6-7}$$

Consequently, for each $\tau \in (0, \delta^*)$ and $\varphi_0$ and $\varphi_\pi$ as above there exist $v_0, v_\pi \in L^2[0,\tau]$ such that

$$M_\tau \begin{bmatrix} v_0 \\ v_\pi \end{bmatrix} = \begin{bmatrix} \varphi_0 \\ \varphi_\pi \end{bmatrix},$$

$$\left\| \begin{bmatrix} v_0 \\ v_\pi \end{bmatrix} \right\|_{(L^2[0,\tau])^2} \leqslant \frac{1}{1-\gamma_\tau} \left\| \begin{bmatrix} \varphi_0 \\ \varphi_\pi \end{bmatrix} \right\|_{A^2(\Delta,\omega_{0,\tau})\times A^2(\tilde{\Delta}.\omega_{\pi,\tau})}. \tag{6-8}$$

We can thus conclude, recalling (4-4) and (4-7), that for every $\psi \in W^{-1,2}(0,\pi)$, $\tau \in (0,\delta^*)$ and $\varphi_0 \in A^2(\Delta,\omega_{0,\tau})$, $\varphi_\pi \in A^2(\tilde{\Delta},\omega_{\pi,\tau})$ satisfying (6-6) there exist $v_0, v_\pi \in L^2[0,\tau]$ satisfying (6-8) such that

$$\Phi_\tau \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} = \mathbb{T}_\tau \psi, \tag{6-9}$$

where

$$u_0(t) = \sqrt{t}\, v_0(t), \quad u_\pi(t) = \sqrt{t}\, v_\pi(t), \quad t \in [0,\tau].$$

With no loss of generality we can assume that $\delta^* < 1$, so that

$$\|u_0\|_{L^2[0,\tau]} \leqslant \|v_0\|_{L^2[0,\tau]}, \quad \|u_\pi\|_{L^2[0,\tau]} \leqslant \|v_\pi\|_{L^2[0,\tau]}.$$

We have shown that for every $\psi \in W^{-1,2}(0,\pi)$, $\tau \in (0,\delta^*)$ and $\varphi_0 \in A^2(\Delta,\omega_{0,\tau})$, $\varphi_\pi \in A^2(\tilde{\Delta},\omega_{\pi,\tau})$ satisfying (6-6) there exist $u_0, u_\pi \in L^2[0,\tau]$ satisfying (6-9), together with

$$\left\| \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} \right\|_{(L^2[0,\tau])^2} \leqslant \frac{1}{1-\gamma_\tau} \left\| \begin{bmatrix} \varphi_0 \\ \varphi_\pi \end{bmatrix} \right\|_{A^2(\Delta,\omega_{0,\tau})\times A^2(\tilde{\Delta},\omega_{\pi,\tau})}, \tag{6-10}$$

where $\gamma_\tau$ was defined in (6-7). Since (6-10) holds for every $\varphi_0 \in A^2(\Delta,\omega_{0,\tau})$ and $\varphi_\pi \in A^2(\tilde{\Delta},\omega_{\pi,\tau})$ satisfying (6-6), using (1-11) and (3-1) it follows that

$$\|\mathbb{T}_\tau \psi\|_{\operatorname{Ran}\Phi_\tau} \leqslant \frac{1}{1-\gamma_\tau} \|\mathbb{T}_\tau \psi\|_\tau, \quad \psi \in W^{-1,2}(0,\pi),\ \tau \in (0,\delta^*). \tag{6-11}$$

Since by Lemma 4.3 we have that $\lim_{\tau\to 0+}\gamma_\tau = 0$, the conclusion (6-5) follows from (6-1) and (6-4). □

**Remark 6.3.** Analyzing the proof of Proposition 6.2 it is easily seen that (6-11) holds with an arbitrary $\eta \in \operatorname{Ran}\Phi_\tau$ instead of $\mathbb{T}_\tau\psi$. We thus have that

$$\limsup_{\tau\to 0+} \frac{\|\eta\|_{\operatorname{Ran}\Phi_\tau}}{\|\eta\|_{X_\tau}} \leqslant 1, \quad \eta \in \operatorname{Ran}\Phi_\tau \setminus \{0\},$$

and the existence of a constant $K^* > 0$ such that

$$\|\eta\|_{\operatorname{Ran}\Phi_\tau} \leqslant K^* \|\eta\|_\tau, \quad \tau \in (0,\delta^*),\ \eta \in \operatorname{Ran}\Phi_\tau.$$

By the closed graph theorem, it follows that for every $\tau \in (0,\delta^*)$ the norms $\|\cdot\|_{\operatorname{Ran}\Phi\tau}$ and $\|\cdot\|_\tau$ are equivalent.

