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Metastability Results for a Class of Linear Boltzmann Equations

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Abstract. We consider a semiclassical linear Boltzmann model with a non-local collision operator. We provide sharp spectral asymptotics for the small spectrum in the low temperature regime from which we deduce the rate of return to equilibrium as well as a metastability result. The main ingredients are resolvent estimates obtained via hypocoercive techniques and the construction of sharp Gaussian quasimodes through an adaptation of the WKB method.

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1. Introduction

1.1. Motivations

We are interested in the linear Boltzmann equation:

$$\begin{cases} h\partial_t u + v\cdot h\partial_x u - \partial_x V\cdot h\partial_v u + Q_{\mathcal{H}}(h,u) = 0 \\ u_{|t=0} = u_0 \end{cases} \tag{1.1}$$

in a semiclassical framework (i.e., in the limit $h\to 0$), where h is a *semiclassical parameter* and corresponds to the temperature of the system. Here, we denoted for shortness ∂_x and ∂_v the partial gradients with respect to x and v. This equation is used to model the evolution of a system of charged particles in a gas on which acts an electrical force associated with the real-valued potential V that only depends on the space variable x. The interactions between the particles are modeled by the linear operator $Q_{\mathcal{H}}$ which is called *collision operator*. Here, the unknown is the function $u:\mathbb{R}_+\to L^1(\mathbb{R}^{2d})$ giving the probability density of the system of particles at time $t\in\mathbb{R}_+$, position $x\in\mathbb{R}^d$ and velocity $v\in\mathbb{R}^d$. For our purpose, we introduce the square roots of the usual Maxwellian distributions

$$\mu_h(v) = \frac{\mathrm{e}^{-\frac{v^2}{4h}}}{(2\pi h)^{d/4}} \qquad \text{and} \qquad \mathcal{M}_h = \mathrm{e}^{-\frac{V}{2h}}\mu_h. \tag{1.2}$$

In many models, we have

$$Q_{\mathcal{H}}(h,\mathcal{M}_h^2) = 0 \qquad \text{and} \qquad Q^*_{\mathcal{H}}(h,1) = 0 \tag{1.3}$$

so in particular $\mathcal{M}_h^2$ is a stable state of (1.1). In order to do a perturbative study of the time independent operator associated with (1.1) near $\mathcal{M}_h^2$, we introduce the natural Hilbert space

$$\mathcal{H} = \big\{u\in\mathcal{D}';\ \mathcal{M}_h^{-1}u\in L^2(\mathbb{R}^{2d})\big\}.$$

It is clear from the Cauchy–Schwarz inequality that $\mathcal{H}$ is indeed a subset of $L^1(\mathbb{R}^{2d})$ provided that $\mathrm{e}^{-\frac{V}{2h}}\in L^2(\mathbb{R}^d_x)$. In view of (1.3) and the definition of $\mathcal{H}$, it is more convenient to work with the new unknown

$$f = \mathcal{M}_h^{-1}u\ :\mathbb{R}_+\to L^2(\mathbb{R}^{2d})$$

for which the new equation becomes

$$\begin{cases} h\partial_t f + v\cdot h\partial_x f - \partial_x V\cdot h\partial_v f + Q_h(f) = 0 \\ f_{|t=0} = f_0 \end{cases} \tag{1.4}$$

where

$$Q_h = \mathcal{M}_h^{-1}\circ Q_{\mathcal{H}}(h,\cdot)\circ\mathcal{M}_h.$$

Our study will be focused on the new time independent operator

$$\begin{aligned}P_h &= v\cdot h\partial_x - \partial_x V\cdot h\partial_v + Q_h\\ &= X_0^h + Q_h\end{aligned}$$

for some specific choices of the collision operator Q_h, where the notation X_0^h will stand for the operator $v\cdot h\partial_x - \partial_x V\cdot h\partial_v$, but also for the vector field $(x, v) \mapsto h(v, -\partial_x V(x))$. There are plenty of different collision operators studied in the literature, their main properties being that these are symmetric integral operators acting as multiplicators in the position variable x and canceling the Maxwellian distribution. Our work is in particular motivated by the study of the *mild relaxation* operator introduced in [19] and given by $H_0(1+H_0)^{-1}$ with H_0 the harmonic oscillator in velocity defined by

$$H_0 = -h^2\Delta_v + \frac{v^2}{4} - \frac{hd}{2}. \tag{1.5}$$

In this spirit, the collision operators we will be working with will always be bounded and self-adjoint so, $(X_0^h, \mathcal{C}_c^\infty(\mathbb{R}^{2d}))$ being essentially skew-adjoint, the operator P_h (endowed with the appropriate domain) is maximal accretive and (1.4) is well-posed. More generally, some interesting cases of collision operators are given by functions of H_0 (see, for instance, [9,13–15,19]) which is the setting that we will adopt.

This paper is concerned with the spectral study of the operator P_h. This type of questions has recently known some major progress on the impulse of microlocal methods. In the case of the linear Boltzmann equation (1.4), the use of hypocoercive techniques in 2015 in [20] enabled to get some resolvent estimates and establish a rough localization of the small spectrum of P_h which consists of exponentially small eigenvalues in correspondence with the minima of the potential V. This type of result is similar to the one obtained for example for the Witten Laplacian by Helffer and Sjöstrand in [7] in the 1980s. Such a localization already leads to return to equilibrium and metastability results which can be improved as the description of the small spectrum becomes more precise. For example, sharp asymptotics of the small eigenvalues of the Witten Laplacian were obtained later in the 2000s in [2,6] and later again for Kramers–Fokker–Planck-type operators by Hérau et al. in [10]. In these papers, the idea was to exhibit a supersymmetric structure for the operator and then study both the derivative acting from 0-forms into 1-forms and its adjoint with the help of basic quasimodes. In [19], Robbe managed to show that the Boltzmann Eq. (1.4) with mild relaxation enjoys such a supersymmetric structure. However, in that case, the matrix appearing in the modification of the inner product does not obey good estimates with respect to the semiclassical parameter h. This is why our goal here will be to give precise spectral asymptotics for the operator P_h through a more recent approach which consists in directly constructing a family of accurate quasimodes for our operator in the spirit of [1,12].

The aim of this paper is twofold. In a first time, we want to prove a result similar to the one obtained by Robbe in [20] but for a large class of collision operators. The second goal is to provide complete asymptotics of the small eigenvalues of P_h as it was done in [6] for the Witten Laplacian or in [10,11] with recent improvements by Bony et al. in [1] in the case of Fokker–Planck-type differential operators. We manage to establish such results for Eq. (1.4) for a class of pseudo-differential collision operators presenting nice symbol properties as well as a factorized structure.

1.2. Setting and Main Results

For $d' \in \mathbb{N}^*$ and $Z \in \mathbb{C}^{d'}$, we use the standard notation $\langle Z\rangle = (1+|Z|^2)^{1/2}$. In this paper, we will treat the case of collision operators of the form

$$Q_h = \varrho(H_0)$$

with ϱ satisfying the following:

Hypothesis 1.1. *The function $\varrho : \mathbb{R}_+ \to \mathbb{R}_+$ vanishes at the origin and for all $t \geq 0$,*

$$\varrho(t) \geq \frac{1}{C}\frac{t}{\langle t\rangle}.$$

Moreover, it admits an analytic extension to $\{\mathrm{Re}\, z > -\frac{1}{C}\}$ for which there exist $\varrho_\infty \in \mathbb{R}_+$ and $\alpha > 0$ such that $\varrho(z) = \varrho_\infty + O(\langle z\rangle^{-\alpha})$.

In particular, Q_h will be bounded uniformly in h and self-adjoint. An example of such collision operator is the *mild relaxation* operator introduced in [19] and given by $H_0(1+H_0)^{-1}$. In order to state the consequences of Hypothesis 1.1, let us introduce a few notations of semiclassical microlocal analysis which will be used in all this paper. These are mainly extracted from [21], chapter 4. We will denote $\Xi \in \mathbb{R}^{d'}$ the dual variable of X and use the semiclassical Fourier transform

$$\mathcal{F}_h(f)(\Xi) = \int_{\mathbb{R}^{d'}} \mathrm{e}^{-\frac{i}{h}X\cdot\Xi} f(X)\,\mathrm{d}X.$$

We consider the space of semiclassical symbols

$$S^\kappa\big(\langle (X,\Xi)\rangle^k\big) = \Big\{a_h \in \mathcal{C}^\infty(\mathbb{R}^{2d'})\,;\, \forall \alpha \in \mathbb{N}^{2d'}, \exists\, C_\alpha > 0 \textit{ such that } |\partial^\alpha a_h(X,\Xi)| \leq C_\alpha h^{-\kappa|\alpha|}\langle (X,\Xi)\rangle^k\Big\}$$

where $k \in \mathbb{R}$ and $\kappa \in [0,1/2]$. Note that those symbols are allowed to depend on h; however, in order to shorten the notations, we will drop the index h in the rest of the paper when dealing with semiclassical symbols. Given a symbol $a \in S^\kappa(\langle (X,\Xi)\rangle^k)$, we define the associated semiclassical pseudo-differential operator for the Weyl quantization acting on functions $u \in \mathcal{S}(\mathbb{R}^{d'})$ by

$$\mathrm{Op}_h(a)u(X) = (2\pi h)^{-d'}\int_{\mathbb{R}^{d'}}\int_{\mathbb{R}^{d'}} \mathrm{e}^{\frac{i}{h}(X-X')\cdot\Xi} a\Big(\frac{X+X'}{2},\Xi\Big)u(X')\,\mathrm{d}X'\mathrm{d}\Xi$$

where the integrals may have to be interpreted as oscillating integrals. We will denote $\Psi^\kappa(\langle (X,\Xi)\rangle^k)$ the set of such operators. In our setting, we will denote ξ (resp. η) the dual variable of x (resp. v). We also need to introduce the notion of analytic symbols. For our purpose, we almost always consider symbols that do not depend on the variable ξ.

Definition 1.2. For $\tau > 0$, let us introduce the set

$$\Sigma_\tau = \{z \in \mathbb{C}\,;\ |\mathrm{Im}\, z| < \tau\}^d \subset \mathbb{C}^d.$$

For $k \in \mathbb{R}$, we denote $S^0_\tau(\langle (x,v,\eta)\rangle^k)$ the space of symbols $a_h \in S^0(\langle (x,v,\eta)\rangle^k)$ independent of ξ such that:

(i) For all $(x,v) \in \mathbb{R}^{2d}$, $a_h(x,v,\cdot)$ is analytic on Σ_τ

(ii) For all $\beta \in \mathbb{N}^{2d}$, there exists $C_\beta > 0$ such that $|\partial^\beta_{(x,v)} a_h| \leq C_\beta \langle (x,v,\eta)\rangle^k$ on $\mathbb{R}^{2d} \times \Sigma_\tau$.

We will also use the notation $a_h = O_{S^0_\tau(\langle (x,v,\eta)\rangle^k)}(h^N)$ to say that for all $\alpha \in \mathbb{N}^{3d}$, there exists $C_{\alpha,N}$ such that $|\partial^\alpha a_h| \leq C_{\alpha,N}\, h^N \langle (x,v,\eta)\rangle^k$ on $\mathbb{R}^{2d} \times \Sigma_\tau$.

Here again, we will drop the index h in the notations of analytic symbols. Using the Cauchy–Riemann equations, we see that item (i) from Definition 1.2 implies that for all $\beta \in \mathbb{N}^{2d}$ and $(x,v) \in \mathbb{R}^{2d}$, the functions $\partial^\beta_{(x,v)} a(x,v,\cdot)$ are also analytic on Σ_τ. Besides, the Cauchy formula implies that for any $\tilde{\tau} < \tau$, $\alpha \in \mathbb{N}^d$ and $\beta \in \mathbb{N}^{2d}$, there exists $C_{\alpha,\beta}$ such that

$$|\partial^\alpha_\eta \partial^\beta_{(x,v)} a| \leq C_{\alpha,\beta} \langle (x,v,\eta)\rangle^k \qquad \text{on } \mathbb{R}^{2d} \times \Sigma_{\tilde{\tau}}$$

, i.e., up to taking τ smaller, item (ii) from Definition 1.2 can be extended to $\beta \in \mathbb{N}^{3d}$. Let us introduce a notion of expansion where the coefficients are allowed to depend on h: We will say that

$$a \sim_h \sum_{j\geq 0} h^j a_j \tag{1.6}$$

in $S^0(\langle (x,v,\eta)\rangle^k)$ (resp. in $S^0_\tau(\langle (x,v,\eta)\rangle^k)$) if $(a_j)_{j\geq 0} \subset S^0(\langle (x,v,\eta)\rangle^k)$ (resp. $(a_j)_{j\geq 0} \subset S^0_\tau(\langle (x,v,\eta)\rangle^k)$) is a family of symbols which may depend on h and are such that for all $N \in \mathbb{N}$,

$$a - \sum_{j=0}^{N-1} h^j a_j = O_{S^0(\langle (x,v,\eta)\rangle^k)}(h^N) \qquad \big(\text{resp. } O_{S^0_\tau(\langle (x,v,\eta)\rangle^k)}(h^N)\big)$$

Finally, we also have the usual notion of classical expansion for a symbol: $a \sim \sum_{j\geq 0} h^j a_j$ in $S^0(\langle (x,v,\eta)\rangle^k)$ (resp. in $S^0_\tau(\langle (x,v,\eta)\rangle^k)$) means that $a \sim_h \sum_{j\geq 0} h^j a_j$ in $S^0(\langle (x,v,\eta)\rangle^k)$ (resp. in $S^0_\tau(\langle (x,v,\eta)\rangle^k)$) and the $(a_j)_{j\geq 0}$ are independent of h.

We now extend these notions to matrix-valued symbols: If

$$M = (m_{p,q})_{\substack{1\leq p\leq n_1\\ 1\leq q\leq n_2}}$$

is a matrix of functions such that each $m_{p,q} \in S^\kappa(\langle (x,v,\eta)\rangle^k)$ (resp. $m_{p,q} \in S^0_\tau(\langle (x,v,\eta)\rangle^k)$), we say that $M \in \mathcal{M}_{n_1,n_2}\big(S^\kappa(\langle (x,v,\eta)\rangle^k)\big)$ $\big(\text{resp.} M \in \mathcal{M}_{n_1,n_2}\big(S^0_\tau(\langle (x,v,\eta)\rangle^k)\big)\big)$ and we denote

$$\mathrm{Op}_h(M) = \Big(\mathrm{Op}_h(m_{p,q})\Big)_{\substack{1\leq p\leq n_1\\ 1\leq q\leq n_2}}.$$

The notation

$$M = O_{\mathcal{M}_{n_1,n_2}\left(S^0(\langle(x,v,\eta)\rangle^k)\right)}(h^N) \qquad \left(\text{resp. } M = O_{\mathcal{M}_{n_1,n_2}\left(S^0_\tau(\langle(x,v,\eta)\rangle^k)\right)}(h^N)\right)$$

means that for all $(p,q) \in [\![1,n_1]\!]\times[\![1,n_2]\!]$, the symbol $m_{p,q}$ is $O_{S^0(\langle(x,v,\eta)\rangle^k)}(h^N)$ (resp. $O_{S^0_\tau(\langle(x,v,\eta)\rangle^k)}(h^N)$). Furthermore, the notions of expansions $M \sim_h \sum_{n\geq 0} h^n M_n$ and $M \sim \sum_{n\geq 0} h^n M_n$ in $\mathcal{M}_{n_1,n_2}\big(S^0(\langle(x,v,\eta)\rangle^k)\big)$ $\big($resp.$\mathcal{M}_{n_1,n_2}$ $\big(S^0_\tau(\langle(x,v,\eta)\rangle^k)\big)\big)$ are straightforward adaptations of the ones for scalar symbols.

These notions enable us to introduce a new class of collision operators which appears to be more general that the one given by Hypothesis 1.1. Let us denote b_h the twisted derivative

$$b_h = h\partial_v + v/2 \tag{1.7}$$

so that in particular with the notation (1.5) we have $H_0 = b_h^* b_h$. We also use the standard notation $\mathcal{M}_d(\mathbb{R})$ for the set of all d-by-d real matrices.

Hypothesis 1.3. *There exist $\tau > 0$ and a symmetric matrix of analytic symbols*

$$M^h(x,v,\eta) = \big(m_{p,q}(x,v,\eta)\big)_{1\leq p,q\leq d} \in \mathcal{M}_d\big(S^0_\tau(\langle(v,\eta)\rangle^{-2})\big)$$

sending $\mathbb{R}^{3d}$ into $\mathcal{M}_d(\mathbb{R})$ and such that, with the notation (1.7), the collision operator Q_h satisfies

- (a) $Q_h = b_h^* \circ Op_h(M^h) \circ b_h$
- (b) $M^h \sim \sum_{n\geq 0} h^n M_n$ *in* $\mathcal{M}_d\big(S^0_\tau(\langle(v,\eta)\rangle^{-2})\big)$
- (c) *For all* $(x,v,\eta) \in \mathbb{R}^{3d}$, $M^h(x,v,\eta) = M^h(x,v,-\eta)$
- (d) *For all* $(x,v,\eta) \in \mathbb{R}^{3d}$, $M_0(x,v,\eta) \geq \frac{1}{C}\langle(v,\eta)\rangle^{-2}\,\mathrm{Id}$.

Since the $(M_n)_n$ do not depend on h, we easily get that these matrices of symbols are also even in η, symmetric, independent of ξ and with values in $\mathcal{M}_d(\mathbb{R})$; so in particular item d) makes sense. This will enable us to establish Lemma 2.1 which is sometimes referred to as *microscopic coercivity* (see, for instance, [5]). As announced, we have the following lemma which is proven in Appendix 6:

Lemma 1.4. *Hypothesis 1.1 implies Hypothesis 1.3.*

We will also make a few confining assumptions on the function V, assuring, for instance, that the bottom spectrum of the associated Witten Laplacian is discrete. In particular, our potential will satisfy Assumption 2 from [12] and Hypothesis 1.1 from [20].

Hypothesis 1.5. *The potential V is a smooth Morse function depending only on the space variable $x \in \mathbb{R}^d$ with values in $\mathbb{R}$ which is bounded from below and such that*

$$|\partial_x V(x)| \geq \frac{1}{C} \qquad \text{for } |x| > C.$$

Moreover, for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \geq 2$, there exists C_α such that

$$|\partial_x^\alpha V| \leq C_\alpha.$$

In particular, for every $0 \leq k \leq d$, the set of critical points of index k of V that we denote $\mathcal{U}^{(k)}$ is finite and we set

$$n_0 = \#\mathcal{U}^{(0)}. \tag{1.8}$$

Finally, we will suppose that $n_0 \geq 2$.

The last assumption comes from the fact that when $n_0 = 1$, the so-called *small spectrum* of the operator P_h (i.e., its eigenvalues with exponentially small modulus) is trivial, so there is nothing to study. It is shown in [16], Lemma 3.14 that for a function V satisfying Hypothesis 1.5, we have $V(x) \geq |x|/C$ outside of a compact. In particular, under Hypothesis 1.5, it holds $\mathrm{e}^{-V/2h} \in L^2(\mathbb{R}^d_x)$. Moreover, in our setting, X_0^h is a smooth vector field whose differential is bounded on $\mathbb{R}^{2d}$, so the operator X_0^h endowed with the domain

$$D = \{u \in L^2(\mathbb{R}^{2d})\,;\, X_0^h u \in L^2(\mathbb{R}^{2d})\} \tag{1.9}$$

is skew-adjoint on $L^2(\mathbb{R}^{2d})$ and the set $\mathcal{S}(\mathbb{R}^{2d})$ is a core for this operator. Therefore, $(P_h, D)^* = (-X_0^h + Q_h, D)$ and (P_h, D) is m-accretive on $L^2(\mathbb{R}^{2d})$.

For an operator such as P_h, which is not, for instance, self-adjoint with compact resolvent, we do not have any information a priori on its spectrum (except here that it is contained in $\{z \in \mathbb{C}\,;\, \mathrm{Re}\, z \geq 0\}$). Section 2 is thus devoted to establishing a first description of the spectrum of P_h near 0 which, in the spirit of the case of other non self-adjoint operators studied in [10,20], appears in particular to be discrete:

Theorem 1.6. *Assume that Hypotheses 1.3 and 1.5 are satisfied and recall the notation (1.8). Then, the operator (P_h, D) admits 0 as a simple eigenvalue. Moreover, there exist $c > 0$ and $h_0 > 0$ such that for all $0 < h \leq h_0$, $\mathrm{Spec}(P_h) \cap \{\mathrm{Re}\, z \leq ch^2\}$ consists of exactly n_0 eigenvalues (counted with algebraic multiplicity) that are exponentially small with respect to $1/h$ and for all $0 < \tilde{c} \leq c$, the resolvent estimate*

$$(P_h - z)^{-1} = O(h^{-2})$$

holds uniformly in $\{\mathrm{Re}\, z \leq ch^2\} \backslash B(0, \tilde{c}h^2)$. Finally, except for 0, the real parts of these small eigenvalues are positive.

This result can be seen as a generalization of Theorem 3.0.2 from [19] (up to the h^2 instead of h) as we saw that the *mild relaxation* operator (which is the collision operator studied in this reference) satisfies our hypotheses. In our case, we get a localization of order h^2 because we adopt a simpler proof based on hypocoercivity (inspired by [20]) than the one presented in [19].

In order to study the long time behavior of the solutions of (1.4), we need a precise description of the small spectrum of P_h. To this aim, we construct in Sects. 3 and 4 in the spirit of the WKB method a family of accurate quasi-modes localized around the minima of V that enables us to establish sharp asymptotics of the small eigenvalues of P_h. This leads in Sect. 5 to the establishment of Theorem 1.8 which is the main result of this paper. For the sake of simplicity, we make in the statement an additional assumption (Hypothesis

3.11) on the topology of the potential V that could actually be omitted (see [17] or [1]). It implies in particular that V has a unique global minimum that we denote $\underline{\mathbf{m}}$. In order to be able to state our main result, we give the following lemma which is actually a consequence of Proposition 4.7 and Lemma 4.8.

Lemma 1.7. *Recall the matrix M_0 from Hypothesis 1.3 and let $\mathbf{m} \in \mathcal{U}^{(0)} \backslash \{\underline{\mathbf{m}}\}$ and $\mathbf{s} \in \mathbf{j}(\mathbf{m})$ where $\mathbf{j}$ is the topological map defined in 3.10. The matrix*

$$\Phi^{\mathbf{s}} = \begin{pmatrix} 0 & -\mathrm{Hess}_{\mathbf{s}} V \\ \mathrm{Id} & M_0(\mathbf{s}, 0, 0) \end{pmatrix}$$

has only one eigenvalue in $\{\mathrm{Re}\, z < 0\}$ which is actually real and that we denote $-\alpha_0^{\mathbf{s}}$.

According to Theorem 1.6, we can associate with each $\mathbf{m} \in \mathcal{U}^{(0)} \backslash \{\underline{\mathbf{m}}\}$ a nonzero exponentially small eigenvalue of P_h that we denote $\lambda(\mathbf{m}, h)$.

Theorem 1.8. *Suppose that Hypotheses 1.3, 1.5 and 3.11 are satisfied and recall the notation $\alpha_0^{\mathbf{s}}$ from Lemma 1.7. The exponentially small eigenvalues of P_h satisfy the following formula:*

$$\lambda(\mathbf{m}, h) = h e^{-2\frac{S(\mathbf{m})}{h}} \frac{\det(\mathrm{Hess}_{\mathbf{m}} V)^{1/2}}{2\pi} B_h(\mathbf{m})$$

where $B_h(\mathbf{m})$ admits a classical expansion whose first term is

$$\sum_{\mathbf{s} \in \mathbf{j}(\mathbf{m})} |\det(\mathrm{Hess}_{\mathbf{s}} V)|^{-1/2} \, \alpha_0^{\mathbf{s}}$$

and the maps S and $\mathbf{j}$ are defined in Definition 3.10.

When Hypothesis 1.3 is replaced by Hypothesis 1.1, we can give a slightly more precise statement. In that case, denoting $\boldsymbol{\mu}_{\mathbf{s}}$ the only negative eigenvalue of $\mathrm{Hess}_{\mathbf{s}} V$, the first term of $B_h(\mathbf{m})$ is

$$\frac{1}{2} \sum_{\mathbf{s} \in \mathbf{j}(\mathbf{m})} |\det(\mathrm{Hess}_{\mathbf{s}} V)|^{-1/2} \Big(-\varrho'(0) + \sqrt{\varrho'(0)^2 - 4\boldsymbol{\mu}_{\mathbf{s}}} \Big). \tag{1.10}$$

Indeed, under Hypothesis 1.1, it is shown in Appendix 6, more precisely in (A.13) that $M_0(\mathbf{s}, 0, 0) = \tilde{\varrho}(0)\,\mathrm{Id} = \varrho'(0)\,\mathrm{Id}$. Thanks to Proposition 4.7 from which we keep the notations, we then have

$$\mathrm{Hess}_{\mathbf{s}} V \nu_2 = -\varrho'(0)^2 (1 + \nu_2^2) \nu_2^2 \, \nu_2$$

and consequently

$$\alpha_0^{\mathbf{s}} = \varrho'(0) \nu_2^2 = -\frac{\varrho'(0)}{2} + \frac{\sqrt{\varrho'(0)^2 - 4\boldsymbol{\mu}_{\mathbf{s}}}}{2}$$

so the statement follows.

Remark 1.9. The case of the Fokker–Planck operator, i.e., when $\varrho(t) = t$ and $M^h = \mathrm{Id}$, is not covered by Theorem 1.8 as it does not fit its hypotheses. However, formally applying our formula (1.10) to this case, we still recover the one from [1,10] for this operator. (Be careful that our notation $\boldsymbol{\mu}_{\mathbf{s}}$ and the notation $\mu(\mathbf{s})$ from [1] do not stand for the same object.)