## 7. Sums of Bergman spaces on symmetric sectors

The aim of this section is two-fold. We first prove Proposition 1.2 and thus, consequently, Corollary 1.3. We next connect our results to those obtained recently, with a different methodology, in [Orsoni 2021], where an apparently different characterization of the reachable space was given. More precisely, the main result in [Orsoni 2021] asserts that

$$\operatorname{Ran}\Phi_\tau = A^2(\Delta) + A^2(\tilde{\Delta}), \quad \tau > 0. \tag{7-1}$$

Putting together (7-1) and Corollary 1.3 it follows that:

**Proposition 7.1.** *For every $\delta > 0$ we have*

$$X_\delta = A^2(\Delta) + A^2(\tilde{\Delta}), \tag{7-2}$$

*where the spaces $(X_\delta)_{\delta>0}$ were defined in (1-10).*

The second main aim of this section is to show that Proposition 7.1 follows from Proposition 1.2 and thus providing a new proof of (7-1).

An essential step in proving Proposition 1.2 is the construction of a family of entire functions $(\Theta_{\tau,t})$ having the property that if $0 < \tau < t$ and $t - \tau$ is small enough then the multiplication by $\Theta_{\tau,t}$ defines a bounded linear operator from $X_\tau$ to $X_t$, provided that $\tau$ and $t$ are close enough. To this aim, we need several lemmas involving the families of functions

$$\theta_{\tau,t}(s) := \mathrm{e}^{s^2(t-\tau)/(4t\tau)}, \qquad \tau,\ t \geqslant 0,\ s \in \Delta \cup \tilde{\Delta}, \tag{7-3}$$

$$\tilde{\theta}_{\tau,t}(s) := \mathrm{e}^{(\pi-s)^2(t-\tau)/(4t\tau)}, \quad \tau,\ t \geqslant 0,\ s \in \Delta \cup \tilde{\Delta}, \tag{7-4}$$

$$\Theta_{\tau,t}(s) := \theta_{\tau,t}(s) + \tilde{\theta}_{\tau,t}(s), \quad \tau,\ t \geqslant 0,\ s \in \Delta \cup \tilde{\Delta}. \tag{7-5}$$

The three functions defined above are, for every $\tau,\ t > 0$, holomorphic on $\mathbb{C}$ and for every $s \in \mathbb{C}$ we have

$$\tilde{\theta}_{\tau,t}(s) = \theta_{\tau,t}(\pi - s), \quad \Theta_{\tau,t}(s) = \Theta_{\tau,t}(\pi - s). \tag{7-6}$$

Moreover, we have:

**Lemma 7.2.** *Assume $t,\ \tau,\ > 0$ are such that*

$$\frac{t\tau}{t-\tau} > \frac{\pi}{4}. \tag{7-7}$$

*Then the function $\Theta_{\tau,t}$ defined in (7-5) has no zeros on $\overline{\Delta \cup \tilde{\Delta}}$. Moreover, there exist $\alpha$ , $\beta > 0$ (possibly depending on $\tau$ and $t$) such that the functions $\theta_{\tau,t}$ and $\tilde{\theta}_{\tau,t}$ defined in (7-3) and (7-4), respectively, satisfy*

$$\alpha \leqslant \left| 1 + \frac{\tilde{\theta}_{\tau,t}(s)}{\theta_{\tau,t}(s)} \right| \leqslant \beta, \quad s \in \Delta. \tag{7-8}$$

*Proof.* Using the fact that

$$\Theta_{\tau,t}(s) = \theta_{\tau,t}(s)(1 + \mathrm{e}^{(\pi^2-2\pi s)(t-\tau)/(4t\tau)}), \quad \tau,\ t \geqslant 0,\ s \in \mathbb{C}, \tag{7-9}$$

it can be easily checked that $\Theta_{\tau,t}$ vanishes for some $s \in \mathbb{C}$ if and only if

$$\operatorname{Re} s = \frac{\pi}{2}, \quad \operatorname{Im} s \in \frac{-2t\tau}{t-\tau} + \frac{4t\tau}{t-\tau}\mathbb{Z}. \tag{7-10}$$

On the other hand, for every $\tau, t$ satisfying (7-7) we have

$$\left( \frac{-2t\tau}{t-\tau} + \frac{4t\tau}{t-\tau}\mathbb{Z} \right) \cap \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] = \varnothing,$$

which, together with (7-10), implies that indeed $\Theta_{\tau,t}$ has no zeros in $\overline{\Delta \cup \tilde{\Delta}}$.