Finally, Sect. 6 consists in using the sharp localization obtained in Theorem 1.8 in order to discuss the phenomena of return to equilibrium and metastability for the solutions of (1.4). More precisely, we are able to give a sharp rate of convergence of the semigroup $\mathrm{e}^{-tP_h/h}$ toward $\mathbb{P}_1$, the orthogonal projector on Ker P_h: Denoting λ^* a nonzero eigenvalue of P_h whose real part is minimal, we establish that the rate of return to equilibrium is essentially given by $\operatorname{Re}\lambda^*/h$:

Corollary 1.10. *Under the assumptions of Theorem 1.8, for any $N \geq 1$, there exist $C_N > 0$ and $h_0 > 0$ such that for all $0 < h \leq h_0$ and $t \geq 0$,*

$$\|\mathrm{e}^{-tP_h/h} - \mathbb{P}_1\| \leq C_N \mathrm{e}^{-t\operatorname{Re}\lambda^*(1-C_N h^N)/h}.$$

Moreover, if λ^* does not share its expansion given by Theorem 1.8 with another eigenvalue of P_h (in particular it is a simple eigenvalue), then λ^* is real and we even have

$$\|\mathrm{e}^{-tP_h/h} - \mathbb{P}_1\| \leq C\mathrm{e}^{-t\lambda^*/h}.$$

Besides, in the spirit of [1], we also show the metastable behavior of the solutions of (1.4):

Corollary 1.11. *Suppose that the assumptions of Theorem 1.8 hold true. Let us consider some local minima $\mathbf{m}_1 = \underline{\mathbf{m}}$, $\mathbf{m}_2, \dots, \mathbf{m}_K$ such that*

$$S(\mathcal{U}^{(0)}) = \{+\infty = S(\mathbf{m}_1) > S(\mathbf{m}_2) > \cdots > S(\mathbf{m}_K)\}$$

for the map S from Definition 3.10. For $2 \leq k \leq K$, denote $\mathbb{P}_k$ the spectral projection associated with the eigenvalues that are $O\big(\mathrm{e}^{-2\frac{S(\mathbf{m}_k)}{h}}\big)$. Then for any times $(t_k^{\pm})_{1\leq k\leq K}$ satisfying

$$t_K^- \geq h^{-1}|\ln(h^\infty)| \quad \text{and} \quad t_k^- \geq |\ln(h^\infty)|\mathrm{e}^{2\frac{S(\mathbf{m}_{k+1})}{h}} \quad \text{for} \quad k = 1, \dots, K-1$$

as well as

$$t_1^+ = +\infty \qquad \text{and} \qquad t_k^+ = O\Big(h^\infty \mathrm{e}^{2\frac{S(\mathbf{m}_k)}{h}}\Big) \qquad \text{for} \quad k = 2, \dots, K$$

one has

$$\mathrm{e}^{-tP_h/h} = \mathbb{P}_k + O(h^\infty) \qquad \text{on } [t_k^-, t_k^+].$$

In other words, we have shown the existence of timescales on which, during its convergence toward the global equilibrium, the solution of (1.4) will essentially visit the metastable spaces associated with the small eigenvalues of P_h.

The results presented in this paper should be reasonably easy to adapt to the case of collision operators satisfying Hypothesis 1.3 with the space S^0 replaced by S^κ for $\kappa \in [0, 1/2[$. (We should get some expansions in powers of $h^{1-2\kappa}$ instead of just h.) Another perspective would then be to study the critical case $\kappa = 1/2$ which should in particular cover the *linear relaxation* collision operator corresponding to the linear BGK model

$$Q_h = h(1 - \Pi_h) \tag{1.11}$$

where, using the notation (1.2),

$$\Pi_h : L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^{2d}) \tag{1.12}$$

denotes the orthogonal projection on

$$E_h = \mu_h \, L^2(\mathbb{R}^d_x) \tag{1.13}$$

and for which Robbe gave a first localization of the small spectrum of the associated operator $X_0^h + Q_h$ in [20].

2. Rough Description of the Small Spectrum

Throughout the paper, we assume that Hypotheses 1.3 and 1.5 hold true. This implies in particular that Q_h is bounded uniformly in h and self-adjoint in $L^2(\mathbb{R}^{2d})$. Let us begin with a lemma which consists in comparing our collision operator with the one introduced in (1.11) and studied in [20]. This will in particular enable us to use some computations from [20] later on.

Lemma 2.1. *There exists $h_0 > 0$ such that for all $0 < h < h_0$,*

$$Q_h \geq \frac{h}{C}(1 - \Pi_h)$$

where Π_h is the projection introduced in (1.12). In particular, Q_h is nonnegative.

Proof. Since the space E_h defined in (1.13) is contained in $\operatorname{Ker} Q_h$, it is enough to prove that $\langle Q_h u, u\rangle \geq \frac{h}{C}\|u\|^2$ for $u \in E_h^\perp$. Let $u \in E_h^\perp$ and recall the notations H_0 and H_1 from (1.5) and (A.4). Let us consider an approximate square root A of $(1 + H_1)$ given by

$$A = \operatorname{Op}_h\Big(\big(1 + v^2/4 + \eta^2 + h(1 - d/2)\big)^{1/2}\mathrm{Id}\Big) \in \Psi^0\big(\langle(v,\eta)\rangle\big).$$

By symbolic calculus, we easily have $A^2 = 1 + H_1 + h^2 R_1$ with $R_1 \in \Psi^0\big(\langle(v,\eta)\rangle^2\big)$. Besides, the symbol of A is clearly elliptic so A is invertible and its inverse is also a pseudo-differential operator satisfying $A^{-2} = (1 + H_1)^{-1} + h^2 R_2$ with $R_2 \in \Psi^0\big(\langle(v,\eta)\rangle^{-2}\big)$ (see, for instance, [4], chapter 8). Thus, using the factorization from Hypothesis 1.3 and the self-adjointness of A, we get

$$\langle Q_h u, u\rangle = \big\langle A \operatorname{Op}_h(M^h) A \, A^{-1} b_h u \, , \, A^{-1} b_h u\big\rangle.$$

Now according to Hypothesis 1.3 and symbolic calculus again, the principal symbol of $A \operatorname{Op}_h(M^h) A$ is elliptic so we can use the Gårding inequality to write

$$\begin{aligned}\langle Q_h u, u\rangle &\geq \frac{1}{C}\big\langle A^{-2} b_h u \, , b_h u\big\rangle \\ &\geq \frac{1}{C}\big\langle b_h^*(1 + H_1)^{-1} b_h u \, , u\big\rangle - \frac{h^2}{C}\big|\big\langle b_h^* \, R_2 \, b_h u \, , u\big\rangle\big|.\end{aligned}$$

Still using symbolic calculus, we get $b_h^* R_2 b_h = O(1)$ so applying (A.5) we finally have

$$\langle Q_h u, u\rangle \geq \frac{1}{C}\big\langle H_0(1+H_0)^{-1}u\,,u\big\rangle - O(h^2)\|u\|^2$$

and the conclusion comes from the fact that the spectrum of $H_0(1+H_0)^{-1}|_{E_h^\perp}$ is contained in $[h/C,+\infty[$.

We can already prove that 0 is a simple eigenvalue of (P_h, D) and that the other eigenvalues have positive real part. It is easy to check that $\mathcal{M}_h$ defined in (1.2) is in Ker P_h. Now, let $\lambda \in \mathbb{R}$ and let us prove that for $u \in$ Ker $(P_h - i\lambda)$, one has $u \in \mathbb{C}\,\mathcal{M}_h$. Since X_0^h is skew-adjoint and Q_h is self-adjoint and nonnegative, we have

$$0 = \mathrm{Re}\langle (P_h - i\lambda)u, u\rangle = \|Q_h^{1/2}u\|^2$$

so in particular $u \in \mathrm{Ker}\, Q_h = E_h$ according to Lemma 2.1. Therefore, $u = w\mu_h$ with $w \in L^2(\mathbb{R}_x^d)$ and using that $\mu_h^{-1}X_0^h u = i\lambda w$ does not depend on v, we get in the sense of distributions $\partial_x(\mathrm{e}^{V/2h}w) = 0$ which yields the desired result.

2.1. Hypocoercivity

Let us now use the dilatation operators

$$S_h : \begin{cases} L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^{2d}) \\ u \mapsto h^{-d/2}u\Big(\frac{\cdot}{\sqrt{h}}\,,\,\frac{\cdot}{\sqrt{h}}\Big) \end{cases} \qquad T_h : \begin{cases} L^2(\mathbb{R}_x^d) \to L^2(\mathbb{R}_x^d) \\ u \mapsto h^{-d/4}u\Big(\frac{\cdot}{\sqrt{h}}\Big) \end{cases}$$

that were introduced in [20] in which these were combined with a scaling of Π_h to conjugate P_h to a non-semiclassical operator with h-dependent potential. In our case, it will enable us to use some computations and results already established in [20].

Lemma 2.2. *Recall the notation* (1.9). *Denoting*

$$X_0 = v\cdot\partial_x - \partial_x V_h(x)\cdot\partial_v$$

where $V_h = h^{-1}V(\sqrt{h}\,\cdot)$,

$$\tilde{Q}_1 = h^{-1}S_h^{-1}Q_h S_h$$

and

$$\mathrm{Dom}\,(P) = \{u \in L^2(\mathbb{R}^{2d})\,;\, X_0 u \in L^2(\mathbb{R}^{2d})\}, \qquad P = X_0 + \tilde{Q}_1,$$

one has

$$(hP,\, \mathrm{Dom}(P)) = (S_h^{-1}P_h S_h,\, S_h^{-1}D).$$

Moreover,

$$(hP,\, \mathrm{Dom}(P))^* = (S_h^{-1}P_h^* S_h,\, S_h^{-1}D).$$

Proof. We have for $u \in L^2(\mathbb{R}^{2d})$

$$hX_0 u = S_h^{-1} X_0^h S_h u$$

so using that S_h is bounded we get $\mathrm{Dom}\,(P) = S_h^{-1} D$. Consequently,

$$(hP,\,\mathrm{Dom}(P)) = (S_h^{-1} P_h S_h,\, S_h^{-1} D)$$

and the result for the adjoint follows immediately.

We also recall the notations of the following differential operators from [9,20]:

$$a = \partial_x + \frac{\partial_x V_h}{2}; \qquad b = \partial_v + \frac{v}{2} \qquad \text{and} \qquad \Lambda^2 = a^* a + b^* b + 1.$$

The operator $(\Lambda^2, \mathcal{C}_c^\infty(\mathbb{R}^{2d}))$ is essentially self-adjoint. The Schwartz space $\mathcal{S}(\mathbb{R}^{2d})$ is included in the domain of its self-adjoint extension $(\Lambda^2, D(\Lambda^2))$ which is invertible. We can then define the operator

$$L = \Lambda^{-2} a^* b \tag{2.1}$$

which is bounded uniformly in h (see [20], Lemma 2.7), as well as the perturbation $h\varepsilon(L + L^*) = O(h)$ where $\varepsilon > 0$ will be chosen small enough later.

Besides, notice that $a^* a = -\Delta_x + |\partial_x V_h|^2/4 - \Delta V_h/2 =: \Delta_{V_h/2}$ is the Witten Laplacian in x associated with the potential $V_h/2$ and that

$$\begin{aligned}\Delta_{V/2}^h &:= hT_h a^* a T_h^{-1}\\ &= -h^2 \Delta_x + |\partial_x V|^2/4 - h\Delta V/2\end{aligned}$$

is the semiclassical Witten Laplacian associated with the potential $V/2$. The small spectrum of this operator was first studied by Helffer and Sjöstrand in [7], and we now know (see, for instance, [6], Definition 4.3) that we can construct an orthonormal family $(\varphi_j)_{1\le j\le n_0} \subset \mathcal{C}_c^\infty(\mathbb{R}_x^d)$ of quasimodes associated with this operator given by

$$\varphi_j = \chi_j \mathrm{e}^{-\frac{V - V(x_j)}{2h}}$$

where x_j is one of the local minima of V and χ_j is a cutoff function localizing around x_j. Recall the notation μ_h from (1.2) and let us now define the families of functions

$$g_j^h = \varphi_j \mu_h \qquad \text{and} \qquad g_j = S_h^{-1} g_j^h$$

for $1 \le j \le n_0$. These are actually quasimodes for our operators P_h and P_h^*:

Lemma 2.3. *The family $(g_j^h)_{1\le j\le n_0}$ is orthonormal, and there exists $\alpha > 0$ such that for all $1 \le j \le n_0$,*

$$P_h g_j^h = O_{L^2}(\mathrm{e}^{-\frac{\alpha}{h}}), \qquad P_h^* g_j^h = O_{L^2}(\mathrm{e}^{-\frac{\alpha}{h}}).$$

Moreover, $P_h g_j^h$ and $P_h^* g_j^h$ are in $\mathcal{S}(\mathbb{R}^{2d}) \subseteq D$ and we have

$$P_h^* P_h g_j^h = O_{L^2}(\mathrm{e}^{-\frac{\alpha}{h}}), \qquad P_h P_h^* g_j^h = O_{L^2}(\mathrm{e}^{-\frac{\alpha}{h}}).$$

Proof. The proof is the same as the one of Lemma 2.4 from [20] since with the notation (1.13) and Lemma 2.1 we also have $E_h = \mathrm{Ker}\; Q_h$.

One of the key results of this section is that the real part of the perturbation of our operator is bounded from below on a subspace of finite codimension given by the orthogonal of the quasimodes. In order to state it, recall the notation (2.1) and denote $N^{\pm}_{h,\varepsilon}$ the bounded self-adjoint operator

$$\mathrm{Id} \pm \varepsilon h(L + L^*).$$

Proposition 2.4. *There exists $\varepsilon > 0$ and $h_0 > 0$ such that for all $h \in]0, h_0]$ and $u \in \mathcal{S}(\mathbb{R}^{2d}) \cap (g_j)^{\perp}_{1\le j\le n_0}$, one has*

$$\mathrm{Re}\langle N^+_{h,\varepsilon} Pu, u\rangle \ge \frac{h}{C}\|u\|^2$$

as well as

$$\mathrm{Re}\langle N^-_{h,\varepsilon} P^* u, u\rangle \ge \frac{h}{C}\|u\|^2.$$

Proof. One has for $u \in \mathcal{S}(\mathbb{R}^{2d})$, using the fact that X_0 is skew-adjoint:

$$\begin{aligned}
\mathrm{Re}\langle N^+_{h,\varepsilon} Pu, u\rangle &= \mathrm{Re}\langle Pu, N^+_{h,\varepsilon} u\rangle \\
&= \mathrm{Re}\langle \tilde{Q}_1 u, N^+_{h,\varepsilon} u\rangle + \mathrm{Re}\langle X_0 u, N^+_{h,\varepsilon} u\rangle \\
&= \|\tilde{Q}_1^{1/2} u\|^2 + h\varepsilon \mathrm{Re}\langle \tilde{Q}_1 u, (L + L^*)u\rangle + h\varepsilon \mathrm{Re}\langle X_0 u, (L + L^*)u\rangle \\
&= \|\tilde{Q}_1^{1/2} u\|^2 + h\varepsilon \mathrm{Re}\langle \tilde{Q}_1 u, (L + L^*)u\rangle + h\varepsilon \mathrm{Re}\langle [L, X_0]u, u\rangle \\
&= I + hII + hIII
\end{aligned}$$

Note that if we replace P by P^* and $N^+_{h,\varepsilon}$ by $N^-_{h,\varepsilon}$, we get $I - hII + hIII$. Besides, it is also proven in [20] that

$$[L, X_0] = \mathcal{A} + \Lambda^{-2} a^* a$$

where $\mathcal{A}$ is also bounded uniformly in h. Since $\|Q_h\| \le C$ and $Q_h \ge \frac{h}{C}(1 - \Pi_h)$ according to Lemma 2.1, we get $\|\tilde{Q}_1\| \le \frac{C}{h}$ and $\tilde{Q}_1 \ge \frac{1}{C}(1 - \Pi_1)$. Hence,

$$\begin{aligned}
I \pm hII &\ge I - h|II| \\
&\ge \|\tilde{Q}_1^{1/2} u\|^2 - h\varepsilon \|\tilde{Q}_1 u\| \|(L + L^*)u\| \\
&\ge \|\tilde{Q}_1^{1/2} u\|^2 - \sqrt{C} h^{\frac{1}{2}} \varepsilon \|\tilde{Q}_1^{1/2} u\| \|(L + L^*)u\| \\
&\ge \frac{1}{2}\|\tilde{Q}_1^{1/2} u\|^2 - 2Ch\varepsilon^2 \|L\|^2 \|u\|^2 \\
&\ge \frac{1}{2C}\|(1 - \Pi_1)u\|^2 - 2Ch\varepsilon^2 \|L\|^2 \|u\|^2 \qquad (2.2)
\end{aligned}$$

We can combine this with the following estimate from [20] (proof of Proposition 2.5): There exists $\delta > 0$ such that for $u \in (g_j)^{\perp}_{1\le j\le n_0}$,

$$III \ge -\frac{1}{4}\|(\mathrm{Id} - \Pi_1)u\|^2 - \varepsilon^2 \|\mathcal{A}\|^2 \|u\|^2 + \frac{\varepsilon\delta}{4}\|\Pi_1 u\|^2 - \varepsilon \|(\mathrm{Id} - \Pi_1)u\|^2.$$

This yields for $\varepsilon < \frac{\delta}{4(\|\mathcal{A}\|^2+C\|L\|^2)}$ that

$$\begin{aligned} I \pm hII + hIII &\geq \frac{1}{C}\|(\mathrm{Id}-\Pi_1)u\|^2 + h\frac{\varepsilon\delta}{4}\|\Pi_1 u\|^2 - h\varepsilon^2\Big(\|\mathcal{A}\|^2 + C\|L\|^2\Big)\|u\|^2 \\ &\geq \frac{h}{C}\|u\|^2. \end{aligned} \tag{2.3}$$

so the proof is complete.

This result extends to $u \in (g_j)^{\perp}_{1\leq j\leq n_0} \cap \mathrm{Dom}\,(P)$ since $\mathcal{S}(\mathbb{R}^{2d})$ is a core for both $(P, \mathrm{Dom}\,(P))$ and $(P^*, \mathrm{Dom}\,(P^*))$. It only differs from Proposition 2.5 in [20] by a factor h in the estimate. This comes from the fact that in our case, $\tilde{Q}_1 = O(h^{-1})$ and not $O(1)$ (because $Q_h = O(1)$ and not $O(h)$) so we have to use a perturbation of order h (the operator $N^{\pm}_{h,\varepsilon}$) to obtain the gain in $\|(1-\Pi_1)u\|^2$ in (2.2). As a consequence, the gain in $\|\Pi_1 u\|^2$ from (2.3) is of order h and not of order 1.

Corollary 2.5. *There exists $c > 0$ and $h_0 > 0$ such that for all $h \in]0, h_0]$, $u \in D \cap (g^h_j)^{\perp}_{1\leq j\leq n_0}$ and $z \in \mathbb{C}$ with $\mathrm{Re}\, z \leq ch^2$*

$$\|(P_h - z)u\| \geq ch^2\|u\| \qquad \textit{and} \qquad \|(P_h^* - z)u\| \geq ch^2\|u\|.$$

Proof. Recall that $N^+_{h,\varepsilon} = 1 + O(h)$. Hence, for $u \in D \cap (g^h_j)^{\perp}_{1\leq j\leq n_0}$, we have by putting $u = S_h w$ and using that S_h is unitary

$$\begin{aligned} \|(P_h - z)u\|\|u\| &\geq \frac{1}{2}\|(P_h - z)u\|\|N^+_{h,\varepsilon}w\| \\ &\geq \frac{1}{2}\mathrm{Re}\langle (P_h - z)u, S_h N^+_{h,\varepsilon}w\rangle \\ &= \frac{1}{2}\mathrm{Re}\langle N^+_{h,\varepsilon}(hP - z)w, w\rangle \\ &\geq \frac{h^2}{C}\|u\|^2 - \mathrm{Re}\, z\|N^+_{h,\varepsilon}\|\|u\|^2 \\ &\geq \frac{h^2}{2C}\|u\|^2 \end{aligned}$$

if $\mathrm{Re}\, z \leq h^2/2C$. The same proof holds when replacing P by P^* and $N^+_{h,\varepsilon}$ by $N^-_{h,\varepsilon}$.

2.2. Resolvent Estimates and First Localization of the Small Eigenvalues

Using Lemma 2.3, it is clear that for $u \in \mathrm{Span}\big((g^h_j)_{1\leq j\leq n_0}\big)$ and $A \in \{P_h, P_h^*, P_h^* P_h, P_h P_h^*\}$ we have

$$\|Au\|^2 = O(\mathrm{e}^{-\frac{2\alpha}{h}})\|u\|^2.$$

Now, if we denote $\mathbb{P}$ the orthogonal projection on $\mathrm{Span}\big((g_j^h)_{1\le j\le n_0}\big)$, we get by using Corollary 2.5 that for $z\in\mathbb{C}$ such that $\mathrm{Re}\, z\le ch^2$ and $u\in D$

$$\begin{aligned}\|(P_h-z)u\|^2 &= \|(P_h-z)(\mathrm{Id}-\mathbb{P})u+(P_h-z)\mathbb{P}u\|^2\\ &= \|(P_h-z)(\mathrm{Id}-\mathbb{P})u\|^2+\|(P_h-z)\mathbb{P}u\|^2\\ &\quad +2\mathrm{Re}\langle (P_h-z)(\mathrm{Id}-\mathbb{P})u,(P_h-z)\mathbb{P}u\rangle\\ &\ge c^2h^4\|(\mathrm{Id}-\mathbb{P})u\|^2+|z|^2\|\mathbb{P}u\|^2-O(\mathrm{e}^{-\frac{\alpha}{h}})\|u\|^2\\ &\quad +2\mathrm{Re}\langle (P_h-z)(\mathrm{Id}-\mathbb{P})u,(P_h-z)\mathbb{P}u\rangle.\end{aligned}$$

The last term equals

$$\begin{aligned}&2\mathrm{Re}\Big[\langle(\mathrm{Id}-\mathbb{P})u,P_h^*P_h\mathbb{P}u\rangle - z\langle(\mathrm{Id}-\mathbb{P})u,P_h\mathbb{P}u\rangle-\bar z\langle(\mathrm{Id}-\mathbb{P})u,P_h^*\mathbb{P}u\rangle\Big]\\ &\quad=(1+|z|)O(\mathrm{e}^{-\frac{\alpha}{h}})\|u\|^2.\end{aligned}$$

Therefore choosing $\tilde c\le c$, there exists $h_0>0$ such that for $h\le h_0$ and z such that $\tilde c h^2\le |z|\le ch^2$

$$\|(P_h-z)u\|^2\ge\Big(|z|^2+O(\mathrm{e}^{-\frac{\alpha}{h}})\Big)\|u\|^2\ge\frac{\tilde c^2h^4}{2}\|u\|^2.$$

Once again, the same estimate holds with P_h^* instead of P_h and since the annulus we are working on is invariant by complex conjugation, we also have

$$\|(P_h-z)^*u\|\ge\frac{\tilde c h^2}{2}\|u\|.$$

Therefore, we get the following resolvent estimate on the annulus centered in 0 and of radiuses $\tilde c h^2$ and ch^2:

$$\|(P_h-z)^{-1}\|=O(h^{-2})\quad\text{for } \tilde c h^2\le|z|\le ch^2. \tag{2.4}$$

We can now consider the spectral projection

$$\Pi_0=\frac{1}{2i\pi}\int_{|z|=ch^2}(z-P_h)^{-1}\mathrm{d}z \tag{2.5}$$

and its range that we denote H. This operator will yield some information on $\mathrm{Spec}(P_h)\cap B(0,ch^2)$ and therefore enable us to prove the main statement from Theorem 1.6.

The main point is that H is of dimension n_0. It can be obtained by a direct adaptation of the proof of Proposition 3.1 from [20]. Hence, $\mathrm{Spec}(P_h)\cap B(0,ch^2)$ which is the same as $\mathrm{Spec}(P_h|_H)$ consists of n_0 eigenvalues (counted with algebraic multiplicity). Here again, our result slightly differs from the one in [20] as we do not rule out the possibilities that $P_h|_H$ contains some Jordan blocks and that some of its eigenvalues are not real. It only remains to prove that these are exponentially small with respect to $1/h$. We begin by noticing that thanks to Lemma 2.3, we have $(z-P_h)g_j^h=zg_j^h+O(\mathrm{e}^{-\frac{\alpha}{h}})$ and $(z-P_h^*)g_j^h=zg_j^h+O(\mathrm{e}^{-\frac{\alpha}{h}})$ from which we easily deduce

$$\Pi_0g_j^h=g_j^h+O(\mathrm{e}^{-\frac{\alpha}{h}})\qquad\text{and}\qquad \Pi_0^*g_j^h=g_j^h+O(\mathrm{e}^{-\frac{\alpha}{h}}). \tag{2.6}$$

In particular, $(\Pi_0 g_j^h)_{1\leq j\leq n_0}$ is almost orthonormal so for $u = \sum u_j \Pi_0 g_j^h \in H$, we have

$$\|u\|^2 = \big(1 + O(\mathrm{e}^{-\alpha/h})\big) \sum_{j=1}^{n_0} |u_j|^2.$$

Therefore, it is enough to prove that P_h is exponentially small on $(\Pi_0 g_j^h)_{1\leq j\leq n_0}$. But thanks to the resolvent estimate (2.4), it is easy to see that $\Pi_0 = O(1)$ and since P_h and Π_0 commute, we get the desired result.

To complete the proof of Theorem 1.6, it only remains to show the existence of the resolvent on $\{\mathrm{Re}\, z \leq ch^2\}\backslash B(0, \tilde{c}h^2)$ as well as the estimate in $O(h^{-2})$.

Lemma 2.6. *Denote $\hat{\Pi}_0 = 1 - \Pi_0$. For all $u \in L^2(\mathbb{R}^{2d})$, we have*

$$\hat{\Pi}_0 u = w + r$$

with $w \in (g_j^h)^{\perp}_{1\leq j\leq n_0}$ and $r \in \mathrm{Span}\big((g_j^h)_{1\leq j\leq n_0}\big)$ satisfying $r = O(\mathrm{e}^{-\frac{\alpha}{h}})\|\hat{\Pi}_0 u\|$.

Proof. First we take for r the orthogonal projection of $\hat{\Pi}_0 u$ on $\mathrm{Span}\big((g_j^h)_{1\leq j\leq n_0}\big)$. Then, we notice that using (2.6), we get

$$\langle g_j^h, \hat{\Pi}_0 u\rangle = \langle \hat{\Pi}_0^* g_j^h, \hat{\Pi}_0 u\rangle = O(\mathrm{e}^{-\frac{\alpha}{h}})\|\hat{\Pi}_0 u\|$$

which implies the announced estimate.

Lemma 2.7. *For all $r' \in \mathrm{Span}\big((g_j)_{1\leq j\leq n_0}\big)$, we have $N^{\pm}_{h,\varepsilon} r' \in \mathrm{Dom}\,(P^*) = \mathrm{Dom}\,(P)$. Moreover, the restrictions to the finite dimensional subspace $\mathrm{Span}\big((g_j)_{1\leq j\leq n_0}\big)$ of the operators $PN^{\pm}_{h,\varepsilon}$ and $P^*N^{\pm}_{h,\varepsilon}$ are all $O(1)$.*

Proof. For the first statement, it is sufficient to show that for $1 \leq j \leq n_0$, the functions Lg_j and L^*g_j are both in $\mathrm{Dom}\,(P)$. But we have in the sense of distributions

$$X_0 L g_j = [X_0, L]g_j + LX_0 g_j \tag{2.7}$$

and we saw in the proof of Proposition 2.4 that $[X_0, L]$ is a bounded operator on $L^2(\mathbb{R}^{2d})$ so it is then clear that $X_0 L g_j \in L^2(\mathbb{R}^{2d})$, i.e., $Lg_j \in \mathrm{Dom}\,(P)$. The same goes easily for L^*g_j. For the second statement, using Lemma 2.3 and the fact that $\tilde{Q}_1 = O(h^{-1})$, it suffices to notice that for $1 \leq j \leq n_0$, (2.7) implies that $X_0 L g_j$ and $X_0 L^* g_j$ are both $O(1)$ as we saw that L and $[X_0, L]$ are $O(1)$.