In order to prove (7-8) we introduce the compact set

$$K := \{ s \in \overline{\Delta} \mid \operatorname{Re} s \in [0, \pi] \}.$$

The function

$$s \mapsto |1+e^{(\pi^2-2\pi s)(t-\tau)/(4t\tau)}|, \quad s \in \mathbb{C},$$

is continuous on $K$ and we have shown above that it is nonvanishing on $K$. Consequently, there exists $\alpha_1 > 0$ such that

$$|1+e^{(\pi^2-2\pi s)(t-\tau)/(4t\tau)}| \geqslant \alpha_1, \quad s \in K.$$

For $s \in \bar{\Delta} \setminus K$ we have $\operatorname{Re} s > \pi$; thus

$$|e^{(\pi^2-2\pi s)(t-\tau)/(4t\tau)}| \leqslant e^{-\pi^2(t-\tau)/(4t\tau)} < 1.$$

Hence, there exists $\alpha_2 > 0$ such that for every $s \in \Delta \setminus K$ we have

$$|1+e^{(\pi^2-2\pi s)(t-\tau)/(4t\tau)}| \geqslant 1-e^{-\pi^2(t-\tau)/(4t\tau)} \geqslant \alpha_2 > 0.$$

Setting $\alpha := \min(\alpha_1, \alpha_2) > 0$ , we obtain the first inequality in (7-8).

Finally, using the fact that $\operatorname{Re} s \geqslant 0$ for every $s \in \bar{\Delta}$, it follows that

$$|1+e^{(\pi^2-2\pi s)(t-\tau)/(4t\tau)}| \leqslant 1+e^{\pi^2(t-\tau)/(4t\tau)}, \quad s \in \bar{\Delta},$$

which implies the second inequality in (7-8). □

**Lemma 7.3.** *Let $\tau, t$ satisfy the assumptions in Lemma 7.2 and let $\Theta_{\tau,t}$ be the function defined in (7-9). Then for every $f \in A^2(\Delta; \omega_{0,t})$ and every $\tilde{f} \in A^2(\tilde{\Delta}; \omega_{\pi,t})$ we have*

$$\frac{f}{\Theta_{\tau,t}} \in A^2(\Delta; \omega_{0,\tau}), \qquad \frac{\tilde{f}}{\Theta_{\tau,t}} \in A^2(\tilde{\Delta}; \omega_{\pi,\tau}).$$

*Proof.* Let $f \in A^2(\Delta; \omega_{0,t})$. We know from Lemma 7.2 that $f/\Theta_{\tau,t}$ is holomorphic on $\Delta$. Moreover, by combining (7-9) and Lemma 7.2, it follows that for every $s \in \Delta$ we have

$$\begin{aligned}\frac{|f(s)|^2}{|\Theta_{\tau,t}(s)|^2}\omega_{0,\tau}(s) &= \frac{|f(s)|^2}{|\theta_{\tau,t}(s)|^2|1+e^{(\pi^2-2\pi s)(t-\tau)/(4t\tau)}|^2}\omega_{0,\tau}(s)\\ &\leqslant \frac{1}{\alpha^2}\frac{|f(s)|^2}{|\theta_{\tau,t}(s)|^2}\omega_{0,\tau}(s) = \frac{t}{\tau\alpha^2}|f(s)|^2\omega_{0,t}(s).\end{aligned}$$

Using our assumptions on $f$ it follows that for every $f \in A^2(\Delta, \omega_{0,t})$ we have

$$\frac{f}{\Theta_{\tau,t}} \in A^2(\Delta; \omega_{0,\tau}).$$

Using this fact and (7-6), the corresponding result for $\tilde{f} \in A^2(\tilde{\Delta}; \omega_{\pi,t})$ readily follows. □

**Lemma 7.4.** *Let $\tau, t$ satisfy the assumptions in Lemma 7.2 and let $\Theta_{\tau,t}$ be the function defined in (7-9). Let $\gamma, \gamma' > 0$ be such that*

$$\frac{t-\tau}{t\tau} + \frac{1}{\gamma'} \leqslant \frac{1}{\gamma}. \tag{7-11}$$

*Then for every* $f \in A^2(\Delta; \omega_{0,\gamma})$ *and every* $\tilde{f} \in A^2(\tilde{\Delta}; \omega_{\pi,\gamma})$ *we have*

$$f\Theta_{\tau,t} \in A^2(\Delta; \omega_{0,\gamma'}), \qquad \tilde{f}\Theta_{\tau,t} \in A^2(\tilde{\Delta}; \omega_{\pi,\gamma'}).$$

*Proof.* Let $f \in A^2(\Delta; \omega_{0,\gamma})$. Using Lemma 7.2 it follows that for every $s \in \Delta$ we have

$$\begin{aligned} |f(s)\Theta_{\tau,t}(s)|^2\omega_{0,\gamma'}(s) &= |f(s)|^2|\theta_{t,\tau}(s)|^2\left|1+\frac{\tilde{\theta}_{\tau,t}(s)}{\theta_{\tau,t}(s)}\right|^2\omega_{0,\gamma'}(s)\\ &\leqslant \frac{\beta^2}{\gamma'}|f(s)|^2 \mathrm{e}^{\mathrm{Re}(s^2)((t-\tau)/(2t\tau)+1/(2\gamma'))} \leqslant \frac{\beta^2\gamma}{\gamma'}|f(s)|^2\omega_{0,\gamma}(s), \end{aligned}$$

which shows that indeed $f\Theta_{\tau,t} \in A^2(\Delta; \omega_{0,\gamma'})$. As above, the corresponding result for $A^2(\tilde{\Delta}; \omega_{\pi,\gamma})$ follows by symmetry; see (7-6). □

**Lemma 7.5.** *Let* $\tau, t$ *satisfy the assumptions in Lemma 7.2 such that* $\varepsilon := t-\tau < \tau$, *and let* $\Theta_{\tau,t}$ *be the function defined in (7-9). Moreover, assume that* $\tau > 0$ *is such that*

$$X_\delta = X_\tau, \quad 0 < \delta \leqslant \tau. \tag{7-12}$$

(*This holds, in particular, for* $\tau \in (0, \delta^*)$, *where* $\delta^*$ *is the constant in Theorem 1.1.*) *Then* $X_t = X_\tau$.

*Proof.* As already mentioned, it is obvious that $X_\tau \subset X_t$. To prove that $X_t \subset X_\tau$, let $\varphi \in A^2(\Delta; \omega_{0,t})$ and $\tilde{\varphi} \in A^2(\tilde{\Delta}; \omega_{\pi,t})$. According to Lemma 7.3 and (7-12) we have

$$\frac{\varphi}{\Theta_{\tau,t}} + \frac{\tilde{\varphi}}{\Theta_{\tau,t}} \in X_\tau = X_{\tau-\varepsilon},$$

where $\varepsilon = t-\tau < \tau$. It follows that there exist $f \in A^2(\Delta; \omega_{0,\tau-\varepsilon})$ and $\tilde{f} \in A^2(\tilde{\Delta}; \omega_{\pi,\tau-\varepsilon})$ such that

$$\frac{\varphi}{\Theta_{\tau,t}} + \frac{\tilde{\varphi}}{\Theta_{\tau,t}} = f + \tilde{f}.$$

Let $\gamma = \tau - \varepsilon$ and $\gamma' = \tau$. Since $t \geqslant \tau - \varepsilon$ we have

$$\frac{t-\tau}{t\tau} + \frac{1}{\gamma'} = \frac{\varepsilon}{t\tau} + \frac{1}{\tau} \leqslant \frac{\varepsilon}{(\tau-\varepsilon)\tau} + \frac{1}{\tau} = \frac{1}{\tau-\varepsilon} = \frac{1}{\gamma},$$

so that inequality (7-11) holds. Consequently, using Lemma 7.4 it follows that

$$\varphi + \tilde{\varphi} = f\Theta_{\tau,t} + \tilde{f}\Theta_{\tau,t} \in A^2(\Delta; \omega_{0,\tau}) + A^2(\tilde{\Delta}; \omega_{\pi,\tau}) = X_\tau,$$

which ends the proof. □

We are now in a position to prove Proposition 1.2.

*Proof of Proposition 1.2.* Let

$$\mathcal{I} = \{\tau > 0 \mid X_\delta = X_\tau \text{ for all } \delta \in (0, \tau]\}.$$

It is clear that if $\tau \in \mathcal{I}$ then $(0, \tau] \subset \mathcal{I}$. From Theorem 1.1 we know that $\mathcal{I} \supset (0, \delta^*/2]$, where $\delta^*$ is the constant in Theorem 1.1. Let

$$\tau_0 := \min\left(\frac{\pi}{8}, \frac{\delta^*}{2}\right), \qquad m := \frac{\pi\tau_0}{\pi - 4\tau_0}, \qquad \varepsilon_0 := \frac{m - \tau_0}{2}.$$

We clearly have that $\tau_0 \in \mathcal{I}$, $\varepsilon_0 < \tau_0$ and

$$\frac{(\tau_0 + \varepsilon_0)\tau_0}{\varepsilon_0} > \frac{\pi}{4}.$$

Since the function $t \mapsto (t + \varepsilon_0)t/\varepsilon_0$ is clearly increasing on $(0, \infty)$, it follows that

$$\frac{(\tau + \varepsilon_0)\tau}{\varepsilon_0} > \frac{\pi}{4}, \qquad \tau > \tau_0. \tag{7-13}$$