Proposition 2.8. *Consider $\hat{P}_h$ the restriction of P_h to $\hat{\Pi}_0 D$ acting on $\hat{\Pi}_0 L^2(\mathbb{R}^{2d})$. Then for all $z \in \mathbb{C}$ such that $\mathrm{Re}\ z \leq ch^2$, the resolvent $(\hat{P}_h - z)^{-1}$ exists and we have the uniform estimate*

$$(\hat{P}_h - z)^{-1} = O(h^{-2}).$$

Proof. We actually prove that the result of Proposition 2.4 remains true when replacing the set $(g_j)^{\perp}_{1\leq j\leq n_0} \cap \mathrm{Dom}\,(P)$ by $S_h^{-1}\hat{\Pi}_0 D$. We will deduce that the result of Corollary 2.5 also remains true when taking $u \in \hat{\Pi}_0 D$ instead of

$(g_j^h)^\perp_{1\le j\le n_0}\cap D$, which is precisely the statement that we want to prove. Let $u\in D$, using the notations from Lemma 2.6 we have

$$\begin{aligned}&\operatorname{Re}\langle PS_h^{-1}\hat\Pi_0 u, N^+_{h,\varepsilon}S_h^{-1}\hat\Pi_0 u\rangle\\&\quad=\operatorname{Re}\langle PS_h^{-1}w, N^+_{h,\varepsilon}S_h^{-1}w\rangle+\operatorname{Re}\langle PS_h^{-1}w, N^+_{h,\varepsilon}S_h^{-1}r\rangle\\&\qquad+\operatorname{Re}\langle PS_h^{-1}r, N^+_{h,\varepsilon}S_h^{-1}w\rangle+\operatorname{Re}\langle PS_h^{-1}r, N^+_{h,\varepsilon}S_h^{-1}r\rangle\,.\end{aligned}$$

Now, let us denote $w'=S_h^{-1}w\in(g_j)^\perp_{1\le j\le n_0}\cap\operatorname{Dom}(P)$ and $r'=S_h^{-1}r\in\operatorname{Span}\big((g_j)_{1\le j\le n_0}\big)$. We can use Proposition 2.4 as well as Lemmas 2.6 and 2.7 to get

$$\begin{aligned}\operatorname{Re}\langle N^+_{h,\varepsilon}PS_h^{-1}\hat\Pi_0 u, S_h^{-1}\hat\Pi_0 u\rangle&=\operatorname{Re}\langle Pw', N^+_{h,\varepsilon}w'\rangle\\&\quad+\operatorname{Re}\langle w', P^*N^+_{h,\varepsilon}r'\rangle+\operatorname{Re}\langle N^+_{h,\varepsilon}Pr', w'\rangle\\&\quad+\operatorname{Re}\langle Pr', N^+_{h,\varepsilon}r'\rangle\\&\ge\frac{h}{C}\|w\|^2-O\big(\|w\|\,\|r\|\big)-O(\mathrm{e}^{-\frac{\alpha}{h}}\|r\|)\\&\ge\frac{h}{2C}\|S_h^{-1}\hat\Pi_0 u\|^2.\end{aligned}$$

As usual, all of the above remains true with P^* and $N^-_{h,\varepsilon}$ instead of P and $N^+_{h,\varepsilon}$ so the proof is now complete.

End of Proof of Theorem 1.6: Let $z\in\mathbb{C}$ satisfying $\operatorname{Re} z\le ch^2$ and $|z|\ge\tilde c h^2$ and recall the notation $H=\operatorname{Ran}\Pi_0$. We already know from Proposition 2.8 that $\hat P_h-z$ is invertible, but it is clearly also the case of $P_h|_H-z$ since $P_h|_H=O(\mathrm{e}^{-\alpha/h})$. Therefore, P_h-z is invertible and we have

$$(P_h-z)^{-1}=(\hat P_h-z)^{-1}\hat\Pi_0+(P_h|_H-z)^{-1}\Pi_0. \tag{2.8}$$

Besides, we easily have for such z that $\|(P_h|_H-z)u\|\ge\frac{1}{C}h^2\|u\|$ which combined with (2.8), Proposition 2.8 and the fact that $\|\Pi_0\|=O(1)$ yields the estimate $(P_h-z)^{-1}=O(h^{-2})$. □

3. Accurate Quasimodes

3.1. General Form

Let us denote

$$W(x,v)=\frac{V(x)}{2}+\frac{v^2}{4}$$

the global potential on $\mathbb{R}^{2d}$. Before we can construct our quasimodes, we need to recall the general labeling of the minima which originates from [6] and was generalized in [11], as well as the topological constructions that go with it. In our case, it has to be done for the global potential, i.e., the function W. However, by the definition of W, a strong connection between these constructions for W and the ones for V will appear, leading to simplifications. In order to give a proper statement about this connection, let us construct the labelings for both W and V. To this aim, we consider $d' \in \mathbb{N}^*$ and a smooth Morse function Y on $\mathbb{R}^{d'}$ bounded from below, having at least two local minima and such that $|\nabla Y| \geq 1/C$ outside of a compact. According to Hypothesis 1.5, one can, for instance, take $Y = V/2$ or $Y = W$ and recall that as we discussed following Hypothesis 1.5, it implies that $Y(X) \geq |X|/C$ outside of a compact. We also denote

$$\mathcal{U}^{(k),Y} \text{ the critical points of } Y \text{ of index } k. \tag{3.1}$$

For shortness, we will write "CC" instead of "connected component".

Lemma 3.1. *If $X \in \mathcal{U}^{(1),Y}$, then there exists $r_0 > 0$ such that for all $0 < r < r_0$, X has a connected neighborhood U_r in $B(X, r)$ such that $U_r \cap \{Y < Y(X)\}$ has exactly 2 CCs.*

Proof. Let $X \in \mathcal{U}^{(1),Y}$; according to the Morse lemma, there exists a connected neighborhood U_r of X, $r' > 0$ and $\varphi : U_r \to B(0, r')$ a smooth diffeomorphism such that

$$Y \circ \varphi^{-1} = Y(X) + \frac{1}{2}\langle \mathrm{Hess}_X Y \cdot, \cdot\rangle.$$

Besides, it is easy to see that

$$U_r \cap \{Y < Y(X)\} = \varphi^{-1}\big(\{y \in B(0, r') \,;\, \langle \mathrm{Hess}_X Y\, y, y\rangle < 0\}\big)$$

and $\{y \in B(0, r') \,;\, \langle \mathrm{Hess}_X Y\, y, y\rangle < 0\}$ has exactly 2 CCs.

Lemma 3.2. *Let $X \in \mathbb{R}^{d'}$ and suppose there exists $r_0 > 0$ such that for every neighborhood U of X in $B(X, r_0)$, the set $U \cap \{Y < Y(X)\}$ is not connected. Then, $X \in \mathcal{U}^{(1),Y}$.*

Proof. First we clearly have that $\nabla Y(X) = 0$ since otherwise one could use the implicit function theorem to find a neighborhood U of X in $B(X, r_0)$ such that $U \cap \{Y < Y(X)\}$ is connected. It is also clear that $X \notin \mathcal{U}^{(0),Y}$ so let us assume by contradiction that $X \in \mathcal{U}^{(k),Y}$ with $k \geq 2$. Then using the Morse lemma as in the proof of Lemma 3.1, we would once again get that X has a neighborhood U in $B(X, r_0)$ such that $U \cap \{Y < Y(X)\}$ has the same number of CCs as $\{y \in B(0, r) \,;\, \langle \mathrm{Hess}_X Y\, y, y\rangle < 0\}$ which is connected since $k \geq 2$. Hence, X has to be in $\mathcal{U}^{(1),Y}$.

In view of the result from Lemma 3.1 and following the approach from [6,11], we give the following definition:

Definition 3.3.
1. We say that $X \in \mathcal{U}^{(1),Y}$ is a separating saddle point and we denote $X \in \mathcal{V}^{(1),Y}$ if for every $r > 0$ small enough, the two CCs of $U_r \cap \{Y < Y(X)\}$ are contained in different CCs of $\{Y < Y(X)\}$.
2. We say that $\sigma \in \mathbb{R}$ is a separating saddle value if $\sigma \in Y(\mathcal{V}^{(1),Y})$.
3. Finally, we say that a set $E \subset \mathbb{R}^{d'}$ is critical if there exists $\sigma \in Y(\mathcal{V}^{(1),Y})$ such that E is a CC of $\{Y < \sigma\}$ satisfying $\partial E \cap \mathcal{V}^{(1),Y} \neq \emptyset$.

Lemma 3.4. *Let* $\mathbf{m}$, $\mathbf{m}'$ *be two distinct local minima of* Y. *The real number*

$$\sigma = \sup\big\{a \in \mathbb{R}\,;\, \mathbf{m} \text{ and } \mathbf{m}' \text{ are in two different CCs of } \{Y < a\}\big\}$$

is well defined, and $\{Y < \sigma\}$ *has at least two CCs* $\Omega \ni \mathbf{m}$ *and* $\Omega' \ni \mathbf{m}'$. *Moreover,* σ *is a separating saddle value and* Ω, Ω' *are critical.*

Proof. We can assume that $Y(\mathbf{m}) \leq Y(\mathbf{m}')$ so taking $a := \inf_{\mathcal{A}} Y$ where $\mathcal{A}$ is a well-chosen annulus centered in $\mathbf{m}'$, we see that

$$\big\{a \in \mathbb{R}\,;\, \mathbf{m} \text{ and } \mathbf{m}' \text{ are in two different CCs of } \{Y < a\}\big\} \neq \emptyset \tag{3.2}$$

and it is then clear that σ is well defined. Besides, if $(\sigma_n)_{n\geq 1}$ is an increasing sequence in the set from (3.2) that converges toward σ and $\gamma : [0,1] \to \mathbb{R}^{d'}$ is a continuous path linking $\mathbf{m}$ and $\mathbf{m}'$, then

$$\gamma([0,1]) \cap \big(\mathbb{R}^{d'} \backslash \{Y < \sigma\}\big) = \bigcap_{n\geq 1} \gamma([0,1]) \cap \big(\mathbb{R}^{d'} \backslash \{Y < \sigma_n\}\big)$$

is non-empty by compactness so we can consider $\Omega \ni \mathbf{m}$ and $\Omega' \ni \mathbf{m}'$ two different CCs of $\{Y < \sigma\}$. To prove that σ is a separating saddle value, we will actually show that there exists a CC of $\{Y < \sigma\}$ that we denote Ω'' which is not Ω and satisfies $\overline{\Omega} \cap \overline{\Omega''} \neq \emptyset$. Assume by contradiction that there exists $\varepsilon > 0$ such that $(\Omega + B(0,\varepsilon))\backslash\overline{\Omega}$ is included in $\{Y \geq \sigma\}$. In that case, the points of $(\Omega + B(0,\varepsilon))\backslash\overline{\Omega}$ on which Y takes the value σ are local minima of Y which is a Morse function, so there are finitely many such points. Thus, up to taking ε smaller, we can assume that

$$\Gamma := \mathrm{dist}(\cdot, \Omega)^{-1}(\{\varepsilon\}) \subseteq \{Y > \sigma\}.$$

Hence, there exists $\delta > 0$ such that the minimum of Y on Γ is $\sigma + \delta$. Since any continuous path linking $\mathbf{m}$ and $\mathbf{m}'$ has to cross Γ, $\mathbf{m}$ and $\mathbf{m}'$ are in two different CCs of $\{Y < \sigma + \delta/2\}$. This contradicts the maximality of σ and proves the existence of Ω''. Hence, Lemma 3.2 implies that $\overline{\Omega} \cap \overline{\Omega''} \subseteq \mathcal{U}^{(1),Y}$ and then $\overline{\Omega} \cap \overline{\Omega''} \subseteq \mathcal{V}^{(1),Y}$ follows obviously from the definition of $\mathcal{V}^{(1),Y}$.

Thanks to Lemma 3.4, we know that $\mathcal{V}^{(1),Y} \neq \emptyset$. Let us then denote $\sigma_2 > \cdots > \sigma_N$ where $N \geq 2$ is the different separating saddle values of Y and for convenience we set $\sigma_1 = +\infty$. We call *labeling* of the minima of Y any injection $l : \mathcal{U}^{(0),Y} \to [\![1,N]\!] \times \mathbb{N}^*$. If $l(\mathbf{m}) = (k,j)$, we denote for shortness $\mathbf{m} = \mathbf{m}_{k,j}$. We are going to introduce the usual labeling of the minima for a potential Y (see, for instance, [6,11,12]). We adopt a slightly unusual point of view in order to facilitate the establishment of the correspondence between the constructions for W and the ones for $V/2$ that we will state later on. For $\sigma \in \mathbb{R} \cup \{+\infty\}$, let us denote $\mathcal{C}^Y_\sigma$ the set of all the CCs of $\{Y < \sigma\}$. Given a labeling l of the minima, we denote for $k \in [\![1,N]\!]$

$$\mathcal{U}^{(0),Y}_k = l^{-1}([\![1,k]\!] \times \mathbb{N}^*) \cap \{Y < \sigma_k\}$$

and we say that the labeling is *adapted* to the separating saddle values if for all $k \in [\![1,N]\!]$, each element of $l^{-1}(\{k\} \times \mathbb{N}^*)$ is a global minimum of Y restricted to some CC of $\{Y < \sigma_k\}$ and the map

$$T_k^Y : \mathcal{U}_k^{(0),Y} \to \mathcal{C}_{\sigma_k}^Y \tag{3.3}$$

sending $\mathbf{m} \in \mathcal{U}_k^{(0),Y}$ on the element of $\mathcal{C}_{\sigma_k}^Y$ to which it belongs is bijective. In particular, $l^{-1}(\{k\} \times \mathbb{N}^*)$ is contained in $\mathcal{U}_k^{(0),Y}$. Such labelings exist, one can, for instance, easily check that the usual labeling procedure presented in [11] is adapted to the separating saddle values.

Lemma 3.5. *Under an adapted labeling of the minima of Y, for any $2 \leq k \leq N$, the elements of $T_k^Y\big(l^{-1}(\{k\} \times \mathbb{N}^*)\big)$ are critical.*

Proof. Let $\mathbf{m}_{k,j} \in l^{-1}(\{k\} \times \mathbb{N}^*)$. There exists a CC of $\{Y < \sigma_{k-1}\}$ that we call E which is such that $T_k^Y(\mathbf{m}_{k,j}) \subseteq E$ and E contains some $\mathbf{m}_{k',j'} \in E$ for $1 \leq k' \leq k-1$ and $j' \in \mathbb{N}^*$ by bijectivity of T_{k-1}^Y. Therefore, $\mathbf{m}_{k',j'}$ and $\mathbf{m}_{k,j}$ are in the same CC of $\{Y < \sigma_{k-1}\}$ but are not both in $T_k^Y(\mathbf{m}_{k,j})$ this time by bijectivity of T_k^Y. Applying Lemma 3.4 to $\mathbf{m}_{k',j'}$ and $\mathbf{m}_{k,j}$, we obtain a separating saddle value $\tilde{\sigma}$ which is the maximal real number such that $\mathbf{m}_{k',j'}$ and $\mathbf{m}_{k,j}$ are in two different CCs of $\{Y < \tilde{\sigma}\}$. Therefore, we get $\tilde{\sigma} = \sigma_k$ so $T_k^Y(\mathbf{m}_{k,j})$ is one of the CCs of $\{Y < \tilde{\sigma}\}$ called Ω and Ω' in Lemma 3.4 and which are critical.

Definition 3.6. Recall the notation (3.1) and Definition 3.3. Given an adapted labeling $(\mathbf{m}_{k,j})_{k,j}$, we can now define the following mappings:

- $E^Y : \mathcal{U}^{(0),Y} \longrightarrow \mathcal{P}(\mathbb{R}^{d'})$
  $\mathbf{m}_{k,j} \longmapsto T_k^Y(\mathbf{m}_{k,j})$
  where T_k^Y is the map defined in (3.3).
- $\mathbf{j}^Y : \mathcal{U}^{(0),Y} \to \mathcal{P}\big(\mathcal{V}^{(1),Y} \cup \{\mathbf{s}_1\}\big)$
  given by $\mathbf{j}^Y(\mathbf{m}_{1,1}) = \mathbf{s}_1$ where $\mathbf{s}_1$ is a fictive saddle point such that $Y(\mathbf{s}_1) = \sigma_1 = +\infty$, and for $2 \leq k \leq N$, $\mathbf{j}^Y(\mathbf{m}_{k,j}) = \partial E^Y(\mathbf{m}_{k,j}) \cap \mathcal{V}^{(1),Y}$ which is not empty according to Lemma 3.5 and included in $\{Y = \sigma_k\}$.
- $\boldsymbol{\sigma}^Y : \mathcal{U}^{(0),Y} \to Y(\mathcal{V}^{(1),Y}) \cup \{\sigma_1\}$
  $\mathbf{m} \mapsto Y(\mathbf{j}^Y(\mathbf{m}))$
  where we allow ourselves to identify the set $Y(\mathbf{j}^Y(\mathbf{m}))$ and its unique element in $Y(\mathcal{V}^{(1),Y}) \cup \{\sigma_1\}$.
- $S^Y : \mathcal{U}^{(0),Y} \longrightarrow]0, +\infty]$
  $\mathbf{m} \longmapsto \boldsymbol{\sigma}^Y(\mathbf{m}) - Y(\mathbf{m})$.

Let us now state a lemma that will enable us to show that, roughly speaking, the previous constructions for $Y = V/2$ are the projections on $\mathbb{R}_x^d$ of the ones for $Y = W$. First, we give the following easy observation.

Remark 3.7. By definition of W, we have $V/2 = W(\cdot, 0)$. Moreover, if $(x_0, v_0) \in \{W < \sigma\}$, then $\{x_0\} \times B(0, |v_0|) \subseteq \{W < \sigma\}$.

For shortness, we denote $\mathcal{C}_\sigma = \mathcal{C}_\sigma^{V/2}$ and $\widetilde{\mathcal{C}}_\sigma = \mathcal{C}_\sigma^W$ as well as $\mathcal{U}^{(k)} = \mathcal{U}^{(k),V/2}$ and $\widetilde{\mathcal{U}}^{(k)} = \mathcal{U}^{(k),W}$. (We do similarly with $\mathcal{V}$ or $\mathcal{U}_k$ instead of $\mathcal{U}$.) Notice that $\widetilde{\mathcal{U}}^{(k)} = \mathcal{U}^{(k)} \times \{0\}$. We introduce the natural projection $\pi_x : \mathbb{R}^{2d} \to \mathbb{R}_x^d$ sending (x, v) on x that we also consider as a map from $\mathcal{P}(\mathbb{R}^{2d})$ to $\mathcal{P}(\mathbb{R}_x^d)$.

Lemma 3.8. *For all $\sigma \in \mathbb{R}$, the projection π_x sends $\widetilde{\mathcal{C}}_\sigma$ in $\mathcal{C}_\sigma$. Moreover, the map $\pi_x : \widetilde{\mathcal{C}}_\sigma \to \mathcal{C}_\sigma$ is bijective.*

Proof. The proof of the first statement is an easy consequence of Remark 3.7. For the second statement, let $x \in E \in \mathcal{C}_\sigma$ and denote $\tilde{E}$ the element of $\widetilde{\mathcal{C}}_\sigma$ containing $(x, 0)$. By the first statement, we necessarily have $\pi_x(\tilde{E}) = E$ so we have shown the surjectivity. Now, let $\tilde{E}_1$, $\tilde{E}_2 \in \widetilde{\mathcal{C}}_\sigma$ such that $\pi_x(\tilde{E}_1) = \pi_x(\tilde{E}_2) = E_1$. Let also $(x_1, v_1) \in \tilde{E}_1$ and $(x_2, v_2) \in \tilde{E}_2$. Since $x_1, x_2 \in E_1$, there exists a path $(\gamma(t), 0)$ from $(x_1, 0)$ to $(x_2, 0)$ contained in $\{W < \sigma\}$. Thus, the concatenation of the paths $(x_1, (1-t)v_1)$, $(\gamma(t), 0)$ and (x_2, tv_2) yields a path linking (x_1, v_1) and (x_2, v_2) in $\{W < \sigma\}$. Hence, $\tilde{E}_1 = \tilde{E}_2$ and we get the injectivity.

Proposition 3.9.
1. *We have $\widetilde{\mathcal{V}}^{(1)} = \mathcal{V}^{(1)} \times \{0\}$. In particular, $V/2$ and W have the same separating saddle values.*
2. *A set $\tilde{E} \in \widetilde{\mathcal{C}}_\sigma$ is critical if and only if $\pi_x(\tilde{E})$ is critical.*
3. *A labeling $((\mathbf{m}, 0)_{k,j})_{k,j}$ is adapted to W if and only if $(\mathbf{m}_{k,j})_{k,j}$ is adapted to $V/2$.*

Moreover, given an adapted labeling, the mappings from Definition 3.6 satisfy

$$E^{V/2}(\mathbf{m}_{k,j}) = \pi_x\big(E^W(\mathbf{m}_{k,j}, 0)\big) \quad \textit{and} \quad \mathbf{j}^W(\mathbf{m}_{k,j}, 0) = \mathbf{j}^{V/2}(\mathbf{m}_{k,j}) \times \{0\}.$$

Proof. Let $\tilde{E} \in \widetilde{\mathcal{C}}_\sigma$. Thanks to Remark 3.7, we easily have

$$(x, 0) \in \partial\tilde{E} \iff x \in \partial\big(\pi_x(\tilde{E})\big). \tag{3.4}$$

(a): We already know that $\widetilde{\mathcal{U}}^{(1)} = \mathcal{U}^{(1)} \times \{0\}$. Besides, we easily deduce from (3.4) and Lemma 3.8 that $(\mathbf{s}, 0) \in \widetilde{\mathcal{U}}^{(1)}$ is in the closure of two distinct CCs of $\{W < W(\mathbf{s}, 0)\}$ if and only if $\mathbf{s} \in \mathcal{U}^{(1)}$ is in the closure of two distinct CCs of $\{V < V(\mathbf{s})\}$ so the first item is proven.

(b): This is also a straightforward consequence of (3.4) and Lemma 3.8 combined with item a).

(c): Let $\tilde{E} \in \widetilde{\mathcal{C}}_{\sigma_k}$. By Remark 3.7, we easily have

$$(\mathbf{m}, 0) \text{ is a global minimum of } W|_{\tilde{E}} \iff \mathbf{m} \text{ is a global minimum of } V|_{\pi_x(\tilde{E})}. \tag{3.5}$$

Besides, since $\widetilde{\mathcal{U}}_k^{(0)} = \mathcal{U}_k^{(0)} \times \{0\}$, we have that π^k defined as $\pi_x : \widetilde{\mathcal{U}}_k^{(0)} \to \mathcal{U}_k^{(0)}$ is bijective. We can then conclude as

$$T_k^W = \pi_x^{-1} \circ T_k^{V/2} \circ \pi^k \tag{3.6}$$

where π_x denotes the bijective map from Lemma 3.8.

The last statement is a direct consequence of (3.6), (3.4) and item a).

From now on, we fix a labeling $(\mathbf{m}_{k,j})_{k,j}$ adapted to V.

Definition 3.10. Recall the maps from Definition 3.6. In the rest of the paper, we set

$$\mathbf{j} = \mathbf{j}^{V/2}.$$

Moreover, in view of Proposition 3.9, we can also set

$$\boldsymbol{\sigma}(\mathbf{m}) = \boldsymbol{\sigma}^{V/2}(\mathbf{m}) = \boldsymbol{\sigma}^{W}(\mathbf{m},0) \qquad \text{and} \qquad S(\mathbf{m}) = S^{V/2}(\mathbf{m}) = S^{W}(\mathbf{m},0).$$

However, be careful that we choose to denote $E = \pi_x^{-1} \circ E^{V/2}$ so that the range of E is in $\mathcal{P}(\mathbb{R}^{2d})$. Following [2,6,11,12], we can now state our last assumption that allows us to treat the generic case. As mentioned in the introduction, this assumption could actually be omitted (see [17] or [1]) but this would introduce additional difficulties that are not the main concern of this paper.

Hypothesis 3.11. *For all* $\mathbf{m} \in \mathcal{U}^{(0)}$, *we have*

(a) $\mathbf{m}$ *is the only global minimum of* $V|_{E^{V/2}(\mathbf{m})}$

(b) *for any* $\mathbf{m}' \in \mathcal{U}^{(0)} \backslash \{\mathbf{m}\}$, *the sets* $\mathbf{j}(\mathbf{m})$ *and* $\mathbf{j}(\mathbf{m}')$ *do not intersect.*

According to Proposition 3.9 and (3.5), this hypothesis is equivalent to the facts that $(\mathbf{m},0)$ is the only global minimum of $W|_{E(\mathbf{m})}$ and $\mathbf{j}^W(\mathbf{m},0) \cap \mathbf{j}^W(\mathbf{m}',0) = \emptyset$ which is what we use in practice.

Recall the notation (1.6) and let us extend our notions of asymptotic expansions to smooth functions that are not necessarily symbols. Throughout the paper, for $d' \in \mathbb{N}^*$, $\Omega \subseteq \mathbb{R}^{d'}$ and $a \in \mathcal{C}^\infty(\Omega)$ a function depending on h and such that for all $\beta \in \mathbb{N}^{d'}$ we have $\partial^\beta a = O_{L^\infty}(1)$, we will denote $a \sim_h \sum_{j \geq 0} h^j a_j$, where $(a_j)_{j \geq 0} \subset \mathcal{C}^\infty(\Omega)$ are allowed to depend on h, provided that for all $\beta \in \mathbb{N}^{d'}$ and $N \in \mathbb{N}$, there exists $C_{\beta,N}$ such that

$$\left\| \partial^\beta \left(a - \sum_{j=0}^{N-1} h^j a_j \right) \right\|_{\infty,\Omega} \leq C_{\beta,N} h^N.$$

It implies in particular that $\partial^\beta a_j = O_{L^\infty}(1)$. We will also say that $a \in \mathcal{C}^\infty(\Omega)$ admits a classical expansion on Ω and we will denote $a \sim \sum_{j \geq 0} h^j a_j$ if $a \sim_h \sum_{j \geq 0} h^j a_j$ and the (a_j) are independent of h. From now on, the letter r will denote a small universal positive constant whose value may decrease as we progress in this paper. (One can think of r as $1/C$.) For $x \in \mathbb{R}^d$, we denote $B_0(x,r) = B(x,r) \times B(0,r) \subseteq \mathbb{R}^{2d}$. We essentially follow the quasimodal construction from [1]. We will also denote

$$H_W = h^{-1} X_0^h = \begin{pmatrix} v \\ -\partial_x V \end{pmatrix},$$

where we allowed ourselves to identify the differential operator X_0^h and the vector field representing it.