Consider now the sequence $(t_n)_{n\in\mathbb{Z}_+}$ defined by $t_0 = \tau_0$ and

$$t_{n+1} = t_n + \varepsilon_0, \qquad n \in \mathbb{Z}_+,$$

which obviously satisfies $\lim_{n\to\infty} t_n = +\infty$. Then (7-13) enables us to recursively apply Lemma 7.5 and obtain that for every $n \in \mathbb{N}$ we have $t_n \in \mathcal{I}$ and, consequently, $X_{t_n} = X_{\tau_0}$. This shows that $\mathcal{I} = (0, \infty)$, which ends the proof. □

As mentioned in the beginning of this section, our second aim here is to give a direct proof of Proposition 7.1, which provides an alternative proof of (7-1). To this aim, we need the following result:

**Lemma 7.6.** *Let $\varphi \in A^2(\Delta)$ and $\tilde{\varphi} \in A^2(\tilde{\Delta})$. Then*

$$\frac{\varphi}{\Theta_{\pi/4,\pi/2}} \in A^2(\Delta; \omega_{0,\pi/2}), \qquad \frac{\tilde{\varphi}}{\Theta_{\pi/4,\pi/2}} \in A^2(\tilde{\Delta}; \omega_{\pi,\pi/2}).$$

*Proof.* Let $\varphi \in A^2(\Delta)$ . We know from Lemma 7.2 that the function $\Theta_{\pi/4,\pi/2}$ has no zeros in $\overline{\Delta \cup \tilde{\Delta}}$, so that the function $s \mapsto \varphi/\Theta_{\pi/4,\pi/2}$ is holomorphic on $\Delta$. Moreover, using (7-9) and again Lemma 7.2 it follows that for every $s \in \Delta$ we have

$$\begin{aligned}\frac{|\varphi(s)|^2}{|\Theta_{\pi/4,\pi/2}(s)|^2}\omega_{0,\pi/2}(s) &= \frac{|\varphi(s)|^2}{|\theta_{\pi/4,\pi/2}(s)|^2|1 + \mathrm{e}^{(\pi^2-2\pi s)/(2\pi)}|^2}\omega_{0,\pi/2}(s)\\ &\leqslant \frac{1}{\alpha^2}\frac{|\varphi(s)|^2}{|\theta_{\pi/4,\pi/2}(s)|^2}\omega_{0,\pi/2}(s) = \frac{2}{\pi\alpha^2}|\varphi(s)|^2,\end{aligned}$$

which implies that $\varphi/\Theta_{\pi/4,\pi/2} \in A^2(\Delta; \omega_{0,\pi/2})$. The fact that $\tilde{\varphi}/\Theta_{\pi/4,\pi/2} \in A^2(\tilde{\Delta}; \omega_{\pi,\pi/2})$ is obtained by symmetry using (7-6). □

We are now in a position to prove Proposition 7.1.

*Proof of Proposition 7.1.* Let $\varphi \in A^2(\Delta)$ and $\tilde{\varphi} \in A^2(\tilde{\Delta})$ and let $g = \varphi + \tilde{\varphi}$. According to Lemma 7.6 and Proposition 1.2 there exist $f \in A^2(\Delta; \omega_{0,\pi/4})$ and $\tilde{f} \in A^2(\tilde{\Delta}; \omega_{\pi,\pi/4})$ such that

$$\frac{g}{\Theta_{\pi/4,\pi/2}} = f + \tilde{f}.$$

Using next Lemma 7.4 with $t=\frac{\pi}{2}$, $\tau=\frac{\pi}{4}$, $\gamma=\frac{\pi}{4}$ and $\gamma'=\frac{\pi}{2}$, it follows that

$$g = f\Theta_{\pi/4,\pi/2} + \tilde{f}\Theta_{\pi/4,\pi/2} \in A^2(\Delta;\omega_{0,\pi/2}) + A^2(\tilde{\Delta};\omega_{\pi,\pi/2}) = X_{\pi/2}.$$

We have thus shown that

$$A^2(\Delta) + A^2(\tilde{\Delta}) \subset X_{\pi/2}.$$

Since the inclusion $X_{\pi/2} \subset A^2(\Delta) + A^2(\tilde{\Delta})$ is an obvious one, we have

$$A^2(\Delta) + A^2(\tilde{\Delta}) = X_{\pi/2}.$$

The conclusion follow now by using again Proposition 1.2. □

Finally, we remark that by combining Proposition 5.1 with Propositions 1.2 and 7.1 we obtain:

**Corollary 7.7.** *Given $n \in \mathbb{Z}_+$ and $\tau > 0$ we have*

$$X_{n,\tau} = A^{n,2}(\Delta) + A^{n,2}(\tilde{\Delta}),$$

*where the Banach spaces $X_{n,\tau}$ and $A^{n,2}(\Delta)$ were defined in (5-2) and (5-14), respectively.*

## 8. Comments and related questions

In this section we first discuss the consequences of our results and methods developed in the previous section on the reachable space of the heat equation with slightly different boundary conditions. Moreover, we give, as promised in Remark 6.1, a lower bound for the cost of null controllability for the system (1-1) in terms of the cost of null controllability for the system described by the heat equation on $\left(0,\frac{\pi}{2}\right)$, with control acting only at the left end. We next discuss possible extensions and open problems.

We use repeatedly in this section the following simple notation: for every complex-valued function $f$ defined on $\left(0,\frac{\pi}{2}\right)$ we denote by $Lf$ its extension to a function defined on $(0,\pi)$ obtained by setting

$$(Lf)(x) = \begin{cases} f(x), & x \in \left(0,\frac{\pi}{2}\right), \\ f(\pi - x), & x \in \left(\frac{\pi}{2},\pi\right). \end{cases} \tag{8-1}$$

When there will be no risk of confusion we will identify $f$ and $Lf$.

Concerning the description of the reachable space for other boundary conditions or controls we detail the case of Dirichlet control at one end, with homogeneous Dirichlet boundary condition at the other end. For the sake of convenience, we consider the corresponding heat equation on the space interval $\left(0,\frac{\pi}{2}\right)$, with control acting at the left end.

More precisely, for every $\tau > 0$ we are interested in the range of the operator $\Phi_\tau^{\text{odd}}$ defined as follows: denoting by $y$ the solution of

$$\begin{cases} \dfrac{\partial y}{\partial t}(t,x) = \dfrac{\partial^2 y}{\partial x^2}(t,x) t \geqslant 0, & x \in \left(0,\frac{\pi}{2}\right), \\ y(t,0) = u(t), \quad y\left(t,\frac{\pi}{2}\right) = 0, & t \in [0,\infty), \\ y(0,x) = 0, & x \in \left(0,\frac{\pi}{2}\right), \end{cases} \tag{8-2}$$

$\Phi_\tau^{\text{odd}}$ is defined by

$$\Phi_\tau^{\text{odd}} u = y(\tau,\cdot), \quad u \in L^2[0,\tau]. \tag{8-3}$$

**Proposition 8.1.** *For $\tau > 0$ we have*

$$\operatorname{Ran} \Phi_\tau^{\mathrm{odd}} = \{\eta \in \operatorname{Ran} \Phi_\tau \mid \eta(s) + \eta(\pi - s) = 0 \text{ for } s \in \Delta \cup (\pi - \Delta)\}, \tag{8-4}$$

*where $\Phi_\tau$ and $\Delta$ were defined in (1-2) and (1-7), respectively.*

*Proof.* Let $\eta \in \operatorname{Ran} \Phi_\tau^{\mathrm{odd}}$ and let $u \in L^2[0, \tau]$ be such that $\Phi_\tau^{\mathrm{odd}} u = \eta$. Consequently, the solution $y$ of (8-2) with $u$ introduced above satisfies

$$y(\tau, x) = \eta(x), \quad x \in \left(0, \tfrac{\pi}{2}\right). \tag{8-5}$$

Let

$$w(t, \cdot) = (Ly)(t, \cdot), \quad t \in [0, \tau], \tag{8-6}$$

where $L$ is the operator introduced in (8-1). Thanks to the fact that $y\left(t, \frac{\pi}{2}\right) = 0$, it follows that $w$ satisfies (1-1), with $u_0(t) = u(t)$ and $u_\pi(t) = -u(t)$. We thus have $L\eta = w(\tau, \cdot) \in \operatorname{Ran} \Phi_\tau$ so that, identifying $\eta$ and $L\eta$, we have thus shown that the inclusion

$$\operatorname{Ran} \Phi_\tau^{\mathrm{odd}} \subset \{\eta \in \operatorname{Ran} \Phi_\tau \mid \eta(s) + \eta(\pi - s) = 0 \text{ for } s \in \Delta \cup (\pi - \Delta)\}$$

holds for every $\tau > 0$.