Let $\mathbf{m} \in \mathcal{U}^{(0)} \backslash \{\underline{\mathbf{m}}\}$; for each $\mathbf{s} \in \mathbf{j}(\mathbf{m})$ we introduce a function $\ell^{\mathbf{s},h}$ that will appear in our quasimodes. Note that thanks to item b) from Hypothesis 3.11, each $\ell^{\mathbf{s},h}$ corresponds to a unique $\mathbf{m} \in \mathcal{U}^{(0)} \backslash \{\underline{\mathbf{m}}\}$. Our goal will be to find some functions $\ell^{\mathbf{s},h}$ such that our quasimodes are the most accurate possible. In order to begin the computations that will yield the equations that the function $\ell^{\mathbf{s},h}$ should satisfy, we will for the moment assume that it satisfies the following:

(3.7)
- (a) $\ell^{\mathbf{s},h}$ is a smooth real-valued function on $\mathbb{R}^{2d}$ whose support is contained in $B_0(\mathbf{s}, 3r)$
- (b) $\ell^{\mathbf{s},h}$ admits a classical expansion $\ell^{\mathbf{s},h}(x,v) \sim \sum h^j \ell^{\mathbf{s}}_j(x,v)$ on $B_0(\mathbf{s}, 2r)$
- (c) $\ell^{\mathbf{s}}_0$ vanishes at $(\mathbf{s}, 0)$
- (d) $(\mathbf{s}, 0)$ is a local minimum of the function $W + (\ell^{\mathbf{s}}_0)^2/2$ which is non-degenerate
- (e) the functions $\theta_{\mathbf{m},h}$ (which depends on $\ell^{\mathbf{s},h}$) and $\chi_{\mathbf{m}}$ that we will introduce in (3.9)–(3.12) are such that $\theta_{\mathbf{m},h}$ is smooth on a neighborhood of supp $\chi_{\mathbf{m}}$.

Once we will have found the desired function $\ell^{\mathbf{s},h}$, we will see in Proposition 5.2 that these assumptions are actually satisfied. Denote $\zeta \in \mathcal{C}^\infty_c(\mathbb{R}, [0,1])$ an even cutoff function supported in $[-\gamma, \gamma]$ that is equal to 1 on $[-\gamma/2, \gamma/2]$ where $\gamma > 0$ is a parameter to be fixed later and

$$A_h = \frac{1}{2}\int_{\mathbb{R}} \zeta(s) \mathrm{e}^{-\frac{s^2}{2h}} \mathrm{d}s = \int_0^\gamma \zeta(s) \mathrm{e}^{-\frac{s^2}{2h}} \mathrm{d}s = \frac{\sqrt{\pi h}}{\sqrt{2}}(1 + O(\mathrm{e}^{-\alpha/h})) \quad \text{for some } \alpha > 0. \tag{3.8}$$

We now define for each $\mathbf{m} \in \mathcal{U}^{(0)} \backslash \{\underline{\mathbf{m}}\}$ a function $\theta_{\mathbf{m},h}$ as follows: if $(x,v) \in B_0(\mathbf{s}, r) \cap \{|\ell^{\mathbf{s},h}| \leq 2\gamma\}$ for some $\mathbf{s} \in \mathbf{j}(\mathbf{m})$,

$$\theta_{\mathbf{m},h}(x,v) = \frac{1}{2}\Big(1 + A_h^{-1} \int_0^{\ell^{\mathbf{s},h}(x,v)} \zeta(s) \mathrm{e}^{-s^2/2h} \mathrm{d}s\Big) \tag{3.9}$$

, whereas we set

$$\theta_{\mathbf{m},h} = 1 \quad \text{on } \ (E(\mathbf{m}) + B(0,\varepsilon)) \Bigg\backslash \Bigg(\bigsqcup_{\mathbf{s} \in \mathbf{j}(\mathbf{m})} \big(B_0(\mathbf{s}, r) \cap \{|\ell^{\mathbf{s},h}| \leq 2\gamma\}\big) \Bigg) \tag{3.10}$$

with $\varepsilon(r) > 0$ to be fixed later and

$$\theta_{\mathbf{m},h} = 0 \quad \text{everywhere else.} \tag{3.11}$$

Note that $\theta_{\mathbf{m},h}$ takes values in $[0,1]$. Denote Ω the CC of $\{W \leq \boldsymbol{\sigma}(\mathbf{m})\}$ containing $\mathbf{m}$. The CCs of $\{W \leq \boldsymbol{\sigma}(\mathbf{m})\}$ are separated so for $\varepsilon > 0$ small enough, there exists $\tilde{\varepsilon} > 0$ such that

$$\min \big\{ W(x,v)\,;\, \mathrm{d}\big((x,v), \Omega\big) = \varepsilon \big\} = \boldsymbol{\sigma}(\mathbf{m}) + 2\tilde{\varepsilon}.$$

Thus, the distance between $\{W \leq \boldsymbol{\sigma}(\mathbf{m}) + \tilde{\varepsilon}\} \cap \big(\Omega + B(0,\varepsilon)\big)$ and $\partial\big(\Omega + B(0,\varepsilon)\big)$ is positive and we can consider a cutoff function

$$\chi_{\mathbf{m}} \in \mathcal{C}^\infty_c(\mathbb{R}^{2d}, [0,1]) \tag{3.12}$$

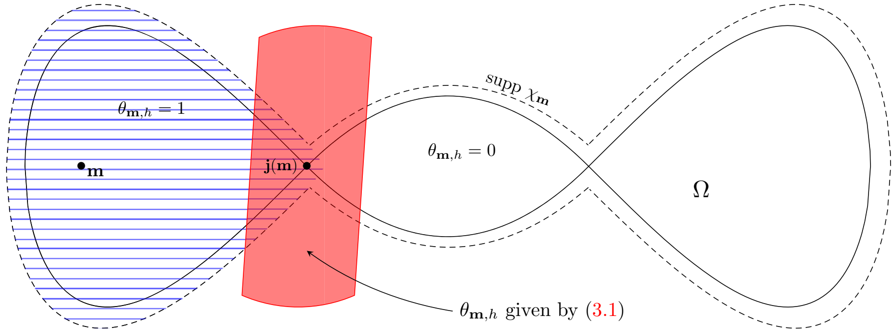
such that

$$\chi_{\mathbf{m}} = 1 \text{ on } \{W \leq \boldsymbol{\sigma}(\mathbf{m}) + \tilde{\varepsilon}\} \cap \big(\Omega + B(0,\varepsilon)\big)$$

and

$$\operatorname{supp}\chi_{\mathbf{m}} \subset \big(\Omega + B(0,\varepsilon)\big).$$

To sum up, we have the following picture:



We also denote

$$W_{\mathbf{m}}(x,v) = W(x,v) - V(\mathbf{m})/2,$$

and it is clear that on the support of $\nabla\chi_{\mathbf{m}}$, we have

$$W_{\mathbf{m}} \geq S(\mathbf{m}) + \tilde{\varepsilon}.$$

Our quasimodes will be the L^2-renormalizations of the functions

$$f_{\mathbf{m},h}(x,v) = \chi_{\mathbf{m}}(x,v)\theta_{\mathbf{m},h}(x,v)\mathrm{e}^{-W_{\mathbf{m}}(x,v)/h} \quad ; \quad \mathbf{m} \in \mathcal{U}^{(0)}\backslash\{\underline{\mathbf{m}}\} \tag{3.13}$$

and for $\mathbf{m} = \underline{\mathbf{m}}$,

$$f_{\underline{\mathbf{m}},h}(x,v) = \mathrm{e}^{-W_{\underline{\mathbf{m}}}(x,v)/h} \in \operatorname{Ker} P_h.$$

Note that these functions belong to $\mathcal{C}_c^\infty(\mathbb{R}^{2d})$ thanks to our assumption on the $(\ell^{\mathbf{s},h})_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}$ and that for $\mathbf{m} \neq \underline{\mathbf{m}}$, we have

$$\operatorname{supp} f_{\mathbf{m},h} \subseteq E(\mathbf{m}) + B(0,\varepsilon') \tag{3.14}$$

where $\varepsilon' = \max(\varepsilon, r)$.

3.2. Action of the Operator P_h

Let us fix $\mathbf{m} \in \mathcal{U}^{(0)}\backslash\{\underline{\mathbf{m}}\}$.

We will denote

$$\widetilde{W}_{\mathbf{m},h} = W_{\mathbf{m}} + \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} (\ell^{\mathbf{s},h})^2/2 \tag{3.15}$$

and

$$\psi^{\mathbf{m},h}(x,v,v') = \int_0^1 \partial_v \widetilde{W}_{\mathbf{m},h}(x, v' + t(v-v'))\mathrm{d}t. \tag{3.16}$$

Remark 3.12. Using Hypothesis 1.3, it is easy to see that $b_h^*\mathrm{Op}_h(M^h) = \mathrm{Op}_h(g^h)$, with

$$g^h = (-i\,{}^t\eta + {}^tv/2)M^h - \frac{h}{2}({}^t\nabla_v - \frac{i}{2}{}^t\nabla_\eta)M^h \in \mathcal{M}_{1,d}\Big(S^0_\tau(\langle(v,\eta)\rangle^{-1})\Big)$$

where

$$ {}^t\nabla_v M^h = \left(\sum_{k=1}^d \partial_{v_k} m_{k,j} \right)_{1\leq j\leq d} $$

and ${}^t\nabla_\eta$ is defined similarly.

Proposition 3.13. *Let $f_{\mathbf{m},h}$ be the quasimode defined in (3.13). With the notations introduced in (3.8) and (3.15), one has*

$$ P_h f_{\mathbf{m},h} = \frac{h}{2} A_h^{-1} \omega^{\mathbf{m},h}\, \mathrm{e}^{\frac{-\widetilde{W}_{\mathbf{m},h}}{h}} \mathbf{1}_{\mathbf{j}^W(\mathbf{m})+B_0(0,2r)} + O_{L^2}\Big(h^\infty \mathrm{e}^{-\frac{S(\mathbf{m})}{h}}\Big) $$

where $\omega^{\mathbf{m},h}$ is a function bounded uniformly in h and defined on $\mathbf{j}^W(\mathbf{m}) + B_0(0,2r)$ by

$$ \omega^{\mathbf{m},h} = \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} \Big(H_W \cdot \nabla \ell^{\mathbf{s},h} + I^{\mathbf{s},h} \Big) $$

with $I^{\mathbf{s},h}(x,v)$ given for $(x,v)\in \mathbf{j}^W(\mathbf{m}) + B_0(0,2r)$ by the oscillatory integral

$$ (2\pi h)^{-d} \int_{\mathbb{R}^d} \int_{|v'|\leq 2r} \mathrm{e}^{\frac{i}{h}\eta\cdot(v-v')} g^h\Big(x, \frac{v+v'}{2}, \eta + i\psi^{\mathbf{m},h}(x,v,v')\Big) \partial_v \ell^{\mathbf{s},h}(x,v')\, \mathrm{d}v'\mathrm{d}\eta. $$

Proof. In order to lighten the notations, we will drop some of the exponents and indexes $\mathbf{m}$, $\mathbf{s}$ and h in the proof. By (3.7), we have on the support of χ that θ is smooth and

$$ \nabla\theta = \frac{A_h^{-1}}{2} \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} \mathrm{e}^{-(\ell^{\mathbf{s}})^2/2h} \zeta(\ell^{\mathbf{s}}) \nabla \ell^{\mathbf{s}}\, \mathbf{1}_{B_0(\mathbf{s},r)}. $$

Here, we have to put the indicator function because $\zeta(\ell)\nabla\ell$ might have some support in $B_0(\mathbf{s},3r)\backslash B_0(\mathbf{s},r)$. We can then begin by computing

$$ \begin{aligned} X_0^h f &= hH_W\cdot\nabla f \\ &= hH_W\cdot\nabla\theta\,\chi\mathrm{e}^{-W_{\mathbf{m}}/h} + hH_W\cdot\nabla\chi\,\theta\mathrm{e}^{-W_{\mathbf{m}}/h} \\ &= \frac{h}{2}A_h^{-1}\chi\mathrm{e}^{-\widetilde{W}/h} \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} \zeta(\ell^{\mathbf{s}}) H_W\cdot\nabla\ell^{\mathbf{s}}\,\mathbf{1}_{B_0(\mathbf{s},r)} + O\Big(h\mathrm{e}^{-\frac{S(\mathbf{m})+\tilde\varepsilon}{h}}\Big). \end{aligned} \tag{3.17} $$

since $W_{\mathbf{m}} \geq S(\mathbf{m}) + \tilde\varepsilon$ on the support of $\nabla\chi$. Now, we can use Remark 3.12 to write

$$ \begin{aligned} Q_h(f) &= h\mathrm{Op}_h(g)\big((\partial_v\theta)\chi\mathrm{e}^{-W_{\mathbf{m}}/h} + (\partial_v\chi)\theta\mathrm{e}^{-W_{\mathbf{m}}/h}\big) \\ &= \frac{h}{2}A_h^{-1} \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} \mathrm{Op}_h(g)\Big(\zeta(\ell^{\mathbf{s}})\chi\mathrm{e}^{-\widetilde{W}/h}\partial_v\ell^{\mathbf{s}}\,\mathbf{1}_{B_0(\mathbf{s},r)}\Big) + O\Big(h\mathrm{e}^{-\frac{S(\mathbf{m})+\tilde\varepsilon}{h}}\Big) \end{aligned} \tag{3.18} $$

since $g \in S(\langle (v,\eta)\rangle^{-1})$ and thus $\mathrm{Op}_h(g)$ is bounded uniformly in h. But since g does not depend on ξ, we have for $\mathbf{s} \in \mathbf{j}(\mathbf{m})$

$$(2\pi h)^d \mathrm{Op}_h(g)\Big(\zeta(\ell)\chi \mathrm{e}^{-\widetilde{W}/h}\partial_v \ell\, \mathbf{1}_{B_0(\mathbf{s},r)}\Big)(x,v)$$
$$= \int_{\mathbb{R}^d}\int_{|v'|\le r} \mathrm{e}^{\frac{i}{h}\eta\cdot(v-v')} g\Big(x, \frac{v+v'}{2}, \eta\Big)$$
$$\times\, \chi(x,v')\zeta\big(\ell(x,v')\big)\mathrm{e}^{-\widetilde{W}(x,v')/h}\partial_v \ell(x,v')\ \mathrm{d}v'\mathrm{d}\eta\, \mathbf{1}_{B(\mathbf{s},r)}(x). \tag{3.19}$$

Let us now treat separately the cases $|v| \ge 2r$ and $|v| < 2r$.

When $|v| \ge 2r$, we have $|v - v'| \ge r$ so we can apply the non-stationary phase to the integral in η to get that for all $x \in B(\mathbf{s}, r)$ and $N \ge 1$, there exists $C_N > 0$ such that

$$\Big|\int_{\mathbb{R}^d}\int_{|v'|\le r} \mathrm{e}^{\frac{i}{h}\eta\cdot(v-v')} g\Big(x, \frac{v+v'}{2}, \eta\Big)\chi(x,v')\zeta\big(\ell(x,v')\big)\mathrm{e}^{-\widetilde{W}(x,v')/h}\partial_v \ell(x,v')\ \mathrm{d}v'\mathrm{d}\eta\Big|$$
$$\le C_N h^N |v|^{-N}\mathrm{e}^{-\frac{S(\mathbf{m})}{h}}$$

where we used item d) from (3.7), the fact that $W_{\mathbf{m}}(\mathbf{s}, 0) + \ell_0^2(\mathbf{s}, 0)/2 = S(\mathbf{m})$ and the estimate $|v - v'| \ge |v|/2$. Hence, we have shown that

$$Q_h f\, \mathbf{1}_{\{|v|\ge 2r\}} = O\Big(h^\infty \mathrm{e}^{-\frac{S(\mathbf{m})}{h}}\Big) \quad \text{and} \quad P_h f\, \mathbf{1}_{\{|v|\ge 2r\}} = O\Big(h^\infty \mathrm{e}^{-\frac{S(\mathbf{m})}{h}}\Big). \tag{3.20}$$

Now for the case $|v| < 2r$, let us denote $J_1^{\mathbf{s}}(x, v)$ the RHS of (3.19). Proceeding as in [18] in order to take the $\mathrm{e}^{-\widetilde{W}(x,v')/h}$ in front of the oscillatory integral, we get that for any $x \in B(\mathbf{s}, r)$,

$$J_1^{\mathbf{s}}(x,v) = \mathrm{e}^{-\widetilde{W}(x,v)/h} J_2^{\mathbf{s}}(x,v) \tag{3.21}$$

where

$$J_2^{\mathbf{s}}(x,v) = \int_{\mathbb{R}^d}\int_{|v'|\le r} \mathrm{e}^{\frac{i}{h}\big(\eta - i\psi(x,v,v')\big)\cdot\big(v-v'\big)} g\Big(x, \frac{v+v'}{2}, \eta\Big)\chi(x,v')\zeta\big(\ell(x,v')\big)$$
$$\partial_v \ell(x,v')\ \mathrm{d}v'\mathrm{d}\eta\, \mathbf{1}_{B(\mathbf{s},r)}(x)$$

and ψ is the function defined in (3.16). For $K \subset \{1, \dots, d\}$ and $z \in \mathbb{C}^d$, denote $z_K = (z_j)_{j\in K}$. We also denote for $d' \in \mathbb{N}$ and $1 \le j \le d'$

$$e_j = (\delta_{k,j})_{1\le k\le d'} \in \mathbb{N}^{d'} \tag{3.22}$$

the elements of the canonical basis of $\mathbb{C}^{d'}$. Now, notice that ψ is a smooth function and that using the expansion of ℓ and (3.15), we get on $B_0(\mathbf{s}, 2r) \times \{|v'| \le 2r\}$,

$$\psi(x,v,v') = \frac{v+v'}{4} + \int_0^1 \big(\ell_0 \partial_v \ell_0\big)(x, v' + t(v - v'))\mathrm{d}t + O(h).$$

In particular, we can choose r small enough so that $|\psi| < \tau$ on $B_0(\mathbf{s}, 2r) \times \{|v'| \le 2r\}$. Besides, since $g \in S^0_\tau(\langle (v,\eta)\rangle^{-1})$, we have for all $K \subset \{1, \dots, d\}$

and $k \in \{1, \dots, d\} \backslash K$ that the symbol

$$\eta_k \mapsto g\left(x, \frac{v+v'}{2}, \eta + i\sum_{j\in K} [\psi(x,v,v')]_j e_j\right)$$

has an analytic continuation to $\{|\eta_k| < \tau\}$ for any $x \in B(\mathbf{s}, r)$, v, $v' \in B(0, 2r)$ and $\eta \in \mathbb{R}^d$. Hence, one can use the Cauchy formula which combined with the decay of g yields

$$\int_{\mathbb{R}} e^{\frac{i}{h}\left(\eta_k - i[\psi(x,v,v')]_k\right)(v_k - v'_k)} g\left(x, \frac{v+v'}{2}, \eta + i\sum_{j\in K}[\psi(x,v,v')]_j e_j\right) \,\mathrm{d}\eta_k$$

$$= \int_{\mathbb{R}} e^{\frac{i}{h}\eta_k(v_k - v'_k)} g\left(x, \frac{v+v'}{2}, \eta + i\sum_{j\in K\cup\{k\}}[\psi(x,v,v')]_j e_j\right) \,\mathrm{d}\eta_k.$$

Applying this successively for each component of η on the integrals in $J_2^{\mathbf{s}}$ finally gives $J_2^{\mathbf{s}} = J_3^{\mathbf{s}}$ where

$$J_3^{\mathbf{s}}(x,v) = \int_{\mathbb{R}^d}\int_{|v'|\le r} e^{\frac{i}{h}\eta\cdot(v-v')} g\Big(x, \frac{v+v'}{2}, \eta + i\psi(x,v,v')\Big)\chi(x,v')\zeta\big(\ell(x,v')\big)$$
$$\partial_v \ell(x,v') \,\mathrm{d}v'\mathrm{d}\eta\, \mathbf{1}_{B(\mathbf{s},r)}(x).$$

Combined with (3.19) and (3.21), this yields for $|v| < 2r$

$$(2\pi h)^d \mathrm{Op}_h(g)\Big(\zeta(\ell)\chi e^{-\widetilde{W}/h}\partial_v \ell\, \mathbf{1}_{B_0(\mathbf{s},r)}\Big)(x,v) = e^{-\widetilde{W}(x,v)/h} J_3^{\mathbf{s}}(x,v). \tag{3.23}$$

Therefore, setting on $\mathbf{j}^W(\mathbf{m}) + B_0(0,2r)$

$$\tilde{\omega} = \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} \Big(\chi\zeta(\ell^{\mathbf{s}}) H_W \cdot \nabla \ell^{\mathbf{s}}\, \mathbf{1}_{B_0(\mathbf{s},r)} + (2\pi h)^{-d} J_3^{\mathbf{s}}(x,v)\Big),$$

we have according to (3.17), (3.18), (3.20) and (3.23)

$$P_h f = \frac{h}{2} A_h^{-1} \tilde{\omega}\, e^{-\widetilde{W}/h} \mathbf{1}_{\mathbf{j}^W(\mathbf{m}) + B_0(0,2r)} + O\Big(h^\infty e^{-\frac{S(\mathbf{m})}{h}}\Big).$$

Hence, it is sufficient to check that on $\mathbf{j}^W(\mathbf{m}) + B_0(0,2r)$

$$(\tilde{\omega} - \omega) e^{-\widetilde{W}/h} = O\Big(h^\infty e^{-\frac{S(\mathbf{m})}{h}}\Big).$$

This can be done easily using again the non-stationary phase on an h-independent neighborhood of $(\mathbf{s}, 0)$ on which $\chi\zeta(\ell) - 1$ vanishes since item d) from (3.7) implies that $e^{-\widetilde{W}/h} = O(e^{-(S(\mathbf{m})+\delta)/h})$ outside of this neighborhood for some $\delta > 0$.

Remark 3.14. Since $P_h^* = -X_0^h + Q_h$, it is clear from the previous proof that

$$P_h^* f_{\mathbf{m},h} = \frac{h}{2} A_h^{-1} \overset{*}{\omega}{}^{\mathbf{m},h}\, e^{\frac{-\widetilde{W}_{\mathbf{m},h}}{h}} \mathbf{1}_{\mathbf{j}^W(\mathbf{m}) + B_0(0,2r)} + O_{L^2}\Big(h^\infty e^{-\frac{S(\mathbf{m})}{h}}\Big)$$

with

$$\overset{*}{\omega}{}^{\mathbf{m},h} = \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} \Big(- H_W \cdot \nabla \ell^{\mathbf{s},h} + I^{\mathbf{s},h}\Big).$$

4. Equations on $\ell^{\mathbf{s},h}$

From now on, we also fix $\mathbf{s} \in \mathbf{j}(\mathbf{m})$.

Lemma 4.1. *The function $\omega^{\mathbf{m},h}$ admits the classical expansion $\omega^{\mathbf{m},h} \sim \sum_{j\geq 0} h^j \omega_j^{\mathbf{m}}$ on $B_0(\mathbf{s}, 2r)$ where*

$$\omega_0^{\mathbf{m}} = H_W \cdot \nabla \ell_0^{\mathbf{s}} + M_0\Big(x, v, i\big(\frac{v}{2} + \ell_0^{\mathbf{s}}\, \partial_v \ell_0^{\mathbf{s}}\big)\Big)\big(v + \ell_0^{\mathbf{s}} \partial_v \ell_0^{\mathbf{s}}\big) \cdot \partial_v \ell_0^{\mathbf{s}}$$

and for $j \geq 1$,

$$\begin{aligned}\omega_j^{\mathbf{m}} = {} & H_W \cdot \nabla \ell_j^{\mathbf{s}} + M_0\Big(x, v, i\big(\frac{v}{2} + \ell_0^{\mathbf{s}}\, \partial_v \ell_0^{\mathbf{s}}\big)\Big)\big(v + 2\ell_0^{\mathbf{s}} \partial_v \ell_0^{\mathbf{s}}\big) \cdot \partial_v \ell_j^{\mathbf{s}} \\ & + i\, \ell_0^{\mathbf{s}} \big({}^t v + \ell_0^{\mathbf{s}}\, {}^t(\partial_v \ell_0^{\mathbf{s}})\big) D_\eta M_0\big(x, v, i(v/2 + \ell_0^{\mathbf{s}} \partial_v \ell_0^{\mathbf{s}})\big)\big(\partial_v \ell_j^{\mathbf{s}}\big)\, \partial_v \ell_0^{\mathbf{s}} \\ & + M_0\Big(x, v, i\big(\frac{v}{2} + \ell_0^{\mathbf{s}}\, \partial_v \ell_0^{\mathbf{s}}\big)\Big) \partial_v \ell_0^{\mathbf{s}} \cdot \partial_v \ell_0^{\mathbf{s}}\, \ell_j^{\mathbf{s}} \\ & + i\big({}^t v + \ell_0^{\mathbf{s}}\, {}^t(\partial_v \ell_0^{\mathbf{s}})\big) D_\eta M_0\big(x, v, i(v/2 + \ell_0^{\mathbf{s}} \partial_v \ell_0^{\mathbf{s}})\big)\big(\partial_v \ell_0^{\mathbf{s}}\big)\, \partial_v \ell_0^{\mathbf{s}}\, \ell_j^{\mathbf{s}} \\ & + R_j(\ell_0^{\mathbf{s}}, \ldots, \ell_{j-1}^{\mathbf{s}}) \end{aligned} \tag{3.1}$$

where $R_j : \big(\mathcal{C}^\infty(B_0(\mathbf{s}, 2r))\big)^j \to \mathcal{C}^\infty(B_0(\mathbf{s}, 2r))$ and D_η denotes the partial differential with respect to the variable η.