To prove the inclusion

$$\{\eta \in \operatorname{Ran} \Phi_\tau \mid \eta(s) + \eta(\pi - s) = 0 \text{ for } s \in \Delta \cup (\pi - \Delta)\} \subset \operatorname{Ran} \Phi_\tau^{\mathrm{odd}}, \tag{8-7}$$

for every $\eta \in \operatorname{Ran} \Phi_\tau$ we denote by $u_0, u_\pi \in L^2[0, \tau]$ two controls such that the solution $w$ of (1-1) satisfies

$$w(\tau, x) = \eta(x), \quad \tau > 0, \ x \in (0, \pi).$$

Let

$$y(t, x) = \tfrac{1}{2}(w(t, x) - w(t, \pi - x)), \quad t \geqslant 0, \ x \in (0, \pi).$$

Then $y$ satisfies (1-1) with $u_0$ and $u_\pi$ replaced by $\frac{1}{2}(u_0 - u_\pi)$ and $\frac{1}{2}(u_\pi - u_0)$, respectively. Since we clearly have $y\left(t, \frac{\pi}{2}\right) = 0$ for every $t \geqslant 0$, it follows that $y$ satisfies (8-2), with $u$ replaced $\frac{1}{2}(u_0 - u_{2\pi})$ and $y(\tau, x) = \eta(x)$ for $x \in \left(0, \frac{\pi}{2}\right)$. Consequently, $\eta = \frac{1}{2}\Phi_\tau^{\mathrm{odd}}(u_0 - u_\pi)$, so that $\eta \in \operatorname{Ran} \Phi_\tau^{\mathrm{odd}}$. We have thus proved (8-7), which ends the proof. □

**Remark 8.2.** The reachable space for the 1-dimensional heat equation with other boundary conditions and controls (at one or both ends) can also be made completely explicit by using arguments fully similar to those above (see also Section 5 of [Hartmann et al. 2020]).

If we introduce, for instance, for every $\tau > 0$, the operator $\Phi_\tau^{\mathrm{even}}$ defined by

$$\Phi_\tau^{\mathrm{even}} u = z(\tau, \cdot), \quad u \in L^2[0, \tau], \tag{8-8}$$

where $z$ satisfies

$$\begin{cases} \dfrac{\partial z}{\partial t}(t, x) = \dfrac{\partial^2 z}{\partial x^2}(t, x), & t \geqslant 0, \ x \in \left(0, \frac{\pi}{2}\right), \\ z(t, 0) = u(t), \quad \dfrac{\partial z}{\partial x}\left(t, \frac{\pi}{2}\right) = 0, & t \in [0, \infty), \\ z(0, x) = 0, & x \in \left(0, \frac{\pi}{2}\right), \end{cases} \tag{8-9}$$

then for every $\tau > 0$ we have

$$\operatorname{Ran} \Phi_\tau^{\text{even}} = \{\eta \in \operatorname{Ran} \Phi_\tau \mid \eta(s) = \eta(\pi - s) \text{ for } s \in \Delta \cup (\pi - \Delta)\}. \tag{8-10}$$

We continue this section with two results leading, as promised in Remark 6.1, to a lower bound for the null controllability cost for the system (1-1) in terms of the cost of the null controllability of the corresponding system on $\left(0, \frac{\pi}{2}\right)$ (that is (8-2)). To this aim, we first note that the following result holds.

**Proposition 8.3.** *Let $\tau > 0$ and $\Phi_\tau$ and $\Phi_\tau^{\text{odd}}$ be the operators defined in (1-2) and (8-3), respectively. Then*

$$\|\eta\|_{\operatorname{Ran} \Phi_\tau^{\text{odd}}} \leqslant \frac{1}{\sqrt{2}} \|L\eta\|_{\operatorname{Ran} \Phi_\tau}, \qquad \tau > 0,\ \eta \in \operatorname{Ran} \Phi_\tau^{\text{odd}}, \tag{8-11}$$

*where the operator $L$ was defined in (8-1).*

*Proof.* Let $\eta \in \operatorname{Ran} \Phi_\tau^{\text{odd}}$. For every $u_0, u_\pi \in L^2[0, \tau]$ with

$$\Phi_\tau \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} = L\eta \tag{8-12}$$

we set

$$y(t, x) = \frac{1}{2} \left( \Phi_t \begin{bmatrix} u_0 - u_\pi \\ u_\pi - u_0 \end{bmatrix} \right)(x), \qquad t > 0,\ x \in \left(0, \tfrac{\pi}{2}\right).$$

Then $y$ clearly satisfies (8-2), with $u$ replaced by $\frac{1}{2}(u_0 - u_\pi)$. This fact, combined with (8-3) and (8-12), implies that

$$\eta = \frac{1}{2} \Phi_\tau^{\text{odd}} (u_0 - u_\pi).$$

Thus for every $\tau > 0$, $\eta \in \operatorname{Ran} \Phi_\tau^{\text{odd}}$ and for every $u_0,\ u_\pi \in L^2[0, \tau]$ satisfying (8-12) we have

$$\|\eta\|_{\operatorname{Ran} \Phi_\tau^{\text{odd}}} \leqslant \frac{1}{2} \|u_0 - u_\pi\|_{L^2[0,\tau]} \leqslant \frac{1}{\sqrt{2}} \sqrt{\|u_0\|_{L^2[0,\tau]}^2 + \|u_\pi\|_{L^2[0,\tau]}^2}.$$

Taking the lower bound of the right-hand side of the above inequality over all the controls $u_0, u_\pi \in L^2[0, \tau]$ satisfying (8-12) we obtain (8-11). □

**Corollary 8.4.** *Let $\mathbb{T}$ and $\mathbb{T}^{\text{odd}}$ be the semigroups on $W^{-1,2}(0, \pi)$ and $W^{-1,2}\left(0, \frac{\pi}{2}\right)$ generated by the Dirichlet Laplacians on $(0, \pi)$ and $\left(0, \frac{\pi}{2}\right)$, respectively. For $\tau > 0$, let $c_\tau$ and $c_\tau^{\text{odd}}$ be the costs of null controllability in time $\tau$ for the systems defined by (1-1) and (8-2), respectively. Recalling (6-1), this means that*

$$c_\tau = \sup_{\|\psi\|_{W^{-1,2}(0,\pi)} \leqslant 1} \|\mathbb{T}_\tau \psi\|_{\operatorname{Ran} \Phi_\tau}, \tag{8-13}$$

$$c_\tau^{\text{odd}} = \sup_{\|\psi\|_{W^{-1,2}(0,\pi/2)} \leqslant 1} \|\mathbb{T}_\tau^{\text{odd}} \psi\|_{\operatorname{Ran} \Phi_\tau^{\text{odd}}}, \tag{8-14}$$

*where $\Phi_\tau$ and $\Phi_\tau^{\text{odd}}$ have been defined in (1-2) and (8-3), respectively.*

*Then*

$$c_\tau^{\text{odd}} \leqslant \sqrt{2} c_\tau, \qquad \tau > 0. \tag{8-15}$$

*Proof.* We first note that the operator $L$ defined in (8-1) satisfies

$$\|Lf\|_{W^{-1,2}(0,\pi)} = 2\|f\|_{W^{-1,2}(0,\pi/2)}, \qquad f \in W^{-1,2}\big(0, \tfrac{\pi}{2}\big). \tag{8-16}$$

Moreover, simple symmetry arguments show that

$$L\mathbb{T}_t^{\mathrm{odd}}\psi = \mathbb{T}_t L\psi \quad t > 0,\ \psi \in W^{-1,2}\big(0, \tfrac{\pi}{2}\big). \tag{8-17}$$

Consequently, using consecutively (8-14), Proposition 8.3, (8-17) and (8-16) we have

$$c_\tau^{\mathrm{odd}} \leqslant \frac{1}{\sqrt{2}} \sup_{\|\psi\|_{W^{-1,2}(0,\pi/2)}\leqslant 1} \|\mathbb{T}_\tau L\psi\|_{\mathrm{Ran}\,\Phi_\tau} \leqslant \frac{1}{\sqrt{2}} \sup_{\|\varphi\|_{W^{-1,2}(0,\pi)}\leqslant 2} \|\mathbb{T}_\tau \varphi\|_{\mathrm{Ran}\,\Phi_\tau}, \tag{8-18}$$

which yields the conclusion (8-15). □

In spite of several recent advances, the study of the reachable space for the 1-dimensional heat equation with various types of controls still has interesting open questions. We can mention, for instance, the case of control supported at a point inside the interval (pointwise control), where the methods developed in the present work might be adapted to describe the dependence of the reachable space on the diophantine approximation properties of the control location. As far as we know, the case of a control acting on a subinterval of $(0, \pi)$ has not been explicitly studied in the literature. However, we think that the methods and results in [Dardé and Ervedoza 2018] can be adapted to this situation, to yield the reachability of functions which are holomorphic in any neighborhood of an appropriate domain. Due to the fact that the use of cut-off functions seems necessary in this situation, we think that the methods developed in our paper would not allow improving this type of result. Finally, concerning the case of a system described by the heat equation in a bounded domain of $\mathbb{R}^n$, with control acting on the whole boundary, interesting advances, which can be seen as generalizations of the main result in [Dardé and Ervedoza 2018], have been obtained in [Strohmaier and Waters 2022]. However, the question of characterizing in this case the reachable space in terms of known spaces of function which are analytic on appropriate domains of $\mathbb{C}^n$ seems still a very interesting open question.

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KARIM KELLAY: kkellay@math.u-bordeaux.fr
*Institut de Mathématiques de Bordeaux, Université de Bordeaux, Talence, France*

THOMAS NORMAND: thomas.normand@math.u-bordeaux.fr
*Institut de Mathématiques de Bordeaux, Université de Bordeaux, Talence, France*

MARIUS TUCSNAK: marius.tucsnak@u-bordeaux.fr
*Institut de Mathématiques de Bordeaux, Université de Bordeaux, Talence, France*

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