Proof. Once again, we drop some of the exponents and indexes $\mathbf{m}$, $\mathbf{s}$ and h in the proof. Denote $B_\infty(0, 2r) = \{v', \eta \in \mathbb{R}^{2d};\ \max(|v'|, |\eta|) < 2r\}$. The first terms of ω_0 and ω_j are both easily obtained thanks to the expansion of ℓ on $B_0(\mathbf{s}, 2r)$. Hence, it remains to get an expansion of $g(x, v/2 + v'/2, \eta + i\psi(x, v, v'))$ that we will then be able to combine with the stationary phase to get an expansion of the whole term $I^{\mathbf{s},h}$ of ω. Let us start with an expansion of ψ: The expansion of ℓ yields

$$\partial_v \widetilde{W} - v/2 \sim \sum_{j \geq 0} h^j \sum_{k=0}^{j} \ell_k \partial_v \ell_{j-k} \quad \text{on } B_0(\mathbf{s}, 2r)$$

so using (3.16), we get

$$\psi \sim \sum_{j\geq 0} h^j \psi_j \quad \text{on } B_0(\mathbf{s}, 2r) \times \{|v'| \leq 2r\}$$

where

$$\psi_0(x, v, v') = \frac{v + v'}{4} + \int_0^1 \big(\ell_0 \partial_v \ell_0\big)(x, v' + t(v - v'))\mathrm{d}t \tag{3.2}$$

and for $j \geq 1$,

$$\psi_j(x, v, v') = \int_0^1 \sum_{k=0}^{j} \big(\ell_k \partial_v \ell_{j-k}\big)(x, v' + t(v - v'))\mathrm{d}t. \tag{3.3}$$

Besides, since $M^h \sim \sum_{n\geq 0} h^n M_n$ in $\mathcal{M}_d\big(S^0_\tau(\langle(v, \eta)\rangle^{-2})\big)$, we deduce thanks to Proposition C.2 and Remark 3.12 that g also has a classical expansion

$g \sim \sum_{n\geq 0} h^n g_n$ in $\mathcal{M}_{1,d}\big(S^0_\tau(\langle(v,\eta)\rangle^{-1})\big)$, where the (g_n) are given by

$$g_0(x,v,\eta) = \Big(-i\,{}^t\eta + \frac{{}^tv}{2}\Big)M_0(x,v,\eta) \tag{3.4}$$

and

$$g_n(x,v,\eta) = \left(-i\,{}^t\eta + \frac{{}^tv}{2}\right)M_n(x,v,\eta) - \frac{1}{2}({}^t\nabla_v - \frac{i}{2}{}^t\nabla_\eta)M_{n-1}(x,v,\eta) \tag{3.5}$$

for $n \geq 1$. According to Corollary C.6, we have

$$g_n\left(x, \frac{v+v'}{2}, \eta + i\psi(x,v,v')\right) \sim \sum_{j\geq 0} h^j g_{n,j}(x,v,v',\eta) \quad \text{on } B_0(\mathbf{s},2r)\times B_\infty(0,2r)$$

with

$$g_{n,0}(x,v,v',\eta) = g_n\Big(x, \frac{v+v'}{2}, \eta + i\psi_0(x,v,v')\Big) \tag{3.6}$$

and for $j \geq 1$

$$g_{n,j}(x,v,v',\eta) = iD_\eta g_n\Big(x, \frac{v+v'}{2}, \eta + i\psi_0(x,v,v')\Big)\big(\psi_j(x,v,v')\big) + R^1_j(\ell_0,\dots,\ell_{j-1}) \tag{3.7}$$

where $R^1_j : \big(\mathcal{C}^\infty(B_0(\mathbf{s},2r))\big)^j \to \mathcal{C}^\infty(B_0(\mathbf{s},2r))$. Using the expansion of g itself and Proposition C.1, we get

$$g\Big(x, \frac{v+v'}{2}, \eta + i\psi(x,v,v')\Big) \sim_h \sum_{n\geq 0} h^n g_n\Big(x, \frac{v+v'}{2}, \eta + i\psi(x,v,v')\Big)$$

on $B_0(\mathbf{s},2r)\times B_\infty(0,2r)$ so we can use Proposition C.3 which yields

$$g\Big(x, \frac{v+v'}{2}, \eta + i\psi(x,v,v')\Big) \sim \sum_{j\geq 0} h^j \sum_{n=0}^{j} g_{n,j-n}(x,v,v',\eta) \tag{3.8}$$

on $B_0(\mathbf{s},2r)\times B_\infty(0,2r)$. Thus, using the expansion (3.8) that we just got, the one of $\partial_v\ell$, and the one for an oscillatory integral given by the stationary phase (see, for instance, [21], Theorem 3.17) as well Proposition C.3, we finally get

$$I^{\mathbf{s},h} \sim \sum_{j\geq 0} h^j I_j \quad \text{on } B_0(\mathbf{s},2r), \tag{3.9}$$

where

$$I_j(x,v) = \sum_{n_1+n_2+n_3+n_4=j} \frac{1}{i^{n_1} n_1!}\big(\partial_{v'}\cdot\partial_\eta\big)^{n_1}\Big(g_{n_2,n_3}(x,v,v',\eta)\partial_v\ell_{n_4}(x,v')\Big)\Big|_{\substack{v'=v\\ \eta=0}}.$$

We can already use (3.6) to deduce the expression of ω_0 by noticing that according to (3.2), $\psi_0(x,v,v) = v/2 + \ell_0\partial_v\ell_0$. For $j \geq 1$, the terms of I_j in which the function ℓ_j appears are obviously the one given by $n_4 = j$, but also the one given by $n_3 = j$ according to (3.7). Indeed, in that case, we have using (3.3) that

$$g_{0,j}(x,v,v,0) = i\ell_0 D_\eta g_0\big(x,v,i(v/2+\ell_0\partial_v\ell_0)\big)\big(\partial_v\ell_j\big) \\ + iD_\eta g_0\big(x,v,i(v/2+\ell_0\partial_v\ell_0)\big)\big(\partial_v\ell_0\big)\,\ell_j + R_j^2(\ell_0,\dots,\ell_{j-1})$$

where $R_j^2 : \big(\mathcal{C}^\infty(B_0(\mathbf{s},2r))\big)^j \to \mathcal{C}^\infty(B_0(\mathbf{s},2r))$. We can now conclude as for any $X\in\mathbb{R}^d$,

$$D_\eta g_0\big(x,v,i(v/2+\ell_0\partial_v\ell_0)\big)(X) = -i\,{}^tXM_0\big(x,v,i(v/2+\ell_0\partial_v\ell_0)\big) \\ + \big({}^tv+\ell_0\,{}^t(\partial_v\ell_0)\big)D_\eta M_0\big(x,v,i(v/2+\ell_0\partial_v\ell_0)\big)(X)$$

according to (3.4).

Denote $(m^n_{p,q})_{p,q}$ the entries of the matrix M_n from Hypothesis 1.3. Since we have for $X\in\mathbb{R}^d$

$$D_\eta M_0\big(x,v,i(v/2+\ell_0\partial_v\ell_0)\big)\big(X\big) = \Big(\partial_\eta m^0_{p,q}\big(x,v,i(v/2+\ell_0\partial_v\ell_0)\big)\cdot X\Big)_{1\le p,q\le d},$$

we get by putting

$$U(x,v) = M_0\left(x,v,i\left(\frac{v}{2}+\ell_0\,\partial_v\ell_0\right)\right)\partial_v\ell_0 \\ + \sum_{1\le p,q\le d}(v_p+\ell_0\partial_{v_p}\ell_0)i\partial_\eta m^0_{p,q}\left(x,v,i\left(\frac{v}{2}+\ell_0\,\partial_v\ell_0\right)\right)\partial_{v_q}\ell_0 \tag{3.10}$$

that equation (3.1) reads

$$\omega_j = \left[H_W + \begin{pmatrix} 0 \\ M_0\Big(x,v,i\big(\frac{v}{2}+\ell_0\,\partial_v\ell_0\big)\Big)(v+\ell_0\partial_v\ell_0)+\ell_0\,U\end{pmatrix}\right] \\ \cdot\nabla\ell_j + U\cdot\partial_v\ell_0\,\ell_j + R_j(\ell_0,\dots,\ell_{j-1}).$$

Lemma 4.2. *Let $(x,v)\in B_0(\mathbf{s},2r)$ and $|v'|<2r$. For any $n\in\mathbb{N}$, $\beta\in\mathbb{N}^d$ and $1\le p,q\le d$, we have*

$$\partial_\eta^\beta m^n_{p,q}\left(x,\frac{v+v'}{2},i\psi_0^{\mathbf{m}}(x,v,v')\right)\in i^{|\beta|}\mathbb{R}$$

and

$$\partial_\eta^\beta g_n\left(x,\frac{v+v'}{2},i\psi_0^{\mathbf{m}}(x,v,v')\right)\in i^{|\beta|}\mathbb{R}^d.$$

In particular, U defined in (3.10) sends $B_0(\mathbf{s},2r)$ in $\mathbb{R}^d$.

Proof. Since ℓ_0 vanishes at $(\mathbf{s},0)$, we can suppose that r is such that $i\psi_0(x,v,v')$ is in

$$D(0,\tau)^d = \{z\in\mathbb{C}\,;\,|z|<\tau\}^d \tag{3.11}$$

so by analyticity and using the parity of $m_{p,q}^n$, we have

$$\partial_\eta^\beta m_{p,q}^n\Big(x, \frac{v+v'}{2}, i\psi_0(x,v,v')\Big) = \sum_{\substack{\gamma\in\mathbb{N}^d;\\ |\gamma|+|\beta|\in 2\mathbb{N}}} i^{|\gamma|}\, \frac{\partial_\eta^{\gamma+\beta} m_{p,q}^n\big(x, \frac{v+v'}{2}, 0\big)}{\gamma!}\, \psi_0(x,v,v')^\gamma \in i^{|\beta|}\mathbb{R}.$$

The result for g_n follows easily using (3.4) and (3.5).

We also have the following result whose proof is postponed to Appendix 6 as it involves tedious calculations.

Lemma 4.3. *The term $R_j(\ell_0^{\mathbf{s}}, \dots, \ell_{j-1}^{\mathbf{s}})$ from Lemma 4.1 is real valued. Moreover, it satisfies $R_j(\ell_0^{\mathbf{s}}, \dots, \ell_{j-1}^{\mathbf{s}}) = -R_j(-\ell_0^{\mathbf{s}}, \dots, -\ell_{j-1}^{\mathbf{s}})$.*

In view of the results from Proposition 3.13 and Lemma 4.1, we want to find ℓ such that on $B_0(\mathbf{s}, 2r)$,

$$H_W \cdot \nabla\ell_0 + M_0\left(x, v, i\left(\frac{v}{2} + \ell_0\,\partial_v\ell_0\right)\right)(v + \ell_0\partial_v\ell_0)\cdot\partial_v\ell_0 = 0 \qquad (3.12)$$

and for $j \geq 1$

$$\left[H_W + \begin{pmatrix} 0 \\ M_0\big(x, v, i\big(\frac{v}{2} + \ell_0\,\partial_v\ell_0\big)\big)(v + \ell_0\partial_v\ell_0) + \ell_0\,U \end{pmatrix}\right]\cdot\nabla\ell_j \qquad (3.13)$$
$$+ \partial_v\ell_0 \cdot U\,\ell_j + R_j(\ell_0, \dots, \ell_{j-1}) = 0$$

where U was introduced in (3.10). Note that Lemmas 4.2 and 4.3 ensure that the fact that the $(\ell_j)_{j\geq 0}$ are real valued is compatible with equations (3.13).

4.1. Solving for $\ell_0^{\mathbf{s}}$

Denote

$$p(x,v,\xi,\eta) = i\xi\cdot v - i\eta\cdot\partial_x V + (-i\,{}^t\eta + {}^t v/2)M_0(x,v,\eta)(i\eta + v/2)$$

the principal symbol of the whole operator P_h and $\tilde{p}(x,v,\xi,\eta) = -p(x,v,i\xi,i\eta)$ its complexification. After computing the Hamiltonian of $\tilde{p}$ which vanishes at $(\mathbf{s},0,0,0)$, we find that its linearization at this point is the matrix

$$F = \begin{pmatrix} 0 & \mathrm{Id} & 0 & 0 \\ -\mathrm{Hess}_{\mathbf{s}}V & 0 & 0 & 2M_0(\mathbf{s},0,0) \\ 0 & 0 & 0 & \mathrm{Hess}_{\mathbf{s}}V \\ 0 & \frac{1}{2}M_0(\mathbf{s},0,0) & -\mathrm{Id} & 0 \end{pmatrix}.$$

One can easily check that for any eigenvector (x,v,ξ,η) of F associated with an eigenvalue λ, the vector $(-x,v,\xi,-\eta)$ is an eigenvector associated with $-\lambda$ so the spectrum of F is centrally symmetric with respect to the origin. Moreover, writing

$$F = \begin{pmatrix} 0 & 0 & \mathrm{Id} & 0 \\ 0 & 0 & 0 & \mathrm{Id} \\ \mathrm{Id} & 0 & 0 & 0 \\ 0 & \mathrm{Id} & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & \mathrm{Hess}_{\mathbf{s}}V \\ 0 & \frac{1}{2}M_0(\mathbf{s},0,0) & -\mathrm{Id} & 0 \\ 0 & \mathrm{Id} & 0 & 0 \\ -\mathrm{Hess}_{\mathbf{s}}V & 0 & 0 & 2M_0(\mathbf{s},0,0) \end{pmatrix}$$

and noticing that

$$F\Big(\{v=\eta=0\}\Big)\cap\{v=\eta=0\}=\operatorname{Ker}F\cap\{v=\eta=0\}=\{0\},$$

we see that F satisfies the assumptions of Lemma B.1. Therefore, F has no eigenvalues in $i\mathbb{R}$ so it has $2d$ eigenvalues (counted with algebraic multiplicity) in $\{\operatorname{Re}z>0\}$, while the $2d$ others are in $\{\operatorname{Re}z<0\}$. Therefore, we can apply the stable manifold theorem to get that the stable manifolds associated with $H_{\tilde{p}}$ given in a neighborhood of $(\mathbf{s},0,0,0)$ by

$$\Lambda_\pm=\Big\{(x,v,\xi,\eta)\,;\lim_{t\to\mp\infty}\mathrm{e}^{tH_{\tilde{p}}}(x,v,\xi,\eta)=(\mathbf{s},0,0,0)\Big\}$$

are both of dimension $2d$ and for all $\rho_\pm\in\Lambda_\pm$, we have

$$H_{\tilde{p}}(\rho_\pm)\in T_{\rho_\pm}\Lambda_\pm \tag{3.14}$$

and for $t>0$,

$$\big\|\mathrm{e}^{\mp tH_{\tilde{p}}}\rho_\pm-(\mathbf{s},0,0,0)\big\|\le C\mathrm{e}^{-t/C}\|\rho_\pm-(\mathbf{s},0,0,0)\|.$$

Moreover, we have (see, for instance, [4] Lemmas 3.2 and 3.3) that

$$\tilde{p}(\Lambda_\pm)=\{0\} \tag{3.15}$$

and $\Lambda_\pm$ are Lagrangian manifolds. In order to get some parameterization for those manifolds, we follow the steps of [10], Lemma 8.1.

Lemma 4.4. *The tangent spaces $T_{(\mathbf{s},0,0,0)}\Lambda_\pm$ that we denote for shortness $T_{\mathbf{s}}\Lambda_\pm$ are transverse to both $\{(\mathbf{s},0)\}\times\mathbb{R}^{2d}$ and $\mathbb{R}^{2d}\times\{(0,0)\}$.*

Proof. We provide an adaptation of the proof from [10] as some simplifications appear in our case. Since we are working in the linearized case, we can assume that $\tilde{p}$ coincides with its quadratic approximation at $(\mathbf{s},0,0,0)$ and for commodity we will work with the variable $x_{\mathbf{s}}=x-\mathbf{s}$ instead of x. Note that if a is a quadratic form, its Hamiltonian H_a is then linear and we denote F_a the associated matrix. We then decompose $\tilde{p}=p_2+p_1-p_0$ where

$$p_2=M_0(\mathbf{s},0,0)\eta\cdot\eta,\quad p_1=v\cdot\xi-\operatorname{Hess}_s V x_{\mathbf{s}}\cdot\eta\quad\text{and}\quad p_0=\frac{1}{4}M_0(\mathbf{s},0,0)v\cdot v.$$

It is clear that p_2+p_0 is positive semidefinite; moreover, the subspace $\{v=\eta=0\}$ on which p_2+p_0 vanishes satisfies $\{v=\eta=0\}\cap F_{p_1}^{-1}\big(\{v=\eta=0\}\big)=\{0\}$. Thus, the quadratic form

$$\tilde{q}=(p_2+p_0)+(p_2+p_0)\circ F_{p_1}$$

is positive definite. Let us denote $L_\pm=\Lambda_\pm\cap\{x_{\mathbf{s}}=v=0\}$. To prove that $L_\pm=\{0\}$, it is sufficient to establish that $\tilde{q}=0$ on $L_\pm$. In order to do so, we will show that $L_\pm$ is an F_{p_1}-invariant subspace on which $p_2+p_0=0$. Indeed, it is clear that $p_0=p_1=0$ on $L_\pm$ and thanks to (3.15) we deduce that p_2 also vanishes on $L_\pm$ so in particular $p_2+p_0=0$ on $L_\pm$. It also implies that $L_\pm$ is included in $\{\eta=0\}$ so $F_{p_2}|_{L_\pm}=0$. Besides, we clearly have $F_{p_0}|_{L_\pm}=0$ so F_{p_1} coincides on $L_\pm$ with $F_{\tilde{p}}$ which leaves $\Lambda_\pm$ invariant according to (3.14). Since it is easy to see that $\{x_{\mathbf{s}}=v=0\}$ is also invariant under F_{p_1}, we can conclude as announced that $L_\pm = \{0\}$. The proof that $\Lambda_\pm \cap \{\xi = \eta = 0\} = \{0\}$ is similar.

Since $\Lambda_\pm$ are Lagrangian manifolds such that $T_{\mathbf{s}}\Lambda_\pm$ are transverse to $\{(\mathbf{s},0)\} \times \mathbb{R}^{2d}$, there exist $\phi_\pm \in \mathcal{C}^\infty(B_0(\mathbf{s},2r),\mathbb{R})$ vanishing together with their gradients at $(\mathbf{s},0)$ and such that

$$\Lambda_\pm = \Big\{\Big((x,v,\nabla\phi_\pm(x,v)\Big)\,;(x,v)\in B_0(\mathbf{s},2r)\Big\}.$$

Therefore, $T_{\mathbf{s}}\Lambda_\pm$ coincide with the graphs of the matrices $\mathrm{Hess}_{(\mathbf{s},0)}\phi_\pm$ which are then invertible according to Lemma 4.4. Now we need a result similar to the one of Proposition 8.2 in [10].

Lemma 4.5. *The Hessian matrix of* $\pm\phi_\pm$ *at* $(\mathbf{s},0)$ *is definite positive.*

Proof. The proof is simply an adaptation of the one found in [10]. Here again, we will assume that $\tilde p$ coincides with its quadratic approximation at $(\mathbf{s},0,0,0)$ and work with the variable $x_{\mathbf{s}} = x - \mathbf{s}$ instead of x. For $\delta \in [0,1]$, let us denote

$$\begin{aligned}\tilde p^\delta &= (1-\delta)\tilde p + \delta\big(\xi^2+\eta^2-(x_{\mathbf{s}}^2+v^2)\big)\\ &= p_2^\delta + (1-\delta)p_1 - p_0^\delta\end{aligned}$$

where

$$p_2^\delta = (1-\delta)p_2 + \delta(\xi^2+\eta^2) \qquad\text{and}\qquad p_0^\delta = (1-\delta)p_0 + \delta(x_{\mathbf{s}}^2+v^2).$$

Note in particular that $\tilde p^0 = \tilde p$ and that $\tilde p^1 = \big(\xi^2+\eta^2-(x_{\mathbf{s}}^2+v^2)\big)$ corresponds to the well know Schrödinger case (see, for instance, [4], chapter 3). Besides, we have that

$$F_{\tilde p^\delta} = \begin{pmatrix} 0 & 0 & \mathrm{Id} & 0\\ 0 & 0 & 0 & \mathrm{Id}\\ \mathrm{Id} & 0 & 0 & 0\\ 0 & \mathrm{Id} & 0 & 0\end{pmatrix}\left[(1-\delta)\begin{pmatrix} 0 & 0 & 0 & \mathrm{Hess}_{\mathbf{s}}V\\ 0 & \frac12 M_0(\mathbf{s},0,0) & -\mathrm{Id} & 0\\ 0 & \mathrm{Id} & 0 & 0\\ -\mathrm{Hess}_{\mathbf{s}}V & 0 & 0 & 2M_0(\mathbf{s},0,0)\end{pmatrix} + 2\delta\,\mathrm{Id}\right]$$

so Lemma B.1 easily yields that the eigenvalues of $F_{\tilde p^\delta}$ cannot cross $i\mathbb{R}$ for some $\delta\in(0,1]$. Moreover, it is clear that for $\delta\in(0,1]$, the quadratic form $p_2^\delta + p_0^\delta$ is positive definite, so the results of Lemma 4.4 are true for the $2d$-dimensional Lagrangian planes

$$\Lambda_\pm^\delta = \Big\{(x_{\mathbf{s}},v,\xi,\eta)\,;\lim_{t\to\mp\infty} \mathrm{e}^{tF_{\tilde p^\delta}}(x,v,\xi,\eta)=0\Big\}$$

for all $\delta\in[0,1]$. In particular, there exist $\phi_\pm^\delta \in \mathcal{C}^\infty(B_0(\mathbf{s},2r),\mathbb{R})$ such that

$$T_{\mathbf{s}}\Lambda_\pm^\delta = \Lambda_\pm^\delta = \left\{\left(x_{\mathbf{s}},v,\mathrm{Hess}_{(\mathbf{s},0)}\phi_\pm^\delta\begin{pmatrix}x_{\mathbf{s}}\\ v\end{pmatrix}\right)\,;(x_{\mathbf{s}},v)\in\mathbb{R}^{2d}\right\}.$$

Hence, the graph of $\mathrm{Hess}_{(\mathbf{s},0)}\phi_\pm^\delta$ is given by $T_{\mathbf{s}}\Lambda_\pm^\delta$ which also corresponds to the sum of the generalized eigenspaces of $F_{\tilde p^\delta}$ associated with eigenvalues in $\{\pm\mathrm{Re}\,z<0\}$ and therefore depends continuously on δ. Besides, by Lemma 4.4, $\mathrm{Hess}_{(\mathbf{s},0)}\phi_\pm^\delta$ is invertible for all $\delta\in[0,1]$ and we know from the Schrödinger case that $\pm\mathrm{Hess}_{(\mathbf{s},0)}\phi_\pm^1 > 0$ so necessarily $\pm\mathrm{Hess}_{(\mathbf{s},0)}\phi_\pm > 0$.

At this point, one can proceed as in [1], Lemma 3.2 to establish the following lemma.

Lemma 4.6. *There exists* $\ell_0^{\mathbf{s}} \in \mathcal{C}^\infty(B_0(\mathbf{s}, 2r), \mathbb{R})$ *such that for* $(x, v) \in B_0(\mathbf{s}, 2r)$,

$$\phi_+(x, v) = W(x, v) - W(\mathbf{s}, 0) + \frac{\ell_0^{\mathbf{s}}(x, v)^2}{2}.$$

In particular, $\ell_0^{\mathbf{s}}$ *vanishes at* $(\mathbf{s}, 0)$. *Moreover,* $\{\ell_0^{\mathbf{s}} \neq 0\}$ *is dense in* $B_0(\mathbf{s}, 2r)$.

This function also appears to solve (3.12) as we see in the next proposition.

Proposition 4.7. *The function* $\ell_0^{\mathbf{s}}$ *from Lemma 4.6 is a solution of (3.12) in* $B_0(\mathbf{s}, 2r)$. *Moreover, the vector* $\nabla \ell_0^{\mathbf{s}}(\mathbf{s}, 0)$ *that we denote* $\nu^{\mathbf{s}} = \begin{pmatrix} \nu_1^{\mathbf{s}} \\ \nu_2^{\mathbf{s}} \end{pmatrix}$ *is not 0 and satisfies* $\Phi^{\mathbf{s}} \nu^{\mathbf{s}} = \big(- M_0(\mathbf{s}, 0, 0) \nu_2^{\mathbf{s}} \cdot \nu_2^{\mathbf{s}} \big) \, \nu^{\mathbf{s}}$, *where*

$$\Phi^{\mathbf{s}} = \begin{pmatrix} 0 & -\mathrm{Hess}_{\mathbf{s}} V \\ \mathrm{Id} & M_0(\mathbf{s}, 0, 0) \end{pmatrix}.$$

In particular, since $\Phi^{\mathbf{s}}$ *is invertible,* $\nu_2^{\mathbf{s}} \neq 0$. *Finally,*

$$\det \left(\mathrm{Hess}_{(\mathbf{s},0)} \left(W + \frac{(\ell_0^{\mathbf{s}})^2}{2} \right) \right) = 2^{-2d} \big| \det(\mathrm{Hess}_{\mathbf{s}} V) \big|.$$

Proof. The proof is the same as in [1], Lemma 3.3 after matching the notations by setting $\Lambda(\mathbf{s}) = \Phi^{\mathbf{s}}$, $b^0 = H_W$,

$$A^0(\mathbf{s}) = \begin{pmatrix} 0 & 0 \\ 0 & M_0(\mathbf{s}, 0, 0) \end{pmatrix} \quad \text{and} \quad B(\mathbf{s}) = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Hess}_{\mathbf{s}} V & 0 \end{pmatrix}.$$

In particular, it is by a Taylor expansion at $(\mathbf{s}, 0)$ in (3.12) that we get

$$\begin{pmatrix} x - \mathbf{s} \\ v \end{pmatrix} \cdot \left[\begin{pmatrix} 0 & -\mathrm{Hess}_{\mathbf{s}} V \\ \mathrm{Id} & 0 \end{pmatrix} \nu^{\mathbf{s}} + \begin{pmatrix} 0 \\ M_0(\mathbf{s}, 0, 0) \nu_2^{\mathbf{s}} \end{pmatrix} + M_0(\mathbf{s}, 0, 0) \nu_2^{\mathbf{s}} \cdot \nu_2^{\mathbf{s}} \, \nu^{\mathbf{s}} \right] = 0$$

from which we deduce that $\nu^{\mathbf{s}}$ is an eigenvector of $\Phi^{\mathbf{s}}$ associated with the eigenvalue $-M_0(\mathbf{s}, 0, 0) \nu_2^{\mathbf{s}} \cdot \nu_2^{\mathbf{s}}$.

4.2. Solving for $(\ell_j^{\mathbf{s}})_{j \geq 1}$

Once again we drop some exponents $\mathbf{s}$ for shortness. Now that ℓ_0 is given by Lemma 4.6 and Proposition 4.7, we can solve the transport equations (3.13) by induction, so we suppose that $\ell_0, \dots, \ell_{j-1}$ are given and we want to find a solution ℓ_j to (3.13). Denote

$$\widetilde{U} = H_W + \begin{pmatrix} 0 \\ M_0\Big(x, v, i\big(\frac{v}{2} + \ell_0 \, \partial_v \ell_0\big)\Big)(v + \ell_0 \partial_v \ell_0) + \ell_0 \, U \end{pmatrix} \in \mathcal{C}^\infty(B_0(\mathbf{s}, 2r))$$

and

$$\alpha = \partial_v \ell_0 \cdot U \in \mathcal{C}^\infty(B_0(\mathbf{s}, 2r))$$

where U was introduced in (3.10). The function ℓ_j must satisfy $(\widetilde{U} \cdot \nabla + \alpha)\ell_j = -R_j(\ell_0, \dots, \ell_{j-1})$ so we are interested in the operator $\mathcal{L} = \widetilde{U} \cdot \nabla + \alpha$ that we decompose as $\mathcal{L} = \mathcal{L}_0^{\mathbf{s}} + \mathcal{L}_>$ with

$$\mathcal{L}_0^{\mathbf{s}} = \widetilde{U}_0^{\mathbf{s}} \begin{pmatrix} x - \mathbf{s} \\ v \end{pmatrix} \cdot \nabla + \alpha_0^{\mathbf{s}}$$

where $\widetilde{U}_0^{\mathbf{s}}$ is the differential of $\widetilde{U}$ at $(\mathbf{s},0)$ and $\alpha_0^{\mathbf{s}}=\alpha(\mathbf{s},0)$, that is

$$\widetilde{U}_0^{\mathbf{s}}=\begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Hess}_{\mathbf{s}}V+2M_0(\mathbf{s},0,0)\nu_2^{\mathbf{s}}\,{}^t\nu_1^{\mathbf{s}} & M_0(\mathbf{s},0,0)(\mathrm{Id}+2\nu_2^{\mathbf{s}}\,{}^t\nu_2^{\mathbf{s}})\end{pmatrix}$$

and

$$\alpha_0^{\mathbf{s}}=M_0(\mathbf{s},0,0)\nu_2^{\mathbf{s}}\cdot\nu_2^{\mathbf{s}}. \tag{3.16}$$

As usual, we will often omit the exponents $\mathbf{s}$ in the notations. Notice that if we denote $\mathcal{P}^n_{hom}$ the space of homogeneous polynomials of degree n in the variables $(x-\mathbf{s},v)$, we have $\mathcal{L}_0\in\mathscr{L}(\mathcal{P}^n_{hom})$ and for $P\in\mathcal{P}^n_{hom}$, $\mathcal{L}_>P(x,v)=O\big((x-\mathbf{s},v)^{n+1}\big)$ near $(\mathbf{s},0)$.

Lemma 4.8. *The negative eigenvalue $-\alpha_0^{\mathbf{s}}$ of the matrix $\Phi^{\mathbf{s}}$ from Proposition 4.7 is its only one (counting multiplicity) in $\{\mathrm{Re}\,z\leq 0\}$. Moreover, all the eigenvalues of $\widetilde{U}_0^{\mathbf{s}}$ have positive real part and the operator $\mathcal{L}_0^{\mathbf{s}}$ is invertible on $\mathcal{P}^n_{hom}$.*

Proof. It is sufficient to prove the first statement. Indeed, if $-\alpha_0$ is the only eigenvalue of Φ in $\{\mathrm{Re}\,\mathrm{z}\leq 0\}$, we can then remark that

$${}^t\widetilde{U}_0=\Phi+2\begin{pmatrix}0 & \nu_1\,{}^t\nu_2M_0(\mathbf{s},0,0)\\ 0 & \nu_2\,{}^t\nu_2M_0(\mathbf{s},0,0)\end{pmatrix}$$

and since the last term has its range included in $\mathbb{C}\nu$ and sends ν on $2\alpha_0\nu$, the matrix of ${}^t\widetilde{U}_0$ in a basis $(\nu,b_2,\dots,b_{2d})$ in which Φ becomes triangular is also triangular and has on its diagonal the eigenvalues of Φ except for $-\alpha_0$ which is replaced by $+\alpha_0$. Hence, $\mathrm{Spec}(\widetilde{U}_0)=\mathrm{Spec}({}^t\widetilde{U}_0)\subset\{\mathrm{Re}\,z>0\}$ and we can conclude thanks to Lemma A.1 from [1]. Let us then prove that $-\alpha_0$ is the only eigenvalue (counting multiplicity) of Φ in $\{\mathrm{Re}\,\mathrm{z}\leq 0\}$. We proceed as in [1], Lemma 2.6. For $t\in[0,1]$, consider the matrix

$$\Phi_t=2\,\mathrm{Hess}_{\mathbf{s}}W\begin{pmatrix}(1-t)\mathrm{Id} & -t\mathrm{Id}\\ t\mathrm{Id} & tM_0(\mathbf{s},0,0)+(1-t)\mathrm{Id}\end{pmatrix}$$

which trivially satisfies the assumptions of Lemma B.1 for $t\in[0,1)$. It is also the case of Φ_1 as $\Phi_1(x,0)=(0,x)$. Hence for every $t\in[0,1]$, Φ_t has no eigenvalues in $i\mathbb{R}$ and since these eigenvalues depend continuously on t, we get that

$$\#\big(\mathrm{Spec}\,\Phi_1\cap\{\mathrm{Re}\,z<0\}\big)=\#\big(\mathrm{Spec}\,\Phi_0\cap\{\mathrm{Re}\,z<0\}\big).$$

But $\Phi_0=2\,\mathrm{Hess}_{\mathbf{s}}W$ has exactly one negative eigenvalue (with multiplicity), while all the others are positive since $\mathbf{s}\in\mathcal{U}^{(1)}$, so we have indeed showed that $-\alpha_0$ is the only eigenvalue of $\Phi=\Phi_1$ (counting multiplicity) in $\{\mathrm{Re}\,z\leq 0\}$.

One can then proceed as in [1], Section 3.3 (see also [4], chapter 3), i.e., use Lemma 4.8 to find an approximate solution of (3.13) using formal power series and then refine it into an actual solution using again Lemma 4.8 as well as the characteristic method. We then get the following result.

Proposition 4.9. *For all $j\geq 1$, there exists $\ell_j^{\mathbf{s}}\in\mathcal{C}^\infty(B_0(\mathbf{s},2r))$ solving (3.13). Moreover, $\ell_j^{\mathbf{s}}$ is real valued in view of Lemmas 4.2 and 4.3.*

5. Computation of the Small Eigenvalues

Now that we have found $(\ell_j)_{j\geq 0} \subset \mathcal{C}^\infty(B_0(\mathbf{s}, 2r), \mathbb{R})$ solving (3.12) and (3.13) with ℓ_0 vanishing at $(\mathbf{s}, 0)$, we can use a Borel procedure to construct $\ell \in \mathcal{C}^\infty(\mathbb{R}^{2d}, \mathbb{R})$ supported in $B_0(\mathbf{s}, 3r)$ and satisfying $\ell \sim \sum_{j\geq 0} \ell_j$ on $B_0(\mathbf{s}, 2r)$.

Remark 5.1. The properties a)-c) from (3.7) are satisfied by both the functions $\ell^{\mathbf{s},h}$ and $-\ell^{\mathbf{s},h}$. Moreover, by Lemma 4.3, $(-\ell_j^{\mathbf{s}})_{j\geq 0}$ also solve (3.12) and (3.13).

We are now in position to prove that all the properties from (3.7) are satisfied.

Proposition 5.2. *We can choose the signs of the functions $(\ell^{\mathbf{s},h})_{\mathbf{j}(\mathbf{m})}$ such that (3.7) holds true and the coefficients from the classical expansion of $\ell^{\mathbf{s},h}$ solve* (3.12) *and* (3.13)*.*

Proof. Recall that by item b) from Hypothesis 3.11, each function $\ell^{\mathbf{s},h}$ corresponds to a unique $\mathbf{m} \in \mathcal{U}^{(0)} \backslash \{\underline{\mathbf{m}}\}$. Thanks to Lemmas 4.5 and 4.6, it is clear that item d) from (3.7) is satisfied by both $\ell^{\mathbf{s},h}$ and $-\ell^{\mathbf{s},h}$. Hence according to Remark 5.1, it is sufficient to prove that the signs of $(\ell^{\mathbf{s},h})_{\mathbf{j}(\mathbf{m})}$ can be chosen so that $\theta_{\mathbf{m},h}$ is smooth on a neighborhood of supp $\chi_{\mathbf{m}}$. From (3.9), (3.10) and (3.11) we see that the only parts on which it is not clear that $\theta_{\mathbf{m},h}$ is smooth are

$$F_1 = \bigsqcup_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} \Big(\{|\ell_0^{\mathbf{s}}| \leq 2\gamma\} \cap \partial B_0(\mathbf{s}, r)\Big), \quad F_2 = \bigsqcup_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} \Big(B_0(\mathbf{s}, r) \cap \{|\ell_0^{\mathbf{s}}| = 2\gamma\}\Big)$$

$$\text{and} \qquad F_3 = \partial\Big(E(\mathbf{m}) + B(0, \varepsilon)\Big) \backslash \Big(\bigsqcup_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} \big(B_0(\mathbf{s}, r) \cap \{|\ell_0^{\mathbf{s}}| \leq 2\gamma\}\big)\Big).$$

Here, the unions are disjoint for r small enough. Let $\mathbf{s} \in \mathbf{j}(\mathbf{m})$ and $(x, v) \in \overline{B_0(\mathbf{s}, r)} \backslash \{(\mathbf{s}, 0)\}$ such that $\ell_0^{\mathbf{s}}(x, v) = 0$. Using Lemma 4.6, we see that if $r > 0$ is small enough,

$$W(x, v) - W(\mathbf{s}, 0) = \phi_+(x, v) > 0 \tag{3.1}$$

because $(\mathbf{s}, 0)$ is a non-degenerate local minimum of ϕ_+. Hence, $\{\ell_0^{\mathbf{s}} = 0\} \cap B_0(\mathbf{s}, r) \subset \{W \geq \boldsymbol{\sigma}(\mathbf{m})\}$. Now, assume by contradiction that for any $r > 0$, the function $\ell_0^{\mathbf{s}}$ takes both positive and negative values on $E(\mathbf{m}) \cap B_0(\mathbf{s}, r)$. Then according to Lemma 3.1, the two CCs of $U_r \cap \{W < \boldsymbol{\sigma}(\mathbf{m})\}$ are both included in $E(\mathbf{m})$ (the one on which $\ell_0^{\mathbf{s}} > 0$ and the one where $\ell_0^{\mathbf{s}} < 0$). This is a contradiction with the fact that $\mathbf{s} \in \mathcal{V}^{(1)}$. Therefore, $\ell_0^{\mathbf{s}}$ has a sign on $E(\mathbf{m}) \cap B_0(\mathbf{s}, r)$ and we can choose it so that $\ell_0^{\mathbf{s}}$ is a positive function on $E(\mathbf{m}) \cap B_0(\mathbf{s}, r)$. By uniform continuity, we can then choose $\varepsilon(\gamma) > 0$ small enough so that

$$\Big(\big(E(\mathbf{m}) + B(0, \varepsilon)\big) \cap B_0(\mathbf{s}, r)\Big) \subseteq \{\ell_0^{\mathbf{s}} \geq -\gamma\}. \tag{3.2}$$

Similarly, if we denote $\Omega_{\mathbf{s}}$ the other CC of $\{W < \boldsymbol{\sigma}(\mathbf{m})\}$ which contains $(\mathbf{s}, 0)$ on its boundary, we have since $(\mathbf{s}, 0)$ is not a critical point of $\ell_0^{\mathbf{s}}$ that this function is negative on $\Omega_{\mathbf{s}} \cap B_0(\mathbf{s}, r)$ and

$$\Big(\big(\Omega_{\mathbf{s}} + B(0, \varepsilon)\big) \cap B_0(\mathbf{s}, r)\Big) \subseteq \{\ell_0^{\mathbf{s}} \leq \gamma\}. \tag{3.3}$$

Choosing once again $\varepsilon(r)$ small enough, we can even assume that

$$\Big(\overline{E(\mathbf{m})+B(0,\varepsilon)}\cap\overline{\Omega_{\mathbf{s}}+B(0,\varepsilon)}\Big)\subseteq B_0(\mathbf{s},r). \tag{3.4}$$

We first prove that F_1 does not meet the support of $\chi_{\mathbf{m}}$. Recall that Ω denotes the CC of $\{W\leq\boldsymbol{\sigma}(\mathbf{m})\}$ containing $\mathbf{m}$. For $\mathbf{s}\in\mathbf{j}(\mathbf{m})$, we can deduce from (3.1) that if $(x,v)\in\partial B_0(\mathbf{s},r)$ such that $\ell_0^{\mathbf{s}}(x,v)=0$, then $(x,v)\notin\Omega$. Hence, $|\ell_0^{\mathbf{s}}|$ must attain a positive minimum on $\partial B_0(\mathbf{s},r)\cap\Omega$, so we can choose $\gamma(r)>0$ such that $\partial B_0(\mathbf{s},r)\cap\{|\ell_0^{\mathbf{s}}|\leq 2\gamma\}$ does not intersect Ω. It follows that we can choose $\varepsilon(\gamma)>0$ such that

$$F_1\subseteq\big(\mathbb{R}^{2d}\backslash\overline{\Omega+B(0,\varepsilon)}\big)\subseteq\big(\mathbb{R}^{2d}\backslash\operatorname{supp}\chi_{\mathbf{m}}\big).$$

Now, we show that $\theta_{\mathbf{m},h}$ is smooth on $F_2\cap(\Omega+B(0,\varepsilon))$: Let $\mathbf{s}\in\mathbf{j}(\mathbf{m})$ and $(x,v)\in B_0(\mathbf{s},r)\cap\{\ell_0^{\mathbf{s}}=2\gamma\}\cap(\Omega+B(0,\varepsilon))$. According to (3.3) and the fact that $\ell^{\mathbf{s},h}=\ell_0^{\mathbf{s}}+O(h)$, there exists a small ball B centered in (x,v) such that

$$B\subset\Big(B_0(\mathbf{s},r)\cap\{\ell^{\mathbf{s},h}>\gamma\}\cap\big(E(\mathbf{m})+B(0,\varepsilon)\big)\Big).$$

Thus, $\theta_{\mathbf{m},h}=1$ on B and $\theta_{\mathbf{m},h}$ is smooth at (x,v). Similarly, for $(x,v)\in B_0(\mathbf{s},r)\cap\{\ell^{\mathbf{s},h}=-2\gamma\}\cap(\Omega+B(0,\varepsilon))$, we can show that $\theta_{\mathbf{m},h}=0$ in a neighborhood of (x,v).

It only remains to prove that, as for F_1, the set F_3 does not meet the support of $\chi_{\mathbf{m}}$. First we remark that thanks to (3.2), we can forget the absolute value in the definition of F_3:

$$F_3=\partial\Big(E(\mathbf{m})+B(0,\varepsilon)\Big)\backslash\Big(\bigsqcup_{\mathbf{j}(\mathbf{m})}\big(B_0(\mathbf{s},r)\cap\{\ell_0^{\mathbf{s}}\leq 2\gamma\}\big)\Big).$$

If $(x,v)\in F_3\cap B_0(\mathbf{s},r)$, we have that $\ell_0^{\mathbf{s}}(x,v)>2\gamma$ so using (3.3), we see that (x,v) is outside $\Omega_{\mathbf{s}}+B(0,\varepsilon)$. Since it is not in $(E(\mathbf{m})+B(0,\varepsilon))$ either, it is outside $\Omega+B(0,\varepsilon)$ which contains the support of $\chi_{\mathbf{m}}$. Now, if $(x,v)\in F_3\backslash\big(\mathbf{j}^W(\mathbf{m})+B_0(0,r)\big)$, (3.4) implies that (x,v) is outside $\cup_{\mathbf{j}(\mathbf{m})}(\Omega_{\mathbf{s}}+B(0,\varepsilon))$ so it is also outside $\Omega+B(0,\varepsilon)$ for ε small enough and the proof is complete.

Lemma 5.3. *Let* $\mathbf{m}\in\mathcal{U}^{(0)}\backslash\{\underline{\mathbf{m}}\}$ *and denote* $\tilde f_{\mathbf{m},h}=f_{\mathbf{m},h}/\|f_{\mathbf{m},h}\|$ *where* $f_{\mathbf{m},h}$ *was defined in* (3.13)*. With the notation* (3.16)*, we have that*

$$\langle P_h\tilde f_{\mathbf{m},h},\tilde f_{\mathbf{m},h}\rangle=h\mathrm{e}^{-2\frac{S(\mathbf{m})}{h}}\frac{\det(\mathrm{Hess}_{\mathbf{m}}V)^{1/2}}{2\pi}\tilde B_h(\mathbf{m})\in\mathbb{R}$$

with $\tilde B_h(\mathbf{m})$ *admitting a classical expansion whose first term equals*

$$\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}|\det(\mathrm{Hess}_{\mathbf{s}}V)|^{-1/2}\,\alpha_0^{\mathbf{s}}.$$

Proof. Since X_0^h is a skew-adjoint differential operator and $f_{\mathbf{m},h}$ is real valued, we have

$$\langle X_0^hf_{\mathbf{m},h},f_{\mathbf{m},h}\rangle=0.$$

Besides, we know from (3.18) that

$$b_hf_{\mathbf{m},h}=h(\partial_v\theta)\chi\mathrm{e}^{-W_{\mathbf{m}}/h}+O_{L^2}(h^\infty\mathrm{e}^{-S(\mathbf{m})/h}) \tag{3.5}$$

so we easily deduce from the fact that $(\partial_v\theta)\chi e^{-W_{\mathbf{m}}/h} = O_{L^2}(e^{-S(\mathbf{m})/h})$ and the boundedness of $\mathrm{Op}_h(M^h)$ that

$$\langle Q_h f_{\mathbf{m},h}, f_{\mathbf{m},h}\rangle = h^2\Big\langle \mathrm{Op}_h(M^h)\big((\partial_v\theta)\chi e^{-W_{\mathbf{m}}/h}\big), (\partial_v\theta)\chi e^{-W_{\mathbf{m}}/h}\Big\rangle + O\Big(h^\infty e^{-\frac{2S(\mathbf{m})}{h}}\Big).$$

Since we have with the notation (3.15)

$$(\partial_v\theta)\chi e^{-W_{\mathbf{m}}/h} = \frac{A_h^{-1}}{2} e^{-\widetilde{W}_{\mathbf{m}}/h}\chi \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} \zeta(\ell^{\mathbf{s}})\partial_v\ell^{\mathbf{s}}\,\mathbf{1}_{B_0(\mathbf{s},r)}$$

and using (3.23) with M instead of g, we get that

$$\langle P_h f_{\mathbf{m},h}, f_{\mathbf{m},h}\rangle = \frac{h^2}{4}A_h^{-2}\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\int_{B_0(\mathbf{s},r)} e^{-2\widetilde{W}_{\mathbf{m}}(x,v)/h}\chi\zeta(\ell^{\mathbf{s}})\tilde{I}^{\mathbf{s}}(x,v)\cdot\partial_v\ell^{\mathbf{s}}\,\mathrm{d}(x,v) + O\Big(h^\infty e^{-2\frac{S(\mathbf{m})}{h}}\Big). \tag{3.6}$$

where

$$\tilde{I}^{\mathbf{s}}(x,v) = (2\pi h)^{-d}\int_{\mathbb{R}^d}\int_{|v'|\le r} e^{\frac{i}{h}\eta\cdot(v-v')}\chi(x,v')\zeta\big(\ell^{\mathbf{s}}(x,v')\big) M\Big(x,\frac{v+v'}{2},\eta+i\psi(x,v,v')\Big)\partial_v\ell^{\mathbf{s}}(x,v')\,\mathrm{d}v'\mathrm{d}\eta.$$

Mimicking the proof of Proposition C.5, one can show that $\zeta(\ell)$ admits a classical expansion whose first term is $\zeta(\ell_0)$. Besides, since M and ψ also have a classical expansion, we could use the stationary phase (see, for instance, [21], Theorem 3.17) as well Proposition C.3 to get an expansion of $\tilde{I}$ similar to the one obtained in (3.9). Thus, we get that $\tilde{I}\cdot\partial_v\ell \sim \sum_{k\ge0} h^k a_k$ where

$$a_0(x,v) = \chi(x,v)\zeta(\ell_0(x,v))\,M_0\left(x,v,i\left(\frac{v}{2}+\ell_0\,\partial_v\ell_0\right)\right)\partial_v\ell_0(x,v)\cdot\partial_v\ell_0(x,v).$$

Hence, using the fact that on $B_0(\mathbf{s},r)$,

$$\widetilde{W} - S(\mathbf{m}) = W_{\mathbf{m}} + \frac{\ell_0^2}{2} - S(\mathbf{m}) + \Big(\frac{\ell^2}{2}-\frac{\ell_0^2}{2}\Big),$$

it is clear that

$$e^{2S(\mathbf{m})/h}\int_{B_0(\mathbf{s},r)} e^{-2\widetilde{W}(x,v)/h}\chi\zeta(\ell)\tilde{I}(x,v)\cdot\partial_v\ell\,\mathrm{d}(x,v) \sim_h \sum_{k\ge0}h^k\int_{B_0(\mathbf{s},r)} e^{-2\frac{W_{\mathbf{m}}(x,v)+\ell_0^2(x,v)/2-S(\mathbf{m})}{h}} e^{-\frac{(\ell^2-\ell_0^2)(x,v)}{h}}\chi\zeta(\ell)a_k\,\mathrm{d}(x,v). \tag{3.7}$$

We would like to apply Proposition C.7 so we need to check that the assumptions are satisfied. First, $\mathrm{Hess}_{(\mathbf{s},0)}(W_{\mathbf{m}}+\ell_0^2/2)$ is definite positive by Lemma 4.5. Besides, $h^{-1}(\ell^2-\ell_0^2)$ admits a classical expansion whose first term is $2(\ell_1\ell_0)$. Therefore, using the expansion of $\zeta(\ell)$ as well as Proposition C.5, one easily gets that the function

$$e^{-\frac{(\ell^2-\ell_0^2)}{h}}(\zeta\circ\ell)$$

admits a classical expansion whose first term is $e^{-2(\ell_1\ell_0)}(\zeta\circ\ell_0)$. Thus, according to Propositions C.7 and 4.7, there exists $(b_{k,j})$ such that

$$\frac{|\det(\mathrm{Hess}_{\mathbf{s}}V)|^{1/2}}{(2\pi h)^d}\int_{B_0(\mathbf{s},r)} e^{-2\frac{W_{\mathbf{m}}(x,v)+\ell_0^2(x,v)/2-S(\mathbf{m})}{h}}e^{-\frac{(\ell^2-\ell_0^2)(x,v)}{h}}\chi\zeta(\ell)a_k\,\mathrm{d}(x,v)$$
$$\sim\sum_{j\geq 0}h^j b_{k,j}$$

where $b_{k,0}=a_k(\mathbf{s},0)$. Hence, using (3.6), (3.7) and Proposition C.3, we deduce that

$$4A_h^2(2\pi)^{-d}h^{-d-2}e^{2S(\mathbf{m})/h}\langle P_h f_{\mathbf{m},h},f_{\mathbf{m},h}\rangle\sim\sum_{k\geq 0}h^k c_k \tag{3.8}$$

with

$$c_0=\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}|\det(\mathrm{Hess}_{\mathbf{s}}V)|^{-1/2}M_0(\mathbf{s},0,0)\nu_2^{\mathbf{s}}\cdot\nu_2^{\mathbf{s}}=\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}|\det(\mathrm{Hess}_{\mathbf{s}}V)|^{-1/2}\,\alpha_0^{\mathbf{s}}.$$

Similarly, thanks to item (a) from Hypothesis 3.11, one can use Proposition C.7 as we already did to see that there exists $(\tilde{c}_k)_{k\geq 0}$ such that

$$\frac{\det(\mathrm{Hess}_{\mathbf{m}}\mathrm{V})^{1/2}}{(2\pi h)^d}\|f_{\mathbf{m},h}\|^2\sim\sum_{k\geq 0}h^k\tilde{c}_k \tag{3.9}$$

with $\tilde{c}_0=1$. The conclusion follows from (3.8), (3.8) and (3.9).

Lemma 5.4. *Let $\mathbf{m}\in\mathcal{U}^{(0)}\backslash\{\underline{\mathbf{m}}\}$. Using the notations from Lemma 5.3, we have*

(i) $\|P_h\tilde{f}_{\mathbf{m},h}\|^2=O(h^\infty\langle P_h\tilde{f}_{\mathbf{m},h},\tilde{f}_{\mathbf{m},h}\rangle)$
(ii) $\|P_h^*\tilde{f}_{\mathbf{m},h}\|^2=O(h\langle P_h\tilde{f}_{\mathbf{m},h},\tilde{f}_{\mathbf{m},h}\rangle)$.

Proof. To prove i), first remark that thanks to (3.17)–(3.20) we have

$$\int_{\mathbb{R}^{2d}\backslash(\mathbf{j}^W(\mathbf{m})+B_0(0,2r))}|P_h f_{\mathbf{m},h}(x,v)|^2\mathrm{d}(x,v)=O\Big(h^\infty e^{-2\frac{S(\mathbf{m})}{h}}\Big). \tag{3.10}$$

Besides, we saw that thanks to Proposition C.7 and Lemma 4.5, we have for $\mathbf{s}\in\mathbf{j}(\mathbf{m})$,

$$\int_{B_0(\mathbf{s},2r)}e^{-2\frac{\widetilde{W}(x,v)}{h}}\mathrm{d}(x,v)=O\Big(h^d e^{-2\frac{S(\mathbf{m})}{h}}\Big).$$

Moreover, the function ω from Proposition 3.13 is $O_{L^\infty(B_0(\mathbf{s},2r))}(h^\infty)$ by Lemma 4.1 and the construction of the $(\ell^{\mathbf{s},h})_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}$. Hence, by Proposition 3.13,

$$\int_{B_0(\mathbf{s},2r)}|P_h f_{\mathbf{m},h}(x,v)|^2\mathrm{d}(x,v)=O\Big(h^\infty e^{-2\frac{S(\mathbf{m})}{h}}\Big). \tag{3.11}$$

The conclusion follows from (3.10), (3.11) as well as (3.9) and Lemma 5.3. The proof of ii) can be obtained similarly with the use of Proposition C.7 and Remark 3.14 after noticing that $\overset{*}{\omega}$ also admits a classical expansion whose first term vanishes on $\mathbf{j}^W(\mathbf{m})$.

From now on, we denote

$$\tilde{\lambda}_{\mathbf{m},h} = \langle P_h \tilde{f}_{\mathbf{m},h}, \tilde{f}_{\mathbf{m},h}\rangle = \langle Q_h \tilde{f}_{\mathbf{m},h}, \tilde{f}_{\mathbf{m},h}\rangle \tag{3.12}$$

for which we computed a classical expansion in Lemma 5.3.

Lemma 5.5. *For* $\mathbf{m}$ *and* $\mathbf{m}'$ *two distinct elements of* $\mathcal{U}^{(0)}$*, we have*

(i) $\langle P_h \tilde{f}_{\mathbf{m},h}, \tilde{f}_{\mathbf{m}',h}\rangle = O\Big(h^\infty \sqrt{\tilde{\lambda}_{\mathbf{m},h}\tilde{\lambda}_{\mathbf{m}',h}}\Big)$

(ii) *There exists* $c > 0$ *such that* $\langle \tilde{f}_{\mathbf{m},h}, \tilde{f}_{\mathbf{m}',h}\rangle = O(\mathrm{e}^{-c/h})$

Proof. (i): The result is obvious when one of the two minima is $\underline{\mathbf{m}}$. Recall the labeling of the minima that we introduced right before Hypothesis 3.11 as well as the map π_x from Lemma 3.8. Let us first suppose that $\mathbf{m} = \mathbf{m}_{k,j}$ and $\mathbf{m}' = \mathbf{m}_{k,j'}$ with $j \neq j'$ and $k \neq 1$ and denote $E = E(\mathbf{m})$ and $E' = E(\mathbf{m}')$. In particular $\boldsymbol{\sigma}(\mathbf{m}) = \boldsymbol{\sigma}(\mathbf{m}')$. Thanks to (3.14) and the fact that P_h is local in x, we have

$$\operatorname{supp} P_h \tilde{f}_{\mathbf{m},h} \subseteq \big(\pi_x(E) \times \mathbb{R}^d_v\big) + B(0,\varepsilon') \qquad \text{and} \qquad \operatorname{supp} \tilde{f}_{\mathbf{m}',h} \subseteq \big(E' + B(0,\varepsilon')\big)$$

so up to taking ε' small enough, it is sufficient to show that $\overline{\pi_x(E) \times \mathbb{R}^d_v}$ and $\overline{E'}$ do not intersect. Since our labeling is adapted, E and E' are two distinct CCs of $\{W < \boldsymbol{\sigma}(\mathbf{m})\}$ so by Lemma 3.8, $\pi_x(E) \times \mathbb{R}^d_v$ and E' are two disjoint open sets. Thus, using successively Remark 3.7 and (3.4), we get

$$\begin{aligned}\overline{\pi_x(E) \times \mathbb{R}^d_v} \cap \overline{E'} &= \Big(\partial\big(\pi_x(E)\big) \times \mathbb{R}^d_v\Big) \cap \partial E' \\ &\subseteq \Big(\partial\big(\pi_x(E)\big) \times \{0\}\Big) \cap \partial E' \\ &\subseteq \Big(\partial\big(\pi_x(E)\big) \cap \partial\big(\pi_x(E')\big)\Big) \times \{0\}.\end{aligned}$$

which is empty thanks to Lemma 3.2 and item b) from Hypothesis 3.11.

Let us now treat the case $\mathbf{m} = \mathbf{m}_{k,j}$ and $\mathbf{m}' = \mathbf{m}_{k',j'}$ with $k, k' \geq 2$ and $k \neq k'$. We can suppose that $k < k'$ (i.e., $\boldsymbol{\sigma}(\mathbf{m}) > \boldsymbol{\sigma}(\mathbf{m}')$) because we can work with P_h^* instead of P_h if needed. We decompose $P_h \tilde{f}_{\mathbf{m},h}$ as in (3.17) and (3.18) and once again we use (3.14) to get

$$\operatorname{supp} \tilde{f}_{\mathbf{m}',h} \subseteq \big(E' + B(0,\varepsilon')\big) \subseteq \Big\{W < \frac{\boldsymbol{\sigma}(\mathbf{m}) + \boldsymbol{\sigma}(\mathbf{m}')}{2}\Big\}$$

as well as the fact that P_h is local in x to get a localization of the support of the first term from (3.18):

$$\begin{aligned}&\operatorname{supp}\Big(\mathrm{Op}_h(g)\big((\partial_v\theta_{\mathbf{m}})\chi_{\mathbf{m}}\mathrm{e}^{-W_{\mathbf{m}}/h}\big)\Big) \subseteq \Big(\big(\mathbf{j}(\mathbf{m}) + B(0,r)\big) \times \mathbb{R}^d_v\Big) \\ &\subseteq \Big\{W > \frac{\boldsymbol{\sigma}(\mathbf{m}) + \boldsymbol{\sigma}(\mathbf{m}')}{2}\Big\}\end{aligned}$$

as W increases with the norm of v. Hence, the support of the first term from (3.18) does not meet the one of $\tilde{f}_{\mathbf{m}',h}$. The same goes easily for the first term of (3.17). For the second term of (3.17), its support is contained in the support of $\nabla\chi_{\mathbf{m}}$ which is itself contained in $\{W \geq \boldsymbol{\sigma}(\mathbf{m}) + \tilde{\varepsilon}\}$ so it clearly does not meet the support of $\tilde{f}_{\mathbf{m}',h}$. It only remains to treat the second term from (3.18), i.e., $\mathrm{Op}_h(g)\big(\theta_{\mathbf{m}}(\partial_v\chi_{\mathbf{m}})\mathrm{e}^{-W_{\mathbf{m}}/h}\big)$. To this aim, notice that (3.5) yields

$b_h f_{\mathbf{m}',h} = O_{L^2}(\mathrm{e}^{-S(\mathbf{m}')/h})$ and since by the support properties of $\nabla\chi_{\mathbf{m}}$ we also have $\theta_{\mathbf{m}}(\partial_v\chi_{\mathbf{m}})\mathrm{e}^{-W_{\mathbf{m}}/h} = O_{L^2}(h^\infty \mathrm{e}^{-S(\mathbf{m})/h})$, we get using the Cauchy–Schwarz inequality and the boundedness of $\mathrm{Op}_h(M)$

$$\begin{aligned}\Big\langle \mathrm{Op}_h(g)\big(\theta_{\mathbf{m}}(\partial_v\chi_{\mathbf{m}})\mathrm{e}^{-W_{\mathbf{m}}/h}\big), f_{\mathbf{m}',h}\Big\rangle &= \Big\langle \mathrm{Op}_h(M)\big(\theta_{\mathbf{m}}(\partial_v\chi_{\mathbf{m}})\mathrm{e}^{-W_{\mathbf{m}}/h}\big), b_h f_{\mathbf{m}',h}\Big\rangle \\ &= O\Big(h^\infty \mathrm{e}^{-\frac{S(\mathbf{m})+S(\mathbf{m}')}{h}}\Big)\end{aligned}$$

which proves the first item.

(ii): Here, we can suppose that $V(\mathbf{m}) \geq V(\mathbf{m}')$. Let us first treat the case where $V(\mathbf{m}) = V(\mathbf{m}')$. Then according to item a) from Hypothesis 3.11, E and E' are two disjoint open sets. Hence, as we saw earlier, Lemma 3.2 and item b) from Hypothesis 3.11 imply that $\overline{E}\cap\overline{E'} = \emptyset$. The conclusion then follows from (3.14).

If $V(\mathbf{m}) > V(\mathbf{m}')$, then item a) from Hypothesis 3.11 implies that $(\mathbf{m}, 0)$ is the only global minimum of $W|_{E+B(0,\varepsilon')}$. Therefore using (3.14), we can easily compute

$$\begin{aligned}\langle f_{\mathbf{m},h}, f_{\mathbf{m}',h}\rangle &= \int_{E+B(0,\varepsilon')} \theta_{\mathbf{m}}\theta_{\mathbf{m}'}\chi_{\mathbf{m}}\chi_{\mathbf{m}'}\mathrm{e}^{-\frac{2V-V(\mathbf{m})-V(\mathbf{m}')+v^2}{2h}}\mathrm{d}(x,v) \\ &= O\Big(\mathrm{e}^{-\frac{V(\mathbf{m})-V(\mathbf{m}')}{2h}}\Big).\end{aligned}$$

The conclusion immediately follows from (3.9).

Let us consider once again the spectral projection introduced in (2.5). We saw in particular that $\Pi_0 = O(1)$.

Lemma 5.6. *For any $\mathbf{m}\in\mathcal{U}^{(0)}$, we have*

$$\|(1-\Pi_0)\tilde{f}_{\mathbf{m},h}\| = O\Big(h^\infty\sqrt{\tilde{\lambda}_{\mathbf{m},h}}\Big) \qquad \text{and} \qquad \|(1-\Pi_0^*)\tilde{f}_{\mathbf{m},h}\| = O\Big(h^{-3/2}\sqrt{\tilde{\lambda}_{\mathbf{m},h}}\Big).$$

Proof. We simply recall the proof from [12]: We write

$$\begin{aligned}(1-\Pi_0)\tilde{f}_{\mathbf{m},h} &= \frac{1}{2i\pi}\int_{|z|=ch^2}\big(z^{-1}-(z-P_h)^{-1}\big)\tilde{f}_{\mathbf{m},h}\mathrm{d}z \\ &= \frac{-1}{2i\pi}\int_{|z|=ch^2} z^{-1}(z-P_h)^{-1}P_h\tilde{f}_{\mathbf{m},h}\mathrm{d}z.\end{aligned}$$

We can then conclude using Lemma 5.4 and the resolvent estimate from Theorem 1.6. The proof for the adjoint is almost identical.

Lemma 5.7. *The family $(\Pi_0\tilde{f}_{\mathbf{m},h})_{\mathbf{m}\in\mathcal{U}^{(0)}}$ is almost orthonormal: There exists $c>0$ such that*

$$\langle \Pi_0\tilde{f}_{\mathbf{m},h}, \Pi_0\tilde{f}_{\mathbf{m}',h}\rangle = \delta_{\mathbf{m},\mathbf{m}'} + O(\mathrm{e}^{-c/h}).$$

In particular, it is a basis of the space $H = \operatorname{Ran}\Pi_0$ introduced in (2.5).

Moreover, we have

$$\langle P_h\Pi_0\tilde{f}_{\mathbf{m},h}, \Pi_0\tilde{f}_{\mathbf{m}',h}\rangle = \delta_{\mathbf{m},\mathbf{m}'}\tilde{\lambda}_{\mathbf{m},h} + O\Big(h^\infty\sqrt{\tilde{\lambda}_{\mathbf{m},h}\tilde{\lambda}_{\mathbf{m}',h}}\Big).$$

Proof. The proof is the same as the one of Proposition 4.10 in [12].

Let us re-label the local minima $\mathbf{m}_1, \ldots, \mathbf{m}_{n_0}$ so that $(S(\mathbf{m}_j))_{j=1,\ldots,n_0}$ is non-increasing in j. For shortness, we will now denote

$$\tilde{f}_j = \tilde{f}_{\mathbf{m}_j,h} \qquad \text{and} \qquad \tilde{\lambda}_j = \tilde{\lambda}_{\mathbf{m}_j,h}$$

which still depend on h. Note in particular that according to Lemma 5.3, $\tilde{\lambda}_j = O(\tilde{\lambda}_k)$ whenever $1 \leq j \leq k \leq n_0$. We also denote $(\tilde{u}_j)_{j=1,\ldots,n_0}$ the orthogonalization by the Gram–Schmidt procedure of the family $(\Pi_0 \tilde{f}_j)_{j=1,\ldots,n_0}$ and

$$u_j = \frac{\tilde{u}_j}{\|\tilde{u}_j\|}.$$

In this setting and with our previous results, we get the following (see [12], Proposition 4.12 for a proof).

Lemma 5.8. *For all $1 \leq j, k \leq n_0$, it holds*

$$\langle P_h u_j, u_k \rangle = \delta_{j,k} \tilde{\lambda}_j + O\Big(h^\infty \sqrt{\tilde{\lambda}_j \tilde{\lambda}_k}\Big).$$

In order to compute the small eigenvalues of P_h, let us now consider the restriction $P_h|_H : H \to H$. We denote $\hat{u}_j = u_{n_0-j+1}$, $\hat{\lambda}_j = \tilde{\lambda}_{n_0-j+1}$ and $\mathcal{M}$ the matrix of $P_h|_H$ in the orthonormal basis $(\hat{u}_1, \ldots, \hat{u}_{n_0})$. Since $\hat{u}_{n_0} = u_1 = \tilde{f}_1$, we have

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}' & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{where} \qquad \mathcal{M}' = \Big(\langle P_h \hat{u}_j, \hat{u}_k \rangle\Big)_{1 \leq j,k \leq n_0-1}$$

and it is sufficient to study the spectrum of $\mathcal{M}'$. We will also denote $\{\hat{S}_1 < \cdots < \hat{S}_p\}$ the set $\{S(\mathbf{m}_j)\,;\, 2 \leq j \leq n_0\}$ and for $1 \leq k \leq p$, E_k the subspace of $L^2(\mathbb{R}^{2d})$ generated by $\{\hat{u}_r\,;\, S(\mathbf{m}_r) = \hat{S}_k\}$. Finally, we set $\varpi_k = \mathrm{e}^{-(\hat{S}_k - \hat{S}_{k-1})/h}$ for $2 \leq k \leq p$ and $\varepsilon_j(\varpi) = \prod_{k=2}^{j} \varpi_k = \mathrm{e}^{-(\hat{S}_j - \hat{S}_1)/h}$ for $2 \leq j \leq p$ (with the convention $\varepsilon_1(\varpi) = 1$).

Proposition 5.9. *There exists a diagonal matrix $M_h^\#$ admitting a classical expansion whose first term is*

$$M_0^\# = \mathrm{diag}\left(\sum_{\mathbf{s} \in \mathbf{j}(\mathbf{m}_{n_0-j+1})} \frac{\det(\mathrm{Hess}_{\mathbf{m}_{n_0-j+1}} V)^{1/2}}{2\pi |\det(\mathrm{Hess}_{\mathbf{s}} V)|^{1/2}} \alpha_0^{\mathbf{s}}\,;\, 1 \leq j \leq n_0 - 1 \right)$$

such that

$$h^{-1} \mathrm{e}^{2\hat{S}_1/h} \mathcal{M}' = \Omega(\varpi)\big(M_h^\# + O(h^\infty)\big)\Omega(\varpi)$$

where $\Omega(\varpi) = \mathrm{diag}\big(\varepsilon_1(\varpi)\mathrm{Id}_{E_1}, \ldots, \varepsilon_p(\varpi)\mathrm{Id}_{E_p}\big)$.

Remark 5.10. In the words of Definition 6.7 from [1], the last Proposition implies that $h^{-1}\mathrm{e}^{2\hat{S}_1/h}\mathcal{M}'$ is a classical graded symmetric matrix.

Proof. According to Lemma 5.8, we can decompose $\mathcal{M}' = \mathcal{M}'_1 + \mathcal{M}'_2$ with

$$\mathcal{M}'_1 = \mathrm{diag}(\hat{\lambda}_j\,;\, 1 \leq j \leq n_0 - 1) \qquad \text{and} \qquad \mathcal{M}'_2 = \Big(O\Big(h^\infty \sqrt{\hat{\lambda}_j \hat{\lambda}_k}\Big)\Big)_{1 \leq j,k \leq n_0-1}.$$

We will take $M_h^{\#} = h^{-1}\mathrm{e}^{2\hat{S}_1/h}\Omega(\varpi)^{-1}\mathcal{M}_1'\Omega(\varpi)^{-1}$ which is clearly diagonal, so we just need to check that it has the proper classical expansion and that $h^{-1}\mathrm{e}^{2\hat{S}_1/h}\Omega(\varpi)^{-1}\mathcal{M}_2'\Omega(\varpi)^{-1} = O(h^\infty)$. It is easy to compute

$$h^{-1}\mathrm{e}^{2\hat{S}_1/h}\Omega(\varpi)^{-1}\mathcal{M}_1'\Omega(\varpi)^{-1} = h^{-1}\mathrm{diag}\Big(\mathrm{e}^{2\hat{S}_{j'}/h}\hat{\lambda}_j\,;\,1 \le j \le n_0 - 1\Big)$$

where $1 \le j' \le p$ is such that $\hat{S}_{j'} = S(\mathbf{m}_{n_0-j+1})$. Hence, Lemma 5.3 yields

$$h^{-1}\mathrm{e}^{2\hat{S}_1/h}\Omega(\varpi)^{-1}\mathcal{M}_1'\Omega(\varpi)^{-1} = \mathrm{diag}\left(\frac{\det(\mathrm{Hess}_{\mathbf{m}_{n_0-j+1}}V)^{1/2}}{2\pi}\tilde{B}_h(\mathbf{m}_{n_0-j+1})\,;\,1 \le j \le n_0 - 1\right)$$

where $\tilde{B}_h(\mathbf{m}_{n_0-j+1})$ was introduced in Lemma 5.3 and admits a classical expansion whose first term is

$$\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m}_{n_0-j+1})} |\det(\mathrm{Hess}_{\mathbf{s}}V)|^{-1/2}\,\alpha_0^{\mathbf{s}}$$

so $M_h^{\#}$ has the desired expansion. Similarly, still using Lemma 5.3, one easily gets

$$\Omega(\varpi)^{-1}\mathcal{M}_2'\Omega(\varpi)^{-1} = \Big(O\Big(h^\infty\sqrt{\hat{\lambda}_j\hat{\lambda}_k}\,\varepsilon_{j'}(\varpi)^{-1}\varepsilon_{k'}(\varpi)^{-1}\Big)\Big)_{1\le j,k\le n_0-1}$$

where $1 \le j' \le p$ and $1 \le k' \le p$ are such that $\sqrt{\hat{\lambda}_j}\,\varepsilon_{j'}(\varpi)^{-1}$ and $\sqrt{\hat{\lambda}_k}\,\varepsilon_{k'}(\varpi)^{-1}$ are both $O(\sqrt{h}\,\mathrm{e}^{-\hat{S}_1/h})$ so the proof is complete.

Proof of Theorem 1.8. According to Remark 5.10, it now suffices to combine the result of Proposition 5.9 with Theorem 4 from [1] which gives a description of the spectrum of classical graded almost symmetric matrices. Indeed, using the notations from this reference, we have for $1 \le j \le p$ that

$$\mathcal{J}\circ\mathcal{R}_j\Big(M_h^{\#} + O(h^\infty)\Big) = \mathcal{J}\circ\mathcal{R}_j\big(M_h^{\#}\big) + O(h^\infty)$$

and the result comes easily since $M_h^{\#}$ is diagonal. Therefore, we have actually proved that $B_h(\mathbf{m})$ from Theorem 1.8 and $\tilde{B}_h(\mathbf{m})$ from Lemma 5.3 have the same classical expansion.

6. Return to Equilibrium and Metastability

The goal of this section is to prove Corollaries 1.10 and 1.11. We assume that the hypotheses of Theorem 1.8 are satisfied and we choose $\mathbf{m}^*$ among the elements of $\mathcal{U}^{(0)}\backslash\{\underline{\mathbf{m}}\}$ for which S is maximal such that the expansion of $\det(\mathrm{Hess}_{\mathbf{m}^*}V)^{1/2}B_h(\mathbf{m}^*)$ is minimal. According to Lemma 5.3 and Theorem 1.8, one can think of $\lambda_{\mathbf{m}^*,h}$ as the nonzero eigenvalue of P_h with the smallest real part modulo $O(h^\infty\mathrm{e}^{-2S(\mathbf{m}^*)/h})$. We will denote $\mathbb{P}_1$ the orthogonal projection on Ker P_h and for shortness λ^* instead of $\lambda_{\mathbf{m}^*,h}$.

Proof of Corollary 1.10. We follow the proof of Theorem 1.11 in [12]. We have that

$$\|\mathrm{e}^{-tP_h/h} - \mathbb{P}_1\| \leq \|\mathrm{e}^{-tP_h/h}\Pi_0 - \mathbb{P}_1\| + \|\mathrm{e}^{-tP_h/h}(1-\Pi_0)\|.$$

and thanks to Proposition 2.8 and Proposition 2.1 from [8], we easily get

$$\mathrm{e}^{-tP_h/h}(1-\Pi_0) = O(\mathrm{e}^{-cht}).$$

Thus, it suffices for the first statement to prove that

$$\|\mathrm{e}^{-tP_h/h}\Pi_0 - \mathbb{P}_1\| \leq C_N \mathrm{e}^{-\mathrm{Re}\,\lambda^*(1-C_N h^N)t/h}.$$

We recall that thanks to the resolvent estimates from Theorem 1.6, $\Pi_0 = O(1)$ and since $\mathbb{P}_1$ is an orthogonal projection on Ker P_h, we have that

$$\mathrm{e}^{-tP_h/h}\Pi_0 - \mathbb{P}_1 = \mathrm{e}^{-tP_h/h}(\Pi_0 - \mathbb{P}_1)$$

and $(\Pi_0 - \mathbb{P}_1) = O(1)$. Therefore, it is sufficient to prove that

$$\|\mathrm{e}^{-tP_h/h}|_{\mathrm{Ran}(\Pi_0-\mathbb{P}_1)}\| \leq C_N \mathrm{e}^{-\mathrm{Re}\,\lambda^*(1-C_N h^N)t/h}. \tag{3.1}$$

Besides, we saw in Sect. 2 that Ker $P_h = \mathbb{C}\mathcal{M}_h$ where $\mathcal{M}_h$ was defined in (1.2) and since the operator Π_0 from (2.5) satisfies $\Pi_0^*\mathcal{M}_h = \mathcal{M}_h$, we get that $\mathcal{M}_h^\perp$ is invariant under Π_0 so $\mathrm{Ran}(\Pi_0 - \mathbb{P}_1) = H \cap \mathcal{M}_h^\perp$. Thus, with the notations from Proposition 5.9 and according to (3.1), it only remains to show that

$$\|\mathrm{e}^{-t\mathcal{M}'/h}\| \leq C_N \mathrm{e}^{-\mathrm{Re}\,\lambda^*(1-C_N h^N)t/h}.$$

This can be done following the steps of [12], proof of Theorem 1.11 as with the notation (3.12) we have $\mathrm{Re}\,\lambda^* \leq \tilde{\lambda}_{\mathbf{m}^*,h}(1+C_N h^N)$. The only difference is that here we have to apply the resolvent estimates given by Theorem 4 from [1] instead of the ones given by Theorem A.4 from [12]. For the last statement, we now assume that for $\mathbf{m} \in \mathcal{U}^{(0)}\backslash\{\mathbf{m}^*\}$, the expansion of $\lambda(\mathbf{m}, h)$ given by Theorem 1.8 differs from the one of $\lambda^* = \lambda(\mathbf{m}^*, h)$. In that case, it is clear that λ^* is a simple eigenvalue but it also happens to be a real one. Indeed, using the fact that X_0^h and b_h are differential operators with real coefficients and that M^h is real valued and even in the variable η, we get that λ is an eigenvalue of P_h if and only if $\overline{\lambda}$ is an eigenvalue of P_h. The rest of the proof is then also similar to the end of the proof of Theorem 1.11 from [12]. □

Finally, the proof of Corollary 1.11 is a straightforward adaptation of the one of Corollary 1.6 from [1]. (Note that our notations t_k^- and t_k^+ differ from that in [1].)

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Appendix A: Proof of Lemma 1.4

Let us begin by showing that there exists a self-adjoint operator A such that

$$\varrho(H_0) = b_h^* \circ A \circ b_h. \tag{A.1}$$

Since $\varrho(0) = 0$, there exists an analytic function $\tilde{\varrho}$ such that $\varrho(z) = z\tilde{\varrho}(z)$ and $|\tilde{\varrho}(z)| \leq C\langle z\rangle^{-1}$. Using Cauchy's formula, one easily gets that for all $z_0 \in \{\operatorname{Re} z > -\frac{1}{2C}\}$ and f an analytic function on $\{\operatorname{Re} z > -\frac{1}{C}\}$ satisfying $f(z) = O(\langle z\rangle^{-\beta})$ for some $\beta > 0$, we have that

$$f(z_0) = \frac{-1}{2i\pi} \int_{\{\operatorname{Re} z = -\frac{1}{2C}\}} f(z)(z_0 - z)^{-1} \mathrm{d}z. \tag{A.2}$$

Working with a Hilbert basis of eigenfunctions of H_0, this identity yields

$$f(H_0) = \frac{-1}{2i\pi} \int_{\{\operatorname{Re} z = -\frac{1}{2C}\}} f(z)(H_0 - z)^{-1} \mathrm{d}z. \tag{A.3}$$

Besides, denoting

$$b_h = \begin{pmatrix} b_h^1 \\ \vdots \\ b_h^d \end{pmatrix},$$

we have $b_h H_0 = (b_h^j H_0)_{1\leq j\leq d}$ and using the identity $b_h^j H_0 = b_h^* b_h b_h^j + h b_h^j$, we get $b_h H_0 = H_1 b_h$ where

$$H_1 = \begin{pmatrix} H_0 + h & & \\ & \ddots & \\ & & H_0 + h \end{pmatrix}. \tag{A.4}$$

In particular, if u is an eigenfunction of H_0 associated with a positive eigenvalue, the function $b_h u$ is an eigenfunction of H_1 associated with the same eigenvalue and therefore

$$H_0(H_0-z)^{-1} = b_h^*(H_1-z)^{-1}b_h. \tag{A.5}$$

It follows using (A.3) with $f=\tilde{\varrho}$ that (A.1) holds with $A=\tilde{\varrho}(H_0+h)\otimes \mathrm{Id}$:

$$\varrho(H_0) = H_0\tilde{\varrho}(H_0) = b_h^*\circ\tilde{\varrho}(H_0+h)\otimes \mathrm{Id}\circ b_h.$$

We can improve the integrability in the integral representation of $\tilde{\varrho}(H_0+h)$ by writing

$$\tilde{\varrho}(z) = \frac{\tilde{\varrho}(z)}{1+z} + \frac{\varrho(z)-\varrho_\infty}{1+z} + \frac{\varrho_\infty}{1+z}$$

which yields always thanks to (A.3)

$$\begin{aligned}\tilde{\varrho}(H_0+h)\otimes \mathrm{Id} = \frac{-1}{2i\pi}\int_{\{\mathrm{Re}\,z=-\frac{1}{2C}\}}\frac{\tilde{\varrho}(z)}{1+z}(H_1-z)^{-1}\mathrm{d}z \\ +\frac{-1}{2i\pi}\int_{\{\mathrm{Re}\,z=-\frac{1}{2C}\}}\frac{\varrho(z)-\varrho_\infty}{1+z}(H_1-z)^{-1}\mathrm{d}z + \varrho_\infty(H_1+1)^{-1}.\end{aligned} \tag{A.6}$$

Besides, it is well known (see, for instance, [4]) that the resolvent $(H_1-z)^{-1}$ is a pseudo-differential operator and we denote its symbol $R_z(v,\eta)$. Thanks to [3], we even have the explicit expression $R_z(v,\eta)=G_z(v^2/2+2\eta^2)\,\mathrm{Id}$ where G_z is an entire function defined by

$$\begin{aligned}G_z(\mu) &= 2h^{-1}\int_0^1(1-s)^{-\frac{z}{h}}(1+s)^{\frac{z}{h}+d-2}\mathrm{e}^{-\frac{s}{h}\mu}\mathrm{d}s \\ &= 2\int_0^{h^{-1}}(1-h\sigma)^{-\frac{z}{h}}(1+h\sigma)^{\frac{z}{h}+d-2}\mathrm{e}^{-\sigma\mu}\mathrm{d}\sigma.\end{aligned}$$

Let us then set in view of (A.6)

$$\begin{aligned}M^h(v,\eta) = \frac{-1}{2i\pi}\int_{\{\mathrm{Re}\,z=-\frac{1}{2C}\}}\frac{\tilde{\varrho}(z)}{1+z}R_z(v,\eta)\mathrm{d}z + \frac{-1}{2i\pi}\int_{\{\mathrm{Re}\,z=-\frac{1}{2C}\}} \\ \frac{\varrho(z)-\varrho_\infty}{1+z}R_z(v,\eta)\mathrm{d}z + \varrho_\infty R_{-1}(v,\eta)\end{aligned} \tag{A.7}$$

and we now want to show that M^h is a matrix of symbols matching the properties listed in Hypothesis 1.3. To this purpose, we need to study more carefully the function R_z for z fixed such that $\mathrm{Re}\,z\le -1/2C$. We already saw that it is analytic in both variables v and η. Now, if we take $(v,\eta)\in\mathbb{R}^d\times\Sigma_\tau$ and put $\mu=v^2/2+2\eta^2$, we get that μ belongs to the sector

$$D_\tau = \{\mu\in\mathbb{C};\ |\mathrm{Im}\,\mu|\le \mathrm{Re}\,\mu+4d\tau^2\}.$$

One can then easily adapt Theorem 10 from [3] to show that for $n \in \mathbb{N}$ and $\mu \in D_\tau$, we have

$$
\begin{aligned}
|\partial_\mu^n G_z(\mu)| &\leq C \int_0^{h^{-1}} \sigma^n (1-h\sigma)^{-\mathrm{Re}\, z/h}(1+h\sigma)^{\mathrm{Re}\, z/h} \mathrm{e}^{-\mathrm{Re}\, \mu\sigma} \mathrm{d}\sigma \\
&\leq C \int_0^{+\infty} \sigma^n \mathrm{e}^{-(\mathrm{Re}\, \mu - 2\mathrm{Re}\, z)\sigma} \mathrm{d}\sigma \leq C_n \langle \mu \rangle^{-(n+1)}
\end{aligned} \tag{A.8}
$$

since $\mathrm{Re}\, \mu - 2\mathrm{Re}\, z > 0$ for τ small enough. From (A.8) we can already conclude that $M^h \in \mathcal{M}_d\big(S^0_\tau(\langle (v,\eta)\rangle^{-2})\big)$. Thus, $\tilde{\varrho}(H_0 + h) \otimes \mathrm{Id} = \mathrm{Op}_h(M^h)$ with M^h sending $\mathbb{R}^{2d}$ in $\mathcal{M}_d(\mathbb{R})$ as H_0 is self-adjoint. Moreover, since R_z is diagonal and even in the variable η, it is also the case of M^h. It only remains to prove that M^h satisfies items b) and d) from Hypothesis 1.3. In order to avoid some tedious computations, instead of proving the whole expansion from item b), we only show that M^h admits a principal term M_0 in $\mathcal{M}_d\big(S^0_\tau(\langle (v,\eta)\rangle^{-2})\big)$ from which we will deduce that item d) is satisfied. One easily gets for $\mathrm{Re}\, z \leq -1/2C$ and $\mu \in D_\tau$ fixed by dominated convergence that

$$
\lim_{h\to 0} G_z(\mu) = 2\int_0^\infty \mathrm{e}^{\sigma(2z-\mu)} \mathrm{d}\sigma = \frac{1}{\mu/2 - z} =: G^0_z(\mu). \tag{A.9}
$$

We would like to get some estimates of the derivatives $\partial_\mu^n (G_z - G^0_z)$ in $O(h\langle\mu\rangle^{-n-1})$ on D_τ uniformly in $z \in \{\mathrm{Re}\, z \leq -1/2C\}$ in order to apply the formula (A.7) to those. We have

$$
\begin{aligned}
\partial_\mu^n (G_z - G^0_z)(\mu) &= 2\int_0^{h^{-1}} \\
&\left[\exp\left(z\left[\frac{1}{h}\ln\left(\frac{1+h\sigma}{1-h\sigma}\right) - 2\sigma\right] + (d-2)\ln(1+h\sigma)\right) - 1\right](-\sigma)^n \mathrm{e}^{\sigma(2z-\mu)}\mathrm{d}\sigma \\
&\quad - 2\int_{h^{-1}}^\infty (-\sigma)^n e^{\sigma(2z-\mu)} \mathrm{d}\sigma \\
&= 2\int_0^{h^{-1}/2} \left[\exp\left(z\left[\frac{1}{h}\ln\left(\frac{1+h\sigma}{1-h\sigma}\right) - 2\sigma\right] + (d-2)\ln(1+h\sigma)\right) - 1\right] \\
&\quad (-\sigma)^n \mathrm{e}^{\sigma(2z-\mu)}\mathrm{d}\sigma + O\Big(\mathrm{e}^{\frac{\mathrm{Re}\,(2z-\mu)}{Ch}}\Big).
\end{aligned} \tag{A.10}
$$

Let us denote

$$
g_{z,h}(\sigma) = \left[\exp\left(z\left[\frac{1}{h}\ln\left(\frac{1+h\sigma}{1-h\sigma}\right) - 2\sigma\right] + (d-2)\ln(1+h\sigma)\right) - 1\right](-\sigma)^n
$$

and observe that for all $0 \leq k \leq n$, one has

$$
\partial_\sigma^k g_{z,h}(0) = 0 \qquad \text{and} \qquad \partial_\sigma^k g_{z,h}(h^{-1}/2) = O(h^{-n}\langle z\rangle^k). \tag{A.11}
$$

Besides, on $\sigma \in [0, h^{-1}/2]$, it holds

$$
\partial_\sigma^{n+1} g_{z,h}(\sigma) = \sum_{j=1}^{n+1} O\big(h\langle z\rangle^j \langle\sigma\rangle^j \sigma^{j-1}\big). \tag{A.12}
$$

Now, let us do $n+1$ integrations by parts in the first term from (A.10). By (A.11), each boundary term is $O(h^{-n}\langle z\rangle^k\langle 2z-\mu\rangle^{-(k+1)}\mathrm{e}^{\mathrm{Re}\,(2z-\mu)/Ch})$, while the remaining integral term satisfies

$$\begin{aligned}&\left|\frac{2}{(\mu-2z)^{n+1}}\int_0^{h^{-1}/2}\partial_\sigma^{n+1}g_{z,h}(\sigma)\mathrm{e}^{\sigma(2z-\mu)}\mathrm{d}\sigma\right|\\&\quad\le C_n h\sum_{j=1}^{n+1}\frac{\langle z\rangle^j}{|2z-\mu|^{n+1}}\int_0^\infty\sigma^{j-1}\langle\sigma\rangle^j\mathrm{e}^{\sigma\,\mathrm{Re}(2z-\mu)}\mathrm{d}\sigma\\&\quad\le C_n h\langle\mu\rangle^{-(n+1)}\end{aligned}$$

thanks to (A.12). Thus, we have shown that for $n\in\mathbb{N}$, $\mu\in D_\tau$ and $\mathrm{Re}\,z\le -1/2C$,

$$|\partial_\mu^n(G_z-G_z^0)(\mu)|\le C_n h\langle\mu\rangle^{-(n+1)}.$$

Putting $R_z^0(v,\eta)=G_z^0(v^2/2+2\eta^2)\,\mathrm{Id}$ and defining $M_0(v,\eta)$ as in (A.7) with R_z replaced by R_z^0, we deduce that

$$|\partial^\alpha(M^h-M_0)(v,\eta)|\le C_\alpha h\langle(v,\eta)\rangle^{-2}\qquad\text{on }\mathbb{R}^d\times\Sigma_\tau$$

so item b) from Hypothesis 1.3 holds true. Finally, by definition of M_0 and thanks to (A.9) and (A.2), we have

$$M_0(v,\eta)=\tilde{\varrho}\big(v^2/4+\eta^2\big)\,\mathrm{Id}\ge\frac{1}{C}\langle(v,\eta)\rangle^{-2}\,\mathrm{Id}\tag{A.13}$$

by assumption on ϱ. Therefore, item d) from Hypothesis 1.3 holds true and the proof is complete.

Appendix B: Linear Algebra Lemma

We use the following lemma which is inspired by [1], Lemma 2.6.

Lemma B.1. *Let $M\in\mathcal{M}_{d'}(\mathbb{C})$ such that $M=S(A+T)$ with S Hermitian and invertible, A skew-Hermitian and T Hermitian positive semidefinite. Suppose moreover that*

$$M(\mathrm{Ker}\,T)\cap\mathrm{Ker}\,T=\mathrm{Ker}\,M\cap\mathrm{Ker}\,T=\{0\}.$$

Then, M has no spectrum in $i\mathbb{R}$.

Proof. Let $\lambda\in\mathbb{R}$ and $X\in\mathrm{Ker}\,[M-i\lambda]$, we first show that $X\in\mathrm{Ker}\,T$. Since T is Hermitian positive semidefinite, it is sufficient to show that $\langle TX,X\rangle=0$. Using the properties of S, A and T, we have

$$\begin{aligned}\langle TX,X\rangle&=\mathrm{Re}\,\big\langle(A+T)X,X\big\rangle\\&=\mathrm{Re}\,\big\langle S^{-1}S(A+T)X,X\big\rangle\\&=\mathrm{Re}\,\big(i\lambda\big\langle S^{-1}X,X\big\rangle\big)\\&=0\end{aligned}$$

so $X \in \operatorname{Ker} T$. Thanks to the assumption, it only remains to prove that $X \in \operatorname{Ker} M$. This can be done easily by noticing that

$$MX = i\lambda X \in M(\operatorname{Ker} T) \cap \operatorname{Ker} T$$

so $MX = 0$ by assumption.

Appendix C: Asymptotic Expansions

Let $d' \in \mathbb{N}^*$. Here, we use the convention $\sum_{j=0}^{-1} a_j = 0$ for any sequence $(a_j)_{j\geq 0}$ in a vector space. For $K \subseteq \mathbb{R}^{d'}$, the notation $a = O_{\mathcal{C}^\infty(K)}(h^N)$ (respectively, $a = O_{L^\infty(K)}(h^N)$) means that for all $\beta \in \mathbb{N}^{d'}$, there exists $C_{\beta,N}$ such that $\|\partial^\beta a\|_{\infty,K} \leq C_{\beta,N} h^N$ (resp. there exists C_N such that $\|a\|_{\infty,K} \leq C_N h^N$). We will also use the notations from Definition 1.2 and (1.6).

Proposition C.1. *Let $m \in \mathbb{N}^*$; $d_1, \ldots, d_m \in \mathbb{N}^*$ and for $1 \leq j \leq m$, $K_j \subset \mathbb{R}^{d_j}$ some compact sets. Let a smooth function*

$$\phi_h : \prod_{j=1}^m K_j \to K \subset \Sigma_\tau$$

such that $\phi_h = O_{\mathcal{C}^\infty(\prod_{j=1}^m K_j)}(1)$. Consider $g^h \sim_h \sum_{n\geq 0} h^n g_n$ in $S^0_\tau(1)$ or in $\mathcal{C}^\infty(K)$ if ϕ_h actually takes values in $\mathbb{R}^d$. Then,

$$g^h \circ \phi_h \sim_h \sum_{n\geq 0} h^n (g_n \circ \phi_h)$$

in $\mathcal{C}^\infty(\prod_{j=1}^m K_j)$.

Proof. Let $N \in \mathbb{N}$ and denote $r_N = g^h - \sum_{n=0}^{N-1} h^n g_n = O_{S^0_\tau(1)}(h^N)$.

$$\begin{aligned} g^h \circ \phi_h &= \left(\sum_{n=0}^{N-1} h^n g_n + r_N \right) \circ \phi_h \\ &= \sum_{n=0}^{N-1} h^n (g_n \circ \phi_h) + r_N \circ \phi_h. \end{aligned}$$

But since all the derivatives of ϕ_h are bounded uniformly in h, and the ones of r_N are $O_{L^\infty(\Sigma_\tau)}(h^N)$, we see that $r_N \circ \phi_h$ is $O_{\mathcal{C}^\infty(\prod_{j=1}^m K_j)}(h^N)$ so we have the announced result.

Proposition C.2. *Since the matrix M^h from Hypothesis 1.3 satisfies $M^h \sim \sum_{n\geq 0} h^n M_n$ in $\mathcal{M}_d\big(S^0_\tau(\langle(v,\eta)\rangle^{-2})\big)$, the vector of symbols g^h defined in Remark 3.12 also admits a classical expansion $g^h \sim \sum_{n\geq 0} h^n g_n$ in $\mathcal{M}_{1,d}\big(S^0_\tau(\langle(v,\eta)\rangle^{-1})\big)$, where the (g_n) are given by*

$$g_0(x,v,\eta) = \Big(-i\,{}^t\eta + \frac{{}^tv}{2} \Big) M_0(x,v,\eta)$$

and

$$g_n(x,v,\eta) = \left(-i\,{}^t\eta + \frac{{}^tv}{2}\right) M_n(x,v,\eta) - \frac{1}{2}\left({}^t\nabla_v - \frac{i}{2}{}^t\nabla_\eta\right) M_{n-1}(x,v,\eta)$$

for $n \geq 1$.

Proof. We have

$$g^h = (-i\,{}^t\eta + {}^tv/2)M^h - \frac{h}{2}\left({}^t\nabla_v - \frac{i}{2}{}^t\nabla_\eta\right) M^h$$

, and the last term clearly admits the expansion

$$-\sum_{n\geq 1} h^n \frac{1}{2}\left({}^t\nabla_v - \frac{i}{2}{}^t\nabla_\eta\right) M_{n-1}$$

in $S^0_\tau(\langle(v,\eta)\rangle^{-2})$. For the first term of g^h, it suffices to notice that for any $N \in \mathbb{N}$,

$$\left(-i\,{}^t\eta + \frac{{}^tv}{2}\right) O_{\mathcal{M}_d\left(S^0_\tau(\langle(v,\eta)\rangle^{-2})\right)}(h^N) = O_{\mathcal{M}_{1,d}\left(S^0_\tau(\langle(v,\eta)\rangle^{-1})\right)}(h^N).$$

Proposition C.3. *Let K a compact set in $\mathbb{R}^{d'}$ and $a \sim_h \sum_{n\geq 0} h^n a_n$ in $\mathcal{C}^\infty(K)$ such that for all $n \geq 0$, $a_n \sim_h \sum_{j\geq 0} h^j a_{n,j}$ in $\mathcal{C}^\infty(K)$. Then,*

$$a \sim_h \sum_{n\geq 0} h^n \sum_{j=0}^{n} a_{j,n-j} \quad \text{in } \mathcal{C}^\infty(K).$$

Proof. It suffices to write for $N \in \mathbb{N}$

$$\begin{aligned} a &= \sum_{n=0}^{N-1} h^n \left(\sum_{j=0}^{N-1-n} h^j a_{n,j} + O_{\mathcal{C}^\infty(K)}(h^{N-n})\right) + O_{\mathcal{C}^\infty(K)}(h^N) \\ &= \sum_{n=0}^{N-1} h^n \sum_{j=0}^{n} a_{j,n-j} + O_{\mathcal{C}^\infty(K)}(h^N). \end{aligned}$$

Proposition C.4. *Let K a compact set in $\mathbb{R}^{d'}$ and $a \in \mathcal{C}^\infty(K)$ such that for all $\beta \in \mathbb{N}^{d'}$, there exists $a_{\beta,j} \in \mathcal{C}^\infty(K)$ such that $\partial^\beta a \sim \sum_{j\geq 0} h^j a_{\beta,j}$ in $L^\infty(K)$. Then, $a_{\beta,j} = \partial^\beta a_{0,j}$, i.e.,*

$$a \sim \sum_{j\geq 0} h^j a_{0,j} \quad \text{in } \mathcal{C}^\infty(K).$$

Proof. For simplicity, we take $d' = 1$. Let us denote $a_j = a_{0,j}$. By induction, it is sufficient to prove the result for $\beta = 1$, i.e., prove that $a_{1,j} = a'_j$. Here again, it suffices to prove the case $j = 0$ which we can then apply to the function $h^{-1}(a - a_0)$ and so on. Let x in the interior of K and $t \in \mathbb{R}^*$ in a neighborhood of 0. We look at the differential fraction

$$\begin{aligned}\frac{a_0(x+t)-a_0(x)}{t} &= \frac{a(x+t)-a(x)}{t} + \frac{O(h)}{t} \\ &= a'(x) + t\int_0^1 (1-s)a''(x+st)\mathrm{d}s + \frac{O(h)}{t} \\ &= a_{1,0}(x) + O(h) + t\int_0^1 (1-s)a''(x+st)\mathrm{d}s + \frac{O(h)}{t} \\ &\xrightarrow[h\to 0]{} a_{1,0}(x) + t\int_0^1 (1-s)a_{2,0}(x+st)\mathrm{d}s.\end{aligned}$$

Taking now the limit $t \to 0$, we get $a_0'(x) = a_{1,0}(x)$ which was the desired result.

Proposition C.5. *Recall the notation* (3.11) *and let* $K \subset \mathbb{R}^{d'}$ *a compact set,* $\Psi : K \to D(0,\tau)^d$ *a smooth function such that* $\Psi \sim \sum_{j\geq 0} h^j \Psi_j$ *in* $\mathcal{C}^\infty(K)$ *and* b *an analytic function on* Σ_τ*. Then,*

$$b \circ \Psi \sim \sum_{j\geq 0} h^j b_j \tag{C.1}$$

in $\mathcal{C}^\infty(K)$*, with*

$$b_0 = b\circ\Psi_0 \quad \text{and for } j\geq 1, \quad b_j = \sum_{|\beta|=1}^{j} \frac{\partial^\beta b\circ\Psi_0}{\beta!} \sum_{s\in S_{\beta,j}} \prod_{k\in K_\beta} \left(\sum_{a\in A_{\beta,s,k}} \prod_{l=1}^{\beta_k} (\Psi_{a_l})_k \right),$$

where $K_\beta = supp\ \beta = \{k \in [\![1,d]\!]\,;\, \beta_k \neq 0\}$, $S_{\beta,j} = \{s \in \mathbb{N}^d\,;\, \operatorname{supp} s = K_\beta,\ |s| = j \text{ and } s \geq \beta\}$ *and* $A_{\beta,s,k} = \{a \in (\mathbb{N}^*)^{\beta_k}\,;\, |a| = s_k\}$.

Proof. We first prove that (C.1) holds in $L^\infty(K)$. Doing a Taylor expansion of b, we have for $N \in \mathbb{N}^*$ that

$$\begin{aligned}b\circ\Psi &= b\circ\Psi_0 + \sum_{|\beta|=1}^{N-1} \frac{\partial^\beta b\circ\Psi_0}{\beta!}(\Psi-\Psi_0)^\beta + O\Big((\Psi-\Psi_0)^N\Big) \\ &= b\circ\Psi_0 + \sum_{|\beta|=1}^{N-1} \frac{\partial^\beta b\circ\Psi_0}{\beta!}(\Psi-\Psi_0)^\beta + O_{L^\infty(K)}(h^N) \qquad (\text{C.2})\end{aligned}$$

since $\Psi - \Psi_0 = O_{\mathcal{C}^\infty(K)}(h)$. Now, one can see that

$$(\Psi-\Psi_0)^\beta \sim \sum_{j\geq|\beta|} h^j \sum_{s\in S_{\beta,j}} \prod_{k\in K_\beta} \left(\sum_{a\in A_{\beta,s,k}} \prod_{l=1}^{\beta_k} (\Psi_{a_l})_k \right)$$

so (C.2) gives

$$\begin{aligned}b\circ\Psi &= b\circ\Psi_0 + \sum_{|\beta|=1}^{N-1} \frac{\partial^\beta b\circ\Psi_0}{\beta!}\left[\sum_{j=|\beta|}^{N-1} h^j \sum_{s\in S_{\beta,j}} \prod_{k\in K_\beta} \left(\sum_{a\in A_{\beta,s,k}} \prod_{l=1}^{\beta_k} (\Psi_{a_l})_k \right) + O_{\mathcal{C}^\infty(K)}(h^N) \right] \\ &\quad + O_{L^\infty(K)}(h^N) \\ &= b\circ\Psi_0 + \sum_{j=1}^{N-1} h^j \sum_{|\beta|=1}^{j} \frac{\partial^\beta b\circ\Psi_0}{\beta!} \sum_{s\in S_{\beta,j}} \prod_{k\in K_\beta} \Big(\sum_{a\in A_{\beta,s,k}} \prod_{l=1}^{\beta_k} (\Psi_{a_l})_k \Big) + O_{L^\infty(K)}(h^N)\end{aligned}$$

which proves that (C.1) holds in $L^\infty(K)$.

Besides, the derivatives of $b \circ \Psi$ are linear combinations of products of some derivatives of Ψ with some $\partial^\gamma b \circ \Psi$ where γ is a integer multi-index. Hence, the expansion of Ψ in $\mathcal{C}^\infty(K)$ and the result that we just proved applied to $\partial^\gamma b \circ \Psi$ instead of $b \circ \Psi$ yield that for all $\beta \in \mathbb{N}^{d'}$, $\partial^\beta(b \circ \Psi)$ admits a classical expansion in $L^\infty(K)$ whose coefficients are smooth. Therefore, Proposition C.4 enables us to conclude that (C.1) holds in $\mathcal{C}^\infty(K)$.

Corollary C.6. *Using the notations from the proof of Lemma 4.1, we have*

$$g_n\Big(x, \frac{v+v'}{2}, \eta + i\psi(x,v,v')\Big) \sim \sum_{j\geq 0} h^j g_{n,j}(x,v,v',\eta) \quad \textit{on } B_0(\mathbf{s},2r) \times B_\infty(0,2r)$$

with

$$g_{n,0}(x,v,v',\eta) = g_n\Big(x, \frac{v+v'}{2}, \eta + i\psi_0(x,v,v')\Big)$$

and for $j \geq 1$

$$g_{n,j}(x,v,v',\eta) = iD_\eta g_n\Big(x, \frac{v+v'}{2}, \eta + i\psi_0(x,v,v')\Big)(\psi_j(x,v,v')) + R_j^1(\ell_0,\dots,\ell_{j-1})$$

where $R_j^1 : \big(\mathcal{C}^\infty(B_0(\mathbf{s},2r))\big)^j \to \mathcal{C}^\infty(B_0(\mathbf{s},2r))$.

Proof. Since $\psi(\mathbf{s},0,0) = O(h)$, we can suppose that r was chosen small enough so that $(x,v,v',\eta) \mapsto \eta + i\psi(x,v,v')$ sends $B_0(\mathbf{s},2r) \times B_\infty(0,2r)$ in $D(0,\tau)^d$ Hence, we can use Proposition C.5 to get that

$$g_n\Big(x, \frac{v+v'}{2}, \eta + i\psi(x,v,v')\Big) \sim \sum_{j\geq 0} h^j g_{n,j}(x,v,v',\eta) \quad \text{on } B_0(\mathbf{s},2r) \times B_\infty(0,2r)$$

with

$$g_{n,0}(x,v,v',\eta) = g_n\Big(x, \frac{v+v'}{2}, \eta + i\psi_0(x,v,v')\Big)$$

and for $j \geq 1$

$$g_{n,j}(x,v,v',\eta) = \sum_{|\beta|=1}^{j} \frac{i^{|\beta|}}{\beta!} \partial_\eta^\beta g_n\Big(x, \frac{v+v'}{2}, \eta + i\psi_0(x,v,v')\Big) \sum_{s\in S_{\beta,j}} \prod_{k\in K_\beta}\Bigg(\sum_{a\in A_{\beta,s,k}} \prod_{l=1}^{\beta_k} (\psi_{a_l})_k\Bigg) \tag{C.3}$$

where $K_\beta = \operatorname{supp} \beta = \{k \in [\![1,d]\!]\,;\, \beta_k \neq 0\}$, $S_{\beta,j} = \{s \in \mathbb{N}^d\,;\, \operatorname{supp} s = K_\beta,\ |s| = j \text{ and } s \geq \beta\}$ and $A_{\beta,s,k} = \{a \in (\mathbb{N}^*)^{\beta_k}\,;\, |a| = s_k\}$. Now, we see thanks to (C.3) that the terms of $g_{n,j}(x,v,v',\eta)$ for which $|\beta| = 1$ yield

$$iD_\eta g_n\Big(x, \frac{v+v'}{2}, \eta + i\psi_0(x,v,v')\Big)(\psi_j(x,v,v')),$$

while the terms for which $|\beta| > 1$ only feature the functions $\ell_0,\dots,\ell_{j-1}$.

Finally, we state the version of Laplace's method for integral approximation that we use in this paper.

Proposition C.7. *Let $x_0 \in \mathbb{R}^{d'}$, K be a compact neighborhood of x_0 and $\varphi \in \mathcal{C}^\infty(K)$ such that x_0 is a non-degenerate minimum of φ and its only global minimum on K. Let also $a_h \sim \sum_{j\geq 0} h^j a_j$ in $\mathcal{C}^\infty(K)$ and denote $H \in \mathcal{M}_{d'}(\mathbb{R})$ the Hessian of φ at x_0. The integral*

$$\frac{\det(H)^{1/2}}{(2\pi h)^{d'/2}} \int_K a_h(x) \mathrm{e}^{-\frac{\varphi(x)-\varphi(x_0)}{h}} \mathrm{d}x$$

admits a classical expansion whose first term is given by $a_0(x_0)$.

Appendix D: Proof of Lemma 4.3

According to the proof of Corollary C.6 and the end of the proof of Lemma 4.1 from which we keep the notations, we have the following expression for R_j:

$$\begin{aligned} &R_j(\ell_0, \dots, \ell_{j-1})(x,v) \\ &\quad = \sum_{\substack{n_1+n_2+n_3+n_4=j \\ n_3,n_4\neq j}} \frac{1}{i^{n_1} n_1!} \big(\partial_{v'} \cdot \partial_\eta\big)^{n_1} \Big(g_{n_2,n_3}(x,v,v',\eta)\partial_v \ell_{n_4}(x,v')\Big)\Big|_{\substack{v'=v \\ \eta=0}} \\ &\qquad + \sum_{|\beta|=2}^{j} \frac{i^{|\beta|}}{\beta!} \partial_\eta^\beta g_0\Big(x, \frac{v+v'}{2}, i\big(\frac{v}{2} + \ell_0(x,v)\,\partial_v \ell_0(x,v)\big)\Big) \\ &\qquad \times \sum_{s\in S_{\beta,j}} \prod_{k\in K_\beta} \Big(\sum_{a\in A_{\beta,s,k}} \prod_{l=1}^{\beta_k} \big(\psi_{a_l}(x,v,v)\big)_k \Big) \partial_v \ell_0(x,v) \\ &\qquad + iD_\eta g_0\big(x,v,i(v/2+\ell_0(x,v)\,\partial_v\ell_0(x,v))\big) \sum_{k=1}^{j-1} \big(\ell_k \partial_v \ell_{j-k}\big)(x,v)\,\partial_v\ell_0(x,v). \end{aligned} \tag{D.1}$$

Using Lemma 4.2 and (C.3), it is clear that the last two terms of $R_j(\ell_0, \cdots, \ell_{j-1})$ given by (D.1) and the terms of the first sum for which $n_1 = 0$ are real valued. For the rest of the first term, we start by noticing that one can establish by induction that for $n_1 \geq 1$,

$$\big(\partial_{v'} \cdot \partial_\eta\big)^{n_1} = \sum_{p\in [\![1,d]\!]^{n_1}} \partial_{v'}^{\gamma(p)} \partial_\eta^{\gamma(p)} \tag{D.2}$$

where using the notation (3.22), we define $\gamma(p) = \sum_{k=1}^{n_1} e_{p_k}$ (note that $|\gamma(p)| = n_1$). Besides, we have for $0 \leq n_2 \leq j$ and $p \in [\![1,d]\!]^{n_1}$

$$\partial_\eta^{\gamma(p)} g_{n_2,0}(x,v,v',0) = \partial_\eta^{\gamma(p)} g_{n_2}\Big(x, \frac{v+v'}{2}, i\psi_0(x,v,v')\Big) \in i^{n_1}\mathbb{R}^d \tag{D.3}$$

according to Lemma 4.2 and in the case $j \geq 2$, for $1 \leq n_3 \leq j-1$

$$\begin{aligned} &\partial_\eta^{\gamma(p)} g_{n_2,n_3}(x,v,v',0) \\ &\quad = \sum_{|\beta|=1}^{n_3} \frac{i^{|\beta|}}{\beta!} \partial_\eta^{\beta+\gamma(p)} g_{n_2}\Big(x, \frac{v+v'}{2}, i\psi_0(x,v,v')\Big) \sum_{s\in S_{\beta,n_3}} \prod_{k\in K_\beta} \\ &\quad \Big(\sum_{a\in A_{\beta,s,k}} \prod_{l=1}^{\beta_k} (\psi_{a_l})_k \Big) \in i^{n_1}\mathbb{R}^d \end{aligned} \tag{D.4}$$

where we used (C.3) and Lemma 4.2 once again. The combination of (D.2), (D.3) and (D.4) enables us to conclude that the term

$$\sum_{\substack{n_1+n_2+n_3+n_4=j \\ n_1\neq 0;\, n_3,n_4\neq j}} \frac{1}{i^{n_1} n_1!} (\partial_{v'}\cdot\partial_\eta)^{n_1} \Big(g_{n_2,n_3}(x,v,v',\eta)\partial_v \ell_{n_4}(x,v')\Big)\Big|_{\substack{v'=v\\ \eta=0}}$$

from (D.1) is also real so $R_j(\ell_0^{\mathbf{s}},\ldots,\ell_{j-1}^{\mathbf{s}})$ is real valued. For the last statement, it suffices to use the formula (D.1) after noticing that ψ (and hence the (g_{n_2,n_3})) remain unchanged when ℓ is replaced by $-\ell$.

